Joakim Edsjö
Fysikum, Stockholms Universitet
Tel.: 08-55 37 87 26
E-post: edsjo@physto.se

# Formelsamling

## Analytisk Mekanik, 5p

### Lagranges ekvationer

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad , \qquad k = 1, \ldots, f$$

### Hamiltons ekvationer

Definiera

$$H(\underset{\sim}{q}, \underset{\sim}{p}, t) = \sum_k \dot{q}_k p_k - L(\underset{\sim}{q}, \underset{\sim}{\dot{q}}, t)$$

med den till $q_k$ konjugerade rörelsemängden $p_k$

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Hamiltons kanoniska ekvationer lyder då

$$\frac{\partial H}{\partial p_k} = \dot{q}_k \quad ; \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k \quad , \quad k = 1, \ldots, f$$

### Kanoniska transformationer

**Typ A.** $F_1 = F_1(\underset{\sim}{q}, \underset{\sim}{Q}, t)$ - genererande funktion

$$p_i = \frac{\partial F_1}{\partial q_i} \quad ; \quad P_j = -\frac{\partial F_1}{\partial Q_j} \quad ; \quad K = H + \frac{\partial F_1}{\partial t}$$

**Typ B.** $F_2 = F_2(\underset{\sim}{q}, \underset{\sim}{P}, t)$ - genererande funktion

$$p_i = \frac{\partial F_2}{\partial q_i} \quad ; \quad Q_j = \frac{\partial F_2}{\partial P_j} \quad ; \quad K = H + \frac{\partial F_2}{\partial t}$$

**Typ C.** $F_3 = F_3(\underset{\sim}{Q}, \underset{\sim}{p}, t)$ - genererande funktion

$$q_i = -\frac{\partial F_3}{\partial p_i} \quad ; \quad P_j = -\frac{\partial F_3}{\partial Q_j} \quad ; \quad K = H + \frac{\partial F_3}{\partial t}$$

**Typ D.** $F_4 = F_4(\underset{\sim}{P}, \underset{\sim}{p}, t)$ - genererande funktion

$$q_i = -\frac{\partial F_4}{\partial p_i} \quad ; \quad Q_j = \frac{\partial F_4}{\partial P_j} \quad ; \quad K = H + \frac{\partial F_4}{\partial t}$$

### Noethers teorem

Om Lagrangefunktionen $L(\underset{\sim}{q}, \underset{\sim}{\dot{q}})$ beskriver ett autonomt system som är invariant under transformationen $\underset{\sim}{q} \to \underset{\sim}{h}^s(\underset{\sim}{q})$ där $s$ är en reell kontinuerlig parameter sådan att $\underset{\sim}{h}^{s=0}(\underset{\sim}{q}) = \underset{\sim}{q}$ är identitetstransformationen så är

$$I(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{ds} h_i^s(\underset{\sim}{q}) \bigg|_{s=0}$$

en rörelsekonstant.

### Eulers dynamiska ekvationer

I ett principalsystem $K'$ fixt i kroppen har vi

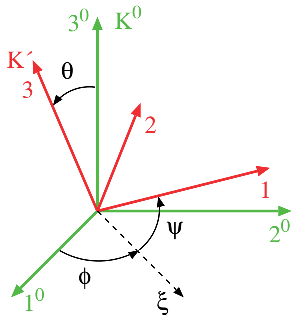
$$\begin{cases} N'_x &= I'_{xx}\dot{\omega}'_x + (I'_{zz} - I'_{yy})\omega'_y\omega'_z \\ N'_y &= I'_{yy}\dot{\omega}'_y + (I'_{xx} - I'_{zz})\omega'_z\omega'_x \\ N'_z &= I'_{zz}\dot{\omega}'_z + (I'_{yy} - I'_{xx})\omega'_x\omega'_y \end{cases}$$

där prim betyder att vi anger våra vektorer och tensorer i det kroppsfixa systemet $K'$.

### Eulervinklarna

Eulervinklarna kan användas för att beskriva en godtycklig rotation av kroppen. Eulervinklarna definieras genom att beskriva hur vi roterar koordinatsystemet $K^0$ (med axlar fixa i rummet) till det kroppsfixa koordinatsystemet $K'$.

- Rotera runt $3^0$-axeln en vinkel $\phi$. $1^0$-axeln roteras då till den tillfälliga $\xi$-axlen.
- Rotera runt $\xi$-axeln en vinkel $\theta$.
- Rotera runt 3-axeln en vinkel $\psi$.




Med hjälp av Eulervinklarna kan vi skriva vinkelhastighetsvektorn (uttryckt i det kroppsfixa systemet $K'$) som

$$\begin{cases} \omega'_x &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega'_y &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega'_z &= \dot{\phi}\cos\theta + \dot{\psi} \end{cases}$$

Joakim Edsjö
Fysikum, Stockholms Universitet
Tel.: 08-55 37 87 26
E-post: edsjo@physto.se

# Collection of formulae

## Analytical Mechanics, 5p

### Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad , \qquad k = 1, \ldots, f$$

### Hamilton's equations

Define

$$H(\underset{\sim}{q}, \underset{\sim}{p}, t) = \sum_k \dot{q}_k p_k - L(\underset{\sim}{q}, \underset{\sim}{\dot{q}}, t)$$

with the to $q_k$ conjugated canonical momentum $p_k$

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Hamilton's canonical equations then read

$$\frac{\partial H}{\partial p_k} = \dot{q}_k \quad ; \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k \quad , \quad k = 1, \ldots, f$$

### Canonical transformations

**Type A.** $F_1 = F_1(\underset{\sim}{q}, \underset{\sim}{Q}, t)$ - generating function

$$p_i = \frac{\partial F_1}{\partial q_i} \quad ; \quad P_j = -\frac{\partial F_1}{\partial Q_j} \quad ; \quad K = H + \frac{\partial F_1}{\partial t}$$

**Type B.** $F_2 = F_2(\underset{\sim}{q}, \underset{\sim}{P}, t)$ - generating function

$$p_i = \frac{\partial F_2}{\partial q_i} \quad ; \quad Q_j = \frac{\partial F_2}{\partial P_j} \quad ; \quad K = H + \frac{\partial F_2}{\partial t}$$

**Type C.** $F_3 = F_3(\underset{\sim}{Q}, \underset{\sim}{p}, t)$ - generating function

$$q_i = -\frac{\partial F_3}{\partial p_i} \quad ; \quad P_j = -\frac{\partial F_3}{\partial Q_j} \quad ; \quad K = H + \frac{\partial F_3}{\partial t}$$

**Type D.** $F_4 = F_4(\underset{\sim}{P}, \underset{\sim}{p}, t)$ - generating function

$$q_i = -\frac{\partial F_4}{\partial p_i} \quad ; \quad Q_j = \frac{\partial F_4}{\partial P_j} \quad ; \quad K = H + \frac{\partial F_4}{\partial t}$$

### Noether's theorem

If the Lagrangian $L(\underset{\sim}{q}, \underset{\sim}{\dot{q}})$ describes an autonomous system which is invariant under the transformation $\underset{\sim}{q} \to \underset{\sim}{h}^s(\underset{\sim}{q})$ where $s$ is a real continuous parameter such that $\underset{\sim}{h}^{s=0}(\underset{\sim}{q}) = \underset{\sim}{q}$ is the identity transformation, then

$$I(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) = \sum_{i=1}^{f} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{ds} h_i^s(\underset{\sim}{q}) \Big|_{s=0}$$

is a constant of motion.

**Euler's dynamical equations**

In a principal system $K'$ fixed in the body we have

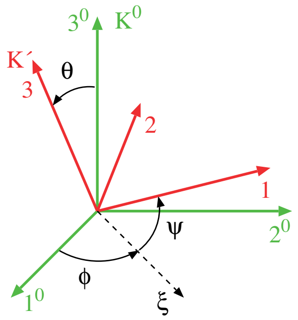
$$\begin{cases} N'_x &= I'_{xx}\dot{\omega}'_x + (I'_{zz} - I'_{yy})\omega'_y\omega'_z \\ N'_y &= I'_{yy}\dot{\omega}'_y + (I'_{xx} - I'_{zz})\omega'_z\omega'_x \\ N'_z &= I'_{zz}\dot{\omega}'_z + (I'_{yy} - I'_{xx})\omega'_x\omega'_y \end{cases}$$

where the prime indicates that we give the vectors and tensors in the body-fixed system $K'$.

**The Euler Angles**

The Euler angles can be used to describe an arbitrary rotation of a rigid body. The Euler angles are defined by describing how we rotate a coordinate system $K^0$ (with axes fixed in space) to the body-fixed coordinate system $K'$.

- Rotate around the $3^0$-axis an angle $\phi$. The $1^0$-axis is then rotated to the temporary $\xi$-axis.
- Rotate around the $\xi$-axis an angle $\theta$.
- Rotate around the 3-axis an angle $\psi$.




With the Euler angles, we can write the angular velocity vector (expressed in the body-fixed system $K'$) as

$$\begin{cases} \omega'_x &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega'_y &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega'_z &= \dot{\phi}\cos\theta + \dot{\psi} \end{cases}$$