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### Solutions to

# Exam in Analytical Mechanics, 5p

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*Solutions are also available on*
*http://www.physto.se/~edsjo/teaching/am/index.html.*

## Problem 1

a) See Scheck, section 3.2.

b) The tensor of inertia is given by

$$\vec{\vec{I}} = \int \left[\vec{x}\cdot\vec{x} - \vec{x}\vec{x}\right]\rho(\vec{x})d^3x$$

where $\vec{x}\cdot\vec{x}$ is a normal scalar product, whereas $\vec{x}\vec{x}$ is a dyadic product. In a cartesian coordinate system, the components of the tensor of inertia are

$$I_{ij} = \int \left[\vec{x}^2\delta_{ij} - x_i x_j\right]\rho(\vec{x})d^3x. \tag{1}$$

According to Eq. (1) we have

$$\begin{aligned} I_{zz} &= \int (x^2+y^2)\rho(\vec{x})d^3x \\ &\leq \int (z^2+x^2+y^2+z^2)\rho(\vec{x})d^3x \\ &= \int (z^2+x^2)\rho(\vec{x})d^3x + \int (y^2+z^2)\rho(\vec{x})d^3x \\ &= I_{yy}+I_{xx}, \end{aligned}$$

i.e. $I_{zz} \leq I_{xx}+I_{yy}$, which is exactly what was to be shown.

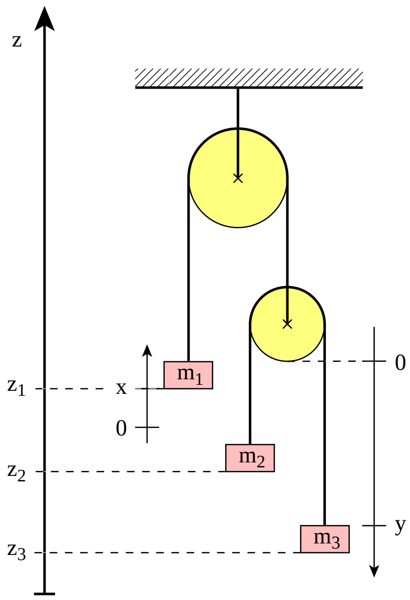
We have equality when

$$\int z^2\rho(\vec{x})d^3x = 0,$$

i.e. when the body does not have any extension in the $z$ direction, i.e. when the body is a thin plate in the $xy$ plane.

**Problem 2**

We realize that the problem has two degrees of freedom and choose $x$ and $y$ as generalized coordinates according to the figure. $x$ is the position of mass 1 with respect to the starting position and $y$ is the position of mass 3 with respect to the right pulley. This means that we can write the heights (or rather the change of the heights relative to the starting position) for the three masses as




$$\begin{cases} z_1 &= x \\ z_2 &= y - x \\ z_3 &= -y - x \end{cases} \tag{2}$$

This gives the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2 + \frac{1}{2}m_3\dot{z}_3^2 \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{y}-\dot{x})^2 + \frac{1}{2}m_3(-\dot{x}-\dot{y})^2 \\ &= \frac{1}{2}(m_1+m_2+m_3)\dot{x}^2 + \frac{1}{2}(m_2+m_3)\dot{y}^2 + (m_3-m_2)\dot{x}\dot{y} \end{aligned}$$

The potential energy is

$$U = m_1gz_1 + m_2gz_2 + m_3gz_3 = (m_1-m_2-m_3)gx + (m_2-m_3)gy$$

which gives us the Lagrangian, $L = T - U$

$$L = \frac{1}{2}(m_1+m_2+m_3)\dot{x}^2 + \frac{1}{2}(m_2+m_3)\dot{y}^2 + (m_3-m_2)\dot{x}\dot{y} - (m_1-m_2-m_3)gx - (m_2-m_3)gy \tag{3}$$

The derivatives of $L$ are given by

$$\begin{cases} \frac{\partial L}{\partial x} &= -(m_1-m_2-m_3)g \\ \frac{\partial L}{\partial y} &= -(m_2-m_3)g \end{cases} \quad ; \quad \begin{cases} \frac{\partial L}{\partial \dot{x}} &= (m_1+m_2+m_3)\dot{x} + (m_3-m_2)\dot{y} \\ \frac{\partial L}{\partial \dot{y}} &= (m_2+m_3)\dot{y} + (m_3-m_2)\dot{x} \end{cases}$$

Lagrange's equations give us the equations of motion

$$(m_1+m_2+m_3)\ddot{x} + (m_3-m_2)\ddot{y} + (m_1-m_2-m_3)g = 0 \tag{4}$$

$$(m_2+m_3)\ddot{y} + (m_3-m_2)\ddot{x} + (m_2-m_3)g = 0 \tag{5}$$

By substitution, we can solve for $\ddot{x}$ in Eq. (4) and put this into Eq. (5). We then get

$$[m_1(m_2+m_3) + 4m_2m_3)]\,\ddot{y} + 2m_1(m_2-m_3)g = 0$$

In the same way, we can solve for $\ddot{y}$ in Eq. (4) and put this into Eq. (5) to get

$$[m_1(m_2+m_3) + 4m_2m_3]\,\ddot{x} + [m_1(m_2+m_3) - 4m_2m_3]\,g = 0$$

Both of these equations are easily integrated twice to get the solution

$$\begin{aligned} x(t) &= -\frac{[m_1(m_2+m_3) - 4m_2m_3]\,g}{m_1(m_2+m_3) + 4m_2m_3}\frac{t^2}{2} + At + B \\ y(t) &= -\frac{2m_1(m_2-m_3)g}{m_1(m_2+m_3) + 4m_2m_3}\frac{t^2}{2} + Ct + D \end{aligned}$$

where $A$, $B$, $C$ and $D$ are constants which are determined from the initial conditions. Inserted into Eq. (2) these equations give us the motion for the three masses.

**Problem 3**

a) The Poisson bracket between two canonical variables $f$ and $g$ is defined by

$$\{f, g\} = \sum_{i}^{f} \left[ \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right]$$

Now consider a transformation from $(\underset{\sim}{q}, \underset{\sim}{p})$ to $(\underset{\sim}{Q}, \underset{\sim}{P})$, where

$$\begin{cases} Q_i &= Q_i(\underset{\sim}{q}, \underset{\sim}{p}, t) \\ P_j &= P_j(\underset{\sim}{q}, \underset{\sim}{p}, t) \end{cases}$$

This transformation is canonical if the following relations hold

$$\begin{cases} \{Q_i, Q_j\} &= 0 \quad ; \quad \forall\, i, j \\ \{P_i, P_j\} &= 0 \quad ; \quad \forall\, i, j \\ \{P_i, Q_j\} &= \begin{cases} 0 & ; \ \forall\, i \neq j \\ 1 & ; \ \forall\, i = j \end{cases} \end{cases}$$

b) We know that the time evaluation for a canonical variable $f$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \tag{6}$$

Now let $f = q_1 p_2 - q_2 p_1$ and insert this into Eq. (6) requiring that $df/dt = 0$,

$$0 = \frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{0} + \{H, f\} = \{q_1 p_2 - q_2 p_1, \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + a_1 q_1^2 + a_2 q_2^2\} \tag{7}$$

Now note that all Poisson brackets between the canonical variables are zero, except

$$\{p_i, q_j\} = 1 \text{ om } i = j.$$

We can further on use the following properties for the Poisson brackets to simplify our expression,

$$\{f, gh\} = g\{f, h\} + \{f, g\}h \quad ; \quad \{fg, h\} = f\{g, h\} + \{f, h\}g \quad ; \quad \{f, g\} = -\{g, f\}$$

Eq. (7) can now be simplified to

$$\begin{aligned} 0 &= \frac{p_2}{2m}\{q_1, p_1^2\} + a_2 q_1 \{p_2, q_2^2\} - \frac{p_1^2}{2m}\{q_2, p_2^2\} - a_1 q_2 \{p_1, q_1^2\} \\ &= \frac{p_2}{2m} 2p_1 \underbrace{\{q_1, p_1\}}_{-1} + a_2 q_1 2q_2 \underbrace{\{p_2, q_2\}}_{1} - \frac{p_1}{2m} 2p_2 \underbrace{\{q_2, p_2\}}_{-1} - a_1 q_2 2q_1 \underbrace{\{p_1, q_1\}}_{1} \\ &= \frac{p_1 p_2}{2m} - \frac{p_1 p_2}{2m} + 2q_1 q_2 (a_2 - a_1) = 2q_1 q_2 (a_2 - a_1) \end{aligned}$$

We thus see that we have to require that $a_1 = a_2$ for $q_1 p_2 - q_2 p_1$ to be a constant of motion.

**Uppgift 4**

The kinetic energy is given by

$$T = \frac{1}{2}m(l\dot{\varphi})^2 + \frac{1}{2}m\dot{l}^2 = \frac{1}{2}m(l\dot{\varphi})^2 + \frac{1}{2}m\alpha^2$$

and the potential energy is givne by

$$U = -mgl(t)\cos\varphi.$$

The Lagrangian is then

$$L = T - U = \frac{1}{2}m(l\dot{\varphi})^2 + \frac{1}{2}m\alpha^2 + mgl\cos\varphi.$$

The canonical momentum is

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = ml^2\dot{\varphi}$$

which finally give us Hamiltonian

$$\begin{aligned} H &= p_\varphi\dot{\varphi} - L = \frac{p_\varphi^2}{ml^2} - \frac{1}{2}ml^2\left(\frac{p_\varphi}{ml^2}\right)^2 - \frac{1}{2}m\alpha^2 - mlg\cos\varphi \\ &= \frac{1}{2}\frac{p_\varphi^2}{ml^2(t)} - \frac{1}{2}m\alpha^2 - mgl(t)\cos\varphi = H(t) \end{aligned} \quad (8)$$

We note that the Hamiltonian depends explicitly on time because of the time dependent constraint (the length of the thread that is reduced as time goes on)
The energy for the system is

$$E = T + U = \frac{1}{2}\frac{p_\varphi^2}{ml^2(t)} + \frac{1}{2}m\alpha^2 - mgl(t)\cos\alpha = E(t) \quad (9)$$

Note that the energy $E \neq H$. If we compare Eq. (8) and (9) we see that

$$E(t) = H(t) + m\alpha^2$$

Because the Hamiltonian depends explicitly on time, it cannot be a constant of motion. We have that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t} \neq 0$$

For the same reason, the energi is not a constant of motion either,

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + \{H, H + m\alpha^2\} = \frac{\partial E}{\partial t} \neq 0$$

So, we have seen that the Hamiltonian is not equal to the total energy and we have also seen that neither the Hamiltonian nor the energy are constants of motion. If our constraints does not depend on time and we write the Lagrangian on its natural form $L = T - U$, the Hamiltonian is given by $H = T + U$, but in this case the constraint is time dependent and then this relation is not valid. We also exchange energy with the system by the external force that pulls the thread and thus the energy is not conserved.

**Problem 5**

a) We introduce a rotating coordinate system $\bar{K}$ where the $x$ and $y$ axes are in the plane of the frisbee and the $z$ axis is along the symmetry axis perpendicular to this plane. This is a principal system with the moments of inertia (see Physics Handbook, or calculate them)

$$I_1 = \frac{1}{4}mr^2 \quad ; \quad I_2 = \frac{1}{4}mr^2 \quad ; \quad I_3 = \frac{1}{2}mr^2$$

Euler's dynamical equation in our system $\bar{K}$ read

$$\begin{cases} \dot{\bar{\omega}}_1 + \frac{I_3-I_2}{I_1}\bar{\omega}_2\bar{\omega}_3 & = & \bar{N}_1 \\ \dot{\bar{\omega}}_2 + \frac{I_1-I_3}{I_2}\bar{\omega}_3\bar{\omega}_1 & = & \bar{N}_2 \\ \dot{\bar{\omega}}_3 + \frac{I_2-I_1}{I_1}\bar{\omega}_1\bar{\omega}_2 & = & \bar{N}_3 \end{cases}$$

In our case $I_1 = I_2$ and all external torques $\bar{N}_i = 0$, which yield

$$\begin{cases} \dot{\bar{\omega}}_1 + \frac{I_3-I_1}{I_1}\bar{\omega}_2\bar{\omega}_3 & = & 0 \\ \dot{\bar{\omega}}_2 - \frac{I_3-I_1}{I_1}\bar{\omega}_3\bar{\omega}_1 & = & 0 \\ \dot{\bar{\omega}}_3 & = & 0 \end{cases}$$

The last of these equations immediately give us that

$$\bar{\omega}_3 = \Omega_{||} = \text{konst.}$$

Now introduce the following constant

$$\Omega_0 = \frac{I_3 - I_1}{I_1}\bar{\omega}_3$$

which enables us to write the first two of Euler's dynamical equations as

$$\begin{cases} \dot{\bar{\omega}}_1 & = & -\Omega_0\bar{\omega}_2 \\ \dot{\bar{\omega}}_2 & = & \Omega_0\bar{\omega}_1 \end{cases}$$

The solution to these equations is easy to find and is given by

$$\begin{cases} \bar{\omega}_1 & = & \Omega_\perp \cos(\Omega_0 t + \beta) \\ \bar{\omega}_2 & = & \Omega_\perp \sin(\Omega_0 t + \beta) \end{cases}$$

The angular velocity vector expressed in the system $\bar{K}$ is thus given by

$$\bar{\boldsymbol{\omega}} = \big(\Omega_\perp \cos(\Omega_0 t + \beta), \Omega_\perp \sin(\Omega_0 t + \beta), \Omega_{||}\big)$$

Expressed in the $\bar{K}$ system, the angular momentum is given by

$$\begin{aligned} \bar{\boldsymbol{L}} &= \boldsymbol{I} \cdot \bar{\boldsymbol{\omega}} = (I_1\bar{\omega}_1, I_2\bar{\omega}_2, I_3\bar{\omega}_3) \\ &= \big(I_1\Omega_\perp \cos(\Omega_0 t + \beta), I_1\Omega_\perp \sin(\Omega_0 t + \beta), I_3\Omega_{||}\big) \end{aligned}$$

From the expressions for $\bar{\boldsymbol{L}}$ and $\bar{\boldsymbol{\omega}}$ it is evident that the symmetri axis (the $\bar{3}$ axis), $\bar{\boldsymbol{L}}$ and $\bar{\boldsymbol{\omega}}$ all lie in the same plane.

b) Since no external torques act on the frisbee, $\boldsymbol{L}$ has to be conserved. Since $\boldsymbol{L}$ rotates around the $\bar{3}$ axis with angular velocity $\Omega_0$ in the $\bar{K}$ system, the $\bar{3}$ axis has to rotate with angular velocity $\Omega_0$ around $\boldsymbol{L}$ in a non-rotating system. The angular velocity we are looking for is thus

$$\Omega_0 = \frac{I_3 - I_1}{I_1}\bar{\omega}_3$$

If $\omega_0$ is the angular velocity for the rotation around the rotation axis, then $\bar{\omega}_3 = \omega_0 \cos\alpha$ and we thus get

$$\Omega_0 = \frac{I_3 - I_1}{I_1}\omega_0 \cos\alpha$$

If we now insert the expressions for $I_3$ and $I_1$ and use the fact that $\omega_0 = 2\pi\nu$ we finally get the sought-after angular velocity

$$\Omega_0 = 2\pi\nu \cos\alpha$$