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Solutions to

Exam in Analytichal Mechanics, 5p

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Solutions are also available on
http://www.physto.se/~edsjo/teaching/am/index.html.

Problem 1

a) The tensor of inertia is given by

$$\vec{\vec{I}} = \int \left[\vec{x} \cdot \vec{x} - \vec{x}\vec{x}\right] \rho(\vec{x}) d^3x$$

where $\vec{x} \cdot \vec{x}$ is a usual scalar product and $\vec{x}\vec{x}$ is a dyadic product. In a cartesian coordinate system, the components are given by

$$I_{ij} = \int \left[\vec{x}^2 \delta_{ij} - x_i x_j\right] \rho(\vec{x}) d^3x. \tag{1}$$

b) According to Eq. (1), I_{xz} is given by

$$I_{xz} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, xz\rho(\vec{x}).$$

Since the body is mirror symmetric in the xy-plane, we have that $\rho(x, y, -z) = \rho(x, y, z)$. Hence, we can write I_{xz} as

$$I_{xz} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[\int_{-\infty}^{0} dz \, xz\rho(x, y, z) + \int_{0}^{\infty} dz \, xz\rho(x, y, z)\right]$$

Now change $(x, y, z) \to (x, y, z') = (x, y, -z)$ in the first integral,

$$\begin{aligned} I_{xz} &= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[\int_{\infty}^{0} dz' \, xz' \underbrace{\rho(x, y, -z')}_{\rho(x,y,z')} + \int_{0}^{\infty} dz \, xz\rho(x, y, z)\right] \\ &= -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \left[-\int_{0}^{\infty} dz' \, xz'\rho(x, y, z') + \int_{0}^{\infty} dz \, xz\rho(x, y, z)\right] = 0 \end{aligned}$$

i.e. for each volume element at (x, y, z) there is a volume element at $(x, y, -z)$ that contributes as much to I_{xz}, but with opposite sign. $I_{zx} = 0$ since $\vec{\vec{I}}$ is symmetric.
That $I_{yz} = I_{zy} = 0$ is shown in the same way.

c) According to Eq. (1) we have that

$$\begin{aligned} I_{zz} &= \int (x^2 + y^2)\rho(\vec{x})d^3x \\ &\leq \int (z^2 + x^2 + y^2 + z^2)\rho(\vec{x})d^3x \\ &= \int (z^2 + x^2)\rho(\vec{x})d^3x + \int (y^2 + z^2)\rho(\vec{x})d^3x \\ &= I_{yy} + I_{xx}, \end{aligned}$$

i.e. $I_{zz} \leq I_{xx} + I_{yy}$, which is exactly what we should prove.
Equality holds when

$$\int z^2 \rho(\vec{x}) d^3x = 0,$$

i.e. when the body does not have any extension in the z direction, i.e. is a thin plate in the xy-plane.

Uppgift 2

a) Problemet har en frihetsgrad och vi kan välja massan ms avstånd r från rotationsaxeln som generaliserad koordinat. Den kinetiska och potentiella energin ges då av

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega_0^2\right) \quad ; \quad U = \frac{1}{2}k\left(r - b\right)^2 = \frac{1}{2}k\left(r^2 + b^2 - 2br\right)$$

Lagrangianen ges således av

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega_0^2\right) - \frac{1}{2}k\left(r^2 + b^2 - 2br\right)$$

Dess derivator är

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{d}{dt}\left(m\dot{r}\right) = m\ddot{r} \\ \frac{\partial L}{\partial r} &= m\omega_0^2 r + kb - kr = (m\omega_0^2 - k)r + kb \end{cases}$$

Lagranges ekvationer ger då rörelseekvationerna

$$\ddot{r} + \left(\frac{k}{m} - \omega_0^2\right) r = \frac{kb}{m} \tag{2}$$

b) Rörelseekvationen, Ekv. (2), har den allmänna lösningen

$$r(t) = r_h(t) + r_p(t)$$

där $r_h(t)$ är lösningen till den homogena ekvationen (med högerledet lika med noll) och $r_p(t)$ är partikulärlösningen för det högerled rörelseekvationen har. Partikulärlösningen ges av

$$r_p(t) = \frac{kb}{m\left(\frac{k}{m} - \omega_0^2\right)}$$

För den homogena ekvationen får vi olika typer av lösningar beroende på tecknet på koefficienten $\frac{k}{m} - \omega_0^2$:

$$\begin{cases} \frac{k}{m} - \omega_0^2 > 0 & \Rightarrow \text{Oscillerande cos- och sin-lösningar} \\ \frac{k}{m} - \omega_0^2 < 0 & \Rightarrow \text{Exponentiellt växande cosh- och sinh-lösningar} \\ \frac{k}{m} - \omega_0^2 = 0 & \Rightarrow \text{Linjärt växande/avtagande (eller konstanta) lösningar} \end{cases}$$

c) Partikulärlösningen ges i detta fall av

$$r_p(t) = 2b$$

Den homogena ekvationen ges av

$$\ddot{r} + \omega_0^2 r = 0$$

vilken har lösningen

$$r_h(t) = A\sin(\omega_0 t + \beta) \quad ; \quad A, \beta = \text{konstanter}$$

vilket ger den fullständiga lösningen

$$r(t) = A\sin(\omega_0 t + \beta) + 2b.$$

Begynnelsevillkoret $\dot{r}(0) = 0$ ger att $\beta = \pi/2$ medan $r(0) = b$ ger att $A = -b$. Lösningen ges således av

$$r(t) = 2b - b\sin\left(\omega_0 t + \frac{\pi}{2}\right) = 2b - b\cos\omega_0 t.$$

Massan m utför med andra ord harmoniska oscillationer runt jämviktsläget $2b$ med amplituden b.

Problem 3

We will now write down the kinetic and potential energy and use Lagrange's equations to get the equations of motion. Since we are interested in small oscillations, we can Taylor expand these expressions and only keep the terms of lowest order.
The problem has two degrees of freedom and we choose x and y as our generalized coordinates. The kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

where $\dot{z}$ is given by

$$\dot{z} = -c\frac{1}{2}\frac{1}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\left(-\frac{2x\dot{x}}{a^2} - \frac{2y\dot{y}}{b^2}\right)$$

If we Taylor expand the square root expression according to

$$\frac{1}{\sqrt{1-\epsilon}} \simeq 1 + \frac{1}{2}\epsilon$$

we see that $\dot{z}^2$ will contain x, y, $\dot{x}$ and $\dot{y}$ (which are all small) in order four and higher, i.e. in higher orders than the other terms in T. We can hence neglect $\dot{z}$ in T for small oscillations and arrive at

$$T \simeq \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right).$$

The potential energy is given by

$$U = mgz = mgc - mgc\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}},$$

which, Taylor expanded to order two in small terms, is

$$U \simeq = mgc - mgc\left(1 - \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \cdots\right)\right) \simeq \frac{1}{2}mgc\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$

Our Lagrangain (to order two in small terms) is thus

$$L = T - U \simeq \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}mgc\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$

The derivatives of the Lagrangian are

$$\begin{cases} \frac{\partial L}{\partial x} &= -\frac{mgc}{a^2}x \\ \frac{\partial L}{\partial y} &= -\frac{mgc}{b^2}y \end{cases} \quad ; \quad \begin{cases} \frac{\partial L}{\partial \dot{x}} &= m\dot{x} \\ \frac{\partial L}{\partial \dot{y}} &= m\dot{y} \end{cases}$$

Inserted into Lagrange's equations, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$ we get the equations of motion,

$$\begin{cases} m\ddot{x} + \frac{mgc}{a^2}x &= 0 \\ m\ddot{y} + \frac{mgc}{b^2}y &= 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x} + \frac{gc}{a^2}x &= 0 \\ \ddot{y} + \frac{gc}{b^2}y &= 0 \end{cases}$$

We recognize these as the usual equations for harmonic oscillators and the angular frequencies can be read-off directly,

$$\begin{cases} \omega_x &= \sqrt{\frac{gc}{a^2}} \\ \omega_y &= \sqrt{\frac{gc}{b^2}} \end{cases}$$

To check if these expressions are reasonable, we let $a = b = c = l$ and recover the usual angular frequency for a planar mathematical pendulum with length l, i.e. what we should get.

Problem 4

Hamilton's principle says that the functional

$$I[\underset{\sim}{q}] = \int_{t_1}^{t_2} L(\underset{\sim}{q}(t), \dot{\underset{\sim}{q}}(t), t)dt$$

assumes an extremum when q describes the actual motion of the system. This means that we can require that the variation of I is zero and from that derive the equations of motion. Perform a small variation of $\underset{\sim}{q}$ around the solution which makes $\delta I = 0$. This variation can be parameterized as

$$\underset{\sim}{q}(t, \alpha) = \underset{\sim}{q}(t) + \alpha\underset{\sim}{\eta}(t)$$

where $\underset{\sim}{\eta}(t)$ is a set of arbitrary functions who are zero at the endpoints and α is a parameter that describes how far we are from the solution $\underset{\sim}{q}$ that gives $\delta I = 0$. The variation of I is then given by

$$
\begin{aligned}
\delta I &\equiv \frac{dI}{d\alpha}d\alpha = \int_{t_1}^{t_2} dt \sum_{k=1}^{f} \left(\frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial \alpha} \right) d\alpha = \left\{ \begin{array}{c} \text{Partial} \\ \text{integration} \\ \text{of the 2nd term} \end{array} \right\} \\
&= \int_{t_1}^{t_2} dt \sum_{k=1}^{f} \frac{\partial L}{\partial q_k}\frac{\partial q_k}{\partial \alpha} d\alpha + \underbrace{\left[\sum_{k=1}^{f} \frac{\partial L}{\partial \dot{q}_k} \underbrace{\frac{\partial q_k}{\partial \alpha}}_{\eta_k(t)} d\alpha \right]_{t_1}^{t_2}}_{=0 \text{ since } \eta_k(t_1) = \eta_k(t_2) = 0} - \int_{t_1}^{t_2} dt \sum_{k=1}^{f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \frac{\partial q_k}{\partial \alpha} d\alpha \\
&= \int_{t_1}^{t_2} dt \sum_{k=1}^{f} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \underbrace{\frac{\partial q_k}{\partial \alpha}}_{\eta_k(t)} d\alpha
\end{aligned}
$$

We now require that $\delta I = 0$. Since $\eta_k(t)$ are arbitrary functions, each term (the part in the square brackets) in the sum has to be zero individually[1]. We then finally arrive at

$$
\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0 \qquad ; \qquad \forall\, k = 1, \ldots, f
$$

which are the sought Lagrange's equations.

Uppgift 5

a) For a generating function of the type U we have that

$$
q_i = -\frac{\partial U}{\partial p_i} \quad ; \quad P_j = -\frac{\partial U}{\partial Q_j} \quad ; \quad \tilde{H} = H + \frac{\partial U}{\partial t}.
$$

We want the new Hamiltonian, $\tilde{H}$, to be identically equal to zero, i.e. that

$$
H(\underset{\sim}{q}, \underset{\sim}{p}, t) + \frac{\partial U}{\partial t} = 0
$$

Use that $q_i = -\frac{\partial U}{\partial p_i}$ and we get

$$
H(-\frac{\partial U}{\partial \underset{\sim}{p}_i}, \underset{\sim}{p}, t) + \frac{\partial U}{\partial t} = 0
$$

which is the Hamilton-Jacobi equation in the momentum representation. This is a partial differential equation for U with respect to $\underset{\sim}{p}$ and t.

b) With the given Hamiltonian, Hamilton-Jacobi's equation in the momentum representation yield

$$
\frac{p^2}{2m} - mg\frac{\partial U}{\partial p} + \frac{\partial U}{\partial t} = 0 \tag{3}
$$

Make the Ansatz

$$
U(p, t) = U_1(p) + U_2(t)
$$

[1] One can e.g. choose all but one $\eta_k \equiv 0$. If one does this for each one of the parts in the sum one easily sees that the expression in the square brackets has to be zero for each term individually.

(where the dependence on the constant Q is not explicitly given). Inserted into Eq. (3) we get

$$\underbrace{\frac{p^2}{2m} - mg\frac{\partial U_1}{\partial p}}_{=E} + \underbrace{\frac{\partial U_2}{\partial t}}_{=-E} = 0$$

where we realize that the first two terms only depend on p whereas the last term only depend on t and thus they have to be equal constants (but with opposite sign). We then get one equation for U_1 and one for U_2,

$$\begin{cases} \frac{p^2}{2m} - mg\frac{\partial U_1}{\partial p} = E \\ \frac{\partial U_2}{\partial t} = -E. \end{cases}$$

These are easily solved and we get

$$\begin{cases} U_1 &=& \frac{p^3}{6m^2 g} - \frac{Ep}{mg} + \text{const.} \\ U_2 &=& -Et + \text{const.} \end{cases}$$

which yield the generating function U we looked for,

$$U = \frac{p^3}{6m^2 g} - \frac{Ep}{mg} - Et + \text{const.} \tag{4}$$

U should be a function of Q, p and t though so our separation constant E has to be a function of our constant Q. We choose to define $E = Q$ and let the arbitrary constant in Eq. (4) be zero, which yield

$$U(Q, p, t) = \frac{p^3}{6m^2 g} - \frac{Qp}{mg} - Qt. \tag{5}$$

We can now get the equations that give us the canonical transformation,

$$\begin{cases} q &=& -\frac{\partial U}{\partial p} = -\frac{p^2}{2m^2 g} + \frac{Q}{mg} \\ P &=& -\frac{\partial U}{\partial Q} = \frac{p}{mg} + t \end{cases}$$

The second of these equations gives

$$p = mg\,(P - t) \tag{6}$$

which, if inserted into the first equation, gives

$$q = -\frac{g\,(P-t)^2}{2} + \frac{Q}{mg} \tag{7}$$

Hamilton's canonical equations for the new variables Q and P are trivial,

$$\begin{cases} \dot{Q} &=& \frac{\partial \tilde{H}}{\partial P} = 0 \\ \dot{P} &=& -\frac{\partial \tilde{H}}{\partial Q} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} Q &=& \beta = \text{const.} \\ P &=& \alpha = \text{const.} \end{cases}$$

Inserted into Eqs. (6)–(7) we get the solution

$$\begin{cases} q(t) &=& \frac{\beta}{mg} - \frac{g(\alpha - t)^2}{2} \\ p(t) &=& mg\,(\alpha - t) \end{cases}$$

The initial condition $p(0) = mv_0$ gives $\alpha = v_0/g$ and $q(0) = 0$ gives $\beta = mv_0^2/2$. The solution with the given initial conditions are thus

$$\begin{cases} q(t) &= \frac{v_0^2}{2g} - \frac{g}{2}\left(\frac{v_0}{g} - t\right)^2 \\ p(t) &= mg\left(\frac{v_0}{g} - t\right) \end{cases}$$