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Solutions to

Exam in Analytical Mechanics, 5p

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Solutions will eventually also be available on
http://www.physto.se/~edsjo/teaching/am/index.html.

Problem 1

a) The kinetic energy is given by

$$T = \frac{1}{2}m\left(R\sin\theta\omega\hat{\boldsymbol{\varphi}} + R\dot{\theta}\hat{\boldsymbol{\theta}}\right)^2 = \frac{1}{2}mR^2\left(\omega^2\sin^2\theta + \dot{\theta}^2\right)$$

and the potential energy is given by

$$U = mgR\left(1-\cos\theta\right)$$

The Lagrangian is then given by

$$L = T - U = \frac{1}{2}mR^2\left(\omega^2\sin^2\theta + \dot{\theta}^2\right) - mgR\left(1-\cos\theta\right)$$

and it's derivatives are

$$\begin{cases} \frac{\partial L}{\partial \theta} &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= mR^2\dot{\theta} \end{cases}$$

Inserted into Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

we get

$$mR^2\ddot{\theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \qquad (1)$$

which is the equation of motion for θ.

b) From Eq. (1) we see that $\ddot{\theta} = 0$ for $\theta = 0$ and hence $\theta = 0$ is an equilibrium point. To find if it is stable or not, we Taylor expand the right-hand side in Eq. (1) keeping terms up to linear order in θ, i.e. we set

$$\begin{cases} \sin\theta &\simeq \theta \\ \cos\theta &\simeq 1 \end{cases}$$

which gives

$$mR^2\ddot{\theta} \simeq \left(mR^2\omega^2 - mgR\right)\theta$$

This equation has oscillating cos and sin solutions if the coefficient in front of θ in the right-hand side is negative, otherwise the solution is exponentials. Hence, for the solution to be stable, the coefficient has to be negative, i.e.

$$mR^2\omega^2 - mgR < 0$$
$$\Rightarrow \omega^2 < \tfrac{g}{R}$$
$$\Rightarrow \omega_c = \sqrt{\tfrac{g}{R}}$$

c) From Eq. (1) we see that $\ddot{\theta} = 0$ when

$$\sin\theta\left(mR^2\omega^2\cos\theta - mgR\right) = 0.$$

We see that this equation is fulfilled when

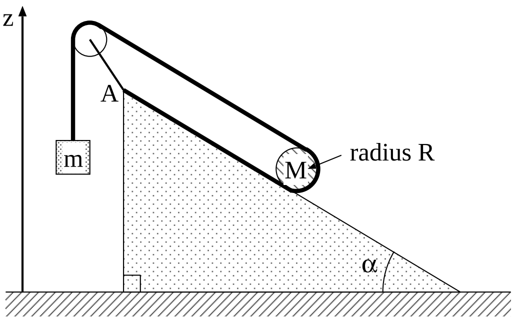
$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{g}{R\omega^2}$$

The first of these gives the two equilibrium points $\theta = 0$ and $\theta = \pi$, whereas the second equation only has a solution when $\omega > \omega_c$ and then the equilibrium point is

$$\theta = \arccos\frac{g}{R\omega^2}$$

One can show that this equilibrium point is stable by Taylor expand the right-hand side in Eq. (1) around $\theta = \arccos\frac{g}{R\omega^2}$.

Uppgift 2



a) Choose z as the height for the mass m as the generalized coordinate. The kinetic energy for the mass m is then given by

$$T_m = \frac{1}{2}m\dot{z}^2$$

When the mass m has moved a distance z, the mass M has moved the distance $z/2$. It's kinetic energy gets a contribution both from the translation of the centre of mass, but also from the rotation around the centre of mass,

$$T_M = \frac{1}{2}M\left(\frac{1}{2}\dot{z}\right)^2 + \frac{1}{2}I\omega^2$$

For a solid cylinder the moment of inertia for rotation around the symmetry axis is given by $I = \frac{1}{2}MR^2$. The angular velocity is given by $\omega = \frac{\dot{z}}{2R}$. The kinetic energy for M is then given by

$$T_M = \frac{1}{8}M\dot{z}^2 + \frac{1}{2}\frac{1}{2}MR^2\left(\frac{\dot{z}}{2R}\right)^2 = \frac{1}{8}M\dot{z}^2 + \frac{1}{16}M\dot{z}^2 = \frac{3}{16}M\dot{z}^2.$$

The potential energy is given by

$$U = mgz - Mg\frac{1}{2}z\sin\alpha$$

which finally gives us the Lagrangian

$$L = T_m + T_M - U = \left(\frac{1}{2}m + \frac{3}{16}M\right)\dot{z}^2 - \left(mg - \frac{1}{2}Mg\sin\alpha\right)z.$$

It's derivatives are

$$\begin{cases} \frac{\partial L}{\partial z} &= \frac{1}{2}Mg\sin\alpha - mg \\ \frac{\partial L}{\partial \dot{z}} &= \left(m + \frac{3}{8}M\right)\dot{z} \end{cases}$$

Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

then yield

$$\left(m + \frac{3}{8}M\right)\ddot{z} = \frac{1}{2}Mg\sin\alpha - mg \tag{2}$$

which is easily integrated to give the solution

$$z(t) = \frac{2Mg\sin\alpha - 4mg}{8m + 3M}t^2 + At + B \quad ; \quad A, B = \text{constants}$$

b) The system is in equilibrium when $\ddot{z} = 0$. Eq. (2) tell us that $\ddot{z} = 0$ when

$$\begin{aligned} &\tfrac{1}{2}Mg\sin\alpha - mg = 0 \\ &\Rightarrow \alpha = \arcsin\tfrac{2m}{M}. \end{aligned}$$

We also see that equilibrium only can be obtained when $M \geq 2m$.

Uppgift 3

a) Vi have that

$$\begin{aligned} \{f, gh\} &= \sum_i \left[\frac{\partial f}{\partial p_i}\frac{\partial (gh)}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial (gh)}{\partial p_i}\right] \\ &= \sum_i \left[\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}h + g\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}h - g\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i}\right] \\ &= g\sum_i \left[\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i}\right] + \sum_i \left[\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right]h = g\{f, h\} + \{f, g\}h. \end{aligned}$$

The second relation is easily shown in the same way.

b) To determine the condition β and γ has to fulfill for L_z to be a constant of motion, we can e.g. use Noether's theorem. Alternatively, we can use Poisson brackets by noting that

$$\frac{dL_z}{dt} = \{H, L_z\} + \underbrace{\frac{\partial L_z}{\partial t}}_{0} = \{H, L_z\}.$$

In other words, we want to determine β and γ such that $\{H, L_z\} = 0$. The Hamiltonian is given by

$$H = \frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2\right] + \alpha z^2 e^{\beta x^2 + \gamma y^2}$$

The z component of the angular momentum is given by

$$L_z = xp_y - yp_x.$$

We are now ready to calculate the Poisson bracket between H and L_z. We then use the relations that were proved in a) and that $\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$, $\{p_i, q_j\} = \delta_{ij}$ and $\{q_i, p_j\} = -\delta_{ij}$ to simplify our expression,

$$\begin{aligned}
\{H, L_z\} &= \{\frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2\right] + \alpha z^2 e^{\beta x^2+\gamma y^2}, xp_y - yp_x\} = \\
&= \frac{1}{2m}\{p_x^2, xp_y\} - \frac{1}{2m}\{p_y^2, yp_x\} + \{\alpha z^2 e^{\beta x^2+\gamma y^2}, xp_y\} - \{\alpha z^2 e^{\beta x^2+\gamma y^2}, yp_x\} \\
&= \frac{1}{2m}p_x \underbrace{\{p_x, x\}}_{1} p_y - \frac{1}{2m}p_y \underbrace{\{p_y, y\}}_{1} p_x + \alpha z^2 x\{e^{\beta x^2+\gamma y^2}, p_y\} - \alpha z^2 y\{e^{\beta x^2+\gamma y^2}, p_x\} \\
&= \frac{1}{2m}\underbrace{[p_x p_y - p_y p_x]}_{0} + \alpha z^2 x \sum_i \left[\frac{\partial e^{\beta x^2+\gamma y^2}}{\partial p_i}\underbrace{\frac{\partial p_y}{\partial q_i}}_{0} - \frac{\partial e^{\beta x^2+\gamma y^2}}{\partial q_i}\underbrace{\frac{\partial p_y}{\partial p_i}}_{\delta_{i2}}\right] \\
&\quad -\alpha z^2 y \sum_i \left[\frac{\partial e^{\beta x^2+\gamma y^2}}{\partial p_i}\underbrace{\frac{\partial p_x}{\partial q_i}}_{0} - \frac{\partial e^{\beta x^2+\gamma y^2}}{\partial q_i}\underbrace{\frac{\partial p_x}{\partial p_i}}_{\delta_{i1}}\right] \\
&= -\alpha z^2 x \frac{\partial e^{\beta x^2+\gamma y^2}}{\partial y} + \alpha z^2 y \frac{\partial e^{\beta x^2+\gamma y^2}}{\partial x} \\
&= -\alpha z^2 x 2\gamma y e^{\beta x^2+\gamma y^2} + \alpha z^2 y 2\beta x e^{\beta x^2+\gamma y^2} = \alpha z^2 xy e^{\beta x^2+\gamma y^2}\left[\beta - \gamma\right]
\end{aligned}$$

$$\Rightarrow \{H, L_z\} = 0 \text{ if } \beta = \gamma$$

L_z is thus conserved if $\beta = \gamma$, which is the condition we looked for.

Remark. $\{H, L_z\} = 0$ is also fulfilled if $x = 0$, $y = 0$ or $z = 0$, but if $\{H, L_z\} = 0$ should be fulfilled for arbitrary initial conditions we have to have $\beta = \gamma$.

Uppgift 4

a) See Scheck, section 2.5 or the lecture notes.

b) This is easy to show with calculus of variations. The distance between (x_0, y_0) and (x_1, y_1) is given by

$$L = \int_{x_0}^{x_1} ds = \int_{x_0}^{x_1} \sqrt{1 + y'^2}dx$$

We can use Euler's equation given in 4a with

$$f(y, y', x) = \sqrt{1 + y'^2}$$

Inserted into Euler's equation, we get

$$0 = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) - 0 = 0$$

which is easily integrated to

$$\frac{y'}{\sqrt{1+y'^2}} = A = \text{const.} \quad \Rightarrow \quad y' = B = \text{const.}$$

Integrating once more, we get

$$y = Bx + C$$

which is the equation for the straight line. The constants B and C are given by the condition that the line has to pass through (x_0, y_0) and (x_1, y_1).

Uppgift 5

a) For a generating function of the type U we have that

$$q_i = -\frac{\partial U}{\partial p_i} \quad ; \quad P_j = -\frac{\partial U}{\partial Q_j} \quad ; \quad \tilde{H} = H + \frac{\partial U}{\partial t}.$$

We want the new Hamiltonian, $\tilde{H}$, to be identically equal to zero, i.e. that

$$H(\underset{\sim}{q}, \underset{\sim}{p}, t) + \frac{\partial U}{\partial t} = 0$$

Use that $q_i = -\frac{\partial U}{\partial p_i}$ and we get

$$H(-\frac{\partial U}{\partial \underset{\sim}{p_i}}, \underset{\sim}{p}, t) + \frac{\partial U}{\partial t} = 0$$

which is the Hamilton-Jacobi equation in the momentum representation. This is a partial differential equation for U with respect to $\underset{\sim}{p}$ and t.

b) With the given Hamiltonian, Hamilton-Jacobi's equation in the momentum representation yield

$$\frac{p^2}{2m} - mg\frac{\partial U}{\partial p} + \frac{\partial U}{\partial t} = 0 \tag{3}$$

Make the Ansatz

$$U(p,t) = U_1(p) + U_2(t)$$

(where the dependence on the constant Q is not explicitly given). Inserted into Eq. (3) we get

$$\underbrace{\frac{p^2}{2m} - mg\frac{\partial U_1}{\partial p}}_{=E} + \underbrace{\frac{\partial U_2}{\partial t}}_{=-E} = 0$$

where we realize that the first two terms only depend on p whereas the last term only depend on t and thus they have to be equal constants (but with opposite sign). We then get one equation for U_1 and one for U_2,

$$\begin{cases} \frac{p^2}{2m} - mg\frac{\partial U_1}{\partial p} = E \\ \frac{\partial U_2}{\partial t} = -E. \end{cases}$$

These are easily solved and we get

$$\begin{cases} U_1 &= \frac{p^3}{6m^2 g} - \frac{Ep}{mg} + \text{const.} \\ U_2 &= -Et + \text{const.} \end{cases}$$

which yield the generating function U we looked for,

$$U = \frac{p^3}{6m^2 g} - \frac{Ep}{mg} - Et + \text{const.} \tag{4}$$

U should be a function of Q, p and t though so our separation constant E has to be a function of our constant Q. We choose to define $E = Q$ and let the arbitrary constant in Eq. (4) be zero, which yield

$$U(Q, p, t) = \frac{p^3}{6m^2 g} - \frac{Qp}{mg} - Qt. \tag{5}$$

We can now get the equations that give us the canonical transformation,

$$\begin{cases} q &= -\frac{\partial U}{\partial p} = -\frac{p^2}{2m^2 g} + \frac{Q}{mg} \\ P &= -\frac{\partial U}{\partial Q} = \frac{p}{mg} + t \end{cases}$$

The second of these equations gives

$$p = mg\,(P - t) \tag{6}$$

which, if inserted into the first equation, gives

$$q = -\frac{g\,(P - t)^2}{2} + \frac{Q}{mg} \tag{7}$$

Hamilton's canonical equations for the new variables Q and P are trivial,

$$\begin{cases} \dot{Q} &= \frac{\partial \tilde{H}}{\partial P} = 0 \\ \dot{P} &= -\frac{\partial \tilde{H}}{\partial Q} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} Q &= \beta = \text{const.} \\ P &= \alpha = \text{const.} \end{cases}$$

Inserted into Eqs. (6)–(7) we get the solution

$$\begin{cases} q(t) &= \frac{\beta}{mg} - \frac{g(\alpha - t)^2}{2} \\ p(t) &= mg\,(\alpha - t) \end{cases}$$

The initial condition $p(0) = mv_0$ gives $\alpha = v_0/g$ and $q(0) = 0$ gives $\beta = mv_0^2/2$. The solution with the given initial conditions are thus

$$\begin{cases} q(t) &= \frac{v_0^2}{2g} - \frac{g}{2}\left(\frac{v_0}{g} - t\right)^2 \\ p(t) &= mg\left(\frac{v_0}{g} - t\right) \end{cases}$$