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Noethers teorem

Noethers teorem ser ut som följer:

> *Om Lagrangefunktionen $L(\underset{\sim}{q}, \dot{\underset{\sim}{q}})$ beskriver ett autonomt system som är invariant under transformationen $\underset{\sim}{q} \to \underset{\sim}{h}^s(\underset{\sim}{q})$ där s är en reell kontinuerlig parameter sådan att $\underset{\sim}{h}^{s=0}(\underset{\sim}{q}) = \underset{\sim}{q}$ är identitetstransformationen så är*
>
> $$I(\underset{\sim}{q}, \dot{\underset{\sim}{q}}) = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{ds} h_i^s(\underset{\sim}{q}) \bigg|_{s=0}$$
>
> *en rörelsekonstant.*

och bevisas t.ex. på följande vis:
Låt $\underset{\sim}{q} = \underset{\sim}{\varphi}$ vara en lösning till Lagranges ekvationer. Eftersom systemet är invariant under transformationen $\underset{\sim}{h}^s$ är

$$\underset{\sim}{q}(s,t) = \underset{\sim}{\phi}(s,t) = \underset{\sim}{h}^s(\underset{\sim}{\varphi}(t))$$

också en lösning till Lagranges ekvationer,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)\right) = \frac{\partial L}{\partial q_i}(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)) \tag{1}$$

Vidare är L invariant under transformationen, d.v.s.

$$0 = \frac{d}{ds} L(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)) = \sum_{i=1}^{f} \left[\frac{\partial L}{\partial q_i}\frac{d\phi_i}{ds} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{\phi}_i}{ds}\right] \tag{2}$$

Använd nu Ekv. (1) för att byta ut $\frac{\partial L}{\partial q_i}$ i Ekv. (2) samt byt ordning på deriveringarna i den sista termen så erhåller vi

$$\sum_{i=1}^{f} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{d\phi_i}{ds} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\frac{d\phi_i}{ds}\right)\right] = 0$$

Detta måste speciellt gälla då $s = 0$ och vi erhåller då slutligen (med $d\phi_i/ds = dh_i^s/ds$)

$$\frac{d}{dt}\left(\sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i}\frac{dh_i^s}{ds}\right)\bigg|_{s=0} = 0 \qquad \Rightarrow \qquad \frac{d}{dt} I = 0$$

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Noether's theorem

Noether's theorem looks like this

> *If the Lagrangian $L(\underset{\sim}{q}, \dot{\underset{\sim}{q}})$ describes an autonomous system which is invariant under the transformation $\underset{\sim}{q} \rightarrow \underset{\sim}{h}^s(\underset{\sim}{q})$ where s is a real continuous parameter such that $\underset{\sim}{h}^{s=0}(\underset{\sim}{q}) = \underset{\sim}{q}$ is the identity transformation, then*
>
> $$I(\underset{\sim}{q}, \dot{\underset{\sim}{q}}) = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{ds} h_i^s(\underset{\sim}{q}) \bigg|_{s=0}$$
>
> *is a constant of motion.*

and it is proven e.g. like this:
Let $\underset{\sim}{q} = \underset{\sim}{\varphi}$ be a solution to Lagrange's equations. Since the system is invariant under the transformation $\underset{\sim}{h}^s$

$$\underset{\sim}{q}(s,t) = \underset{\sim}{\phi}(s,t) = \underset{\sim}{h}^s(\underset{\sim}{\varphi}(t))$$

is also a solution to Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)\right) = \frac{\partial L}{\partial q_i}(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)) \tag{3}$$

Further on, L, is invariant under the transformation, i.e.

$$0 = \frac{d}{ds} L(\underset{\sim}{\phi}(s,t), \dot{\underset{\sim}{\phi}}(s,t)) = \sum_{i=1}^{f} \left[\frac{\partial L}{\partial q_i}\frac{d\phi_i}{ds} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{\phi}_i}{ds}\right] \tag{4}$$

Now use Eq. (3) to replace $\frac{\partial L}{\partial q_i}$ in Eq. (4) and change the order of the derivates in the last term. We then get

$$\sum_{i=1}^{f}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{d\phi_i}{ds} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\frac{d\phi_i}{ds}\right)\right] = 0$$

This must especially hold when $s = 0$ and we then finally get (with $d\phi_i/ds = dh_i^s/ds$)

$$\frac{d}{dt}\left(\sum_{i=1}^{f}\frac{\partial L}{\partial \dot{q}_i}\frac{dh_i^s}{ds}\right)\bigg|_{s=0} = 0 \qquad \Rightarrow \qquad \frac{d}{dt} I = 0$$