# A conformal field theory description of fractional quantum Hall states

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# Contents

# Introduction

More than one hundred years after the discovery of the electron, there is still no complete understanding of the states of matter that can be formed in a system of electrons. In this thesis we consider electron systems under so-called 'quantum Hall conditions'. We propose and study a number of novel states of matter that can be formed by electrons under these conditions. Before we describe our work, we briefly discuss the discovery of the electron and on the various quantum states that electrons can form.

In the last decade of the nineteenth century, the experiments of Joseph Thomson on cathode ray tubes led to the discovery of the electron[1]. He introduced the electron with the following words: "Electrification essentially involves the splitting up of the atom, a part of the mass of the atom getting free and becoming detached from the original atom". Also, the experiments of Pieter Zeeman on the effect of a magnetic field on spectral lines and the subsequent explanation of Hendrik Lorentz corroborated the discovery of Thomson. The (anomalous) Zeeman effect played a crucial role in the determination of the properties of the electron. This splitting of spectral lines under the influence of a magnetic field could eventually be described by Wolfgang Pauli by making the assumption that there was an additional quantum number in the problem. However, he did not clearly state to what this quantum number relates. Independently, Ralph Kronig and George Uhlenbeck and Samuel Goudsmit proposed that this additional quantum number is intrinsic to the electron; they discovered the property of the electron which is nowadays called *spin*. One more ingredient is needed to be able to explain the spectra of atoms in a magnetic field, namely the famous Pauli exclusion principle, which states that no two electrons can be in the same quantum state. The spin of the electron and the Pauli exclusion principle are crucial ingredients in theories of condensed matter systems.

At the Cavendish Laboratory (where Thomson performed his experiments) annual dinner a toast used to be offered on the discovery of the electron: "The electron: may it never be of use to anybody." Nevertheless, the electron changed society in a profound way, as the operation of all electronic equipment is based on electrons.

The state of the electrons in a normal metal is understood in terms of the Fermi liquid, whose properties are largely similar to those of a free electron gas. But, in roughly the same way as ordinary water molecules can form different phases (namely the solid, liquid and gas phases), a system of electrons can exist in states that are entirely different from the free electron gas. In 1911, Heike Kamerlingh Onnes discovered the superconducting state

in mercury by cooling it down to just 4 Kelvin above the absolute minimum. In this state of matter, pairs of electrons can move frictionless through the metal. It took theoretical physicists up to 1957 to explain this type of superconductivity. More states of matter have been discovered. Here, I would like to mention a special class of superconductors, namely the 'high-temperature' superconductors, discovered in 1986. These ceramic materials become superconducting at much higher temperatures than the original ones discovered by Kamerlingh Onnes. The state of matter formed by the electrons in these systems is poorly understood at present.

In this thesis, we will concentrate on yet another class of electron states, the so called *quantum Hall fluids*. In 1980, Klaus von Klitzing, Gerhard Dorda and Michael Pepper made a remarkable discovery. On performing Hall measurements on a system in which the electrons are confined to a plane, they discovered that, at certain values of the magnetic field, the longitudinal resistance of the system vanished and the Hall resistance showed plateaux. The value of the conductance on these plateaux was equal to an integer times the fundamental conductance quantum, given by $\frac{e^2}{h}$. While this result came more or less unexpected, this effect can (naively) be understood in terms of a theory of non-interacting electrons in the presence of disorder.

In 1982, another type of quantum Hall states was discovered. By investigating very clean hetero-junctions, Daniel Tsui, Horst Störmer and Arthur Gossard discovered quantized Hall plateaux with a quantized Hall conductance equal to a fractional value (namely $\frac{1}{3}$) of the conductance quantum. The first big step of explaining this effect was made by Robert Laughlin. He came up with an approximate (though very good) trial wave function for the full, interacting many body problem. One of the most interesting consequences of this theory is the existence of particles which carry $\frac{1}{3}$ of the charge of the electron. In 1995, this fractional charge was observed in *shot-noise* experiments.

Prompted by observations first made in 1987, Gregory Moore and Nicholas Read proposed new states of two-dimensional electron systems which combine pairing of electrons with the properties of the fractional quantum Hall fluids proposed by Laughlin. It is these types of quantum Hall fluids which we will consider in this thesis.

[1]The references for this section are: E.N. da C. Andrade, [2]; J. Bardeen, L.N. Cooper and J.R. Schrieffer, [11]; J.G. Bednorz and K.A. Müller [12]; H. Kamerlingh Onnes, [61]; K. von Klitzing, G. Dorda and M. Pepper, [64]; R.B. Laughlin, [67]; G. Moore and N. Read,[71]; W. Pauli, [77, 78, 79]; R. de Picciotto *et. al.*, [80]; L. Saminadayar *et. al.*, [89]; J.J. Thomson, [97, 98]; D.C. Tsui, H.L. Störmer and A.C. Gossard, [99]; G.E. Uhlenbeck and S. Goudsmit, [100]; B.L. van der Waerden, [101]; P. Zeeman, [117].

# Chapter 1

# The quantum Hall effect

In 1980, K. Von Klitzing, G. Dorda and M. Pepper made a remarkable discovery [64]. In doing Hall measurements on a silicon MOSFET (metal-oxide-semiconductor field effect transistor), they found that the Hall resistance, which is given by $R_{\mathrm{H}} = \frac{V_{\mathrm{H}}}{I}$ (see the left panel of figure 1.1 for a sample setup) did not follow the classical behaviour, which would be linear in the applied magnetic field. Instead, they found that at certain values of the magnetic field, *plateaux* were formed. At these plateaux, the Hall conductance was quantized very precisely (precision nowadays is better than $10^{-8}$) to an integer times the fundamental units of conductance, $\frac{e^2}{h}$. At the values of the magnetic field where the plateaux in the Hall conductance are observed, the longitudinal voltage goes to zero. This effect is called the *integer* quantum Hall effect.

Though this observation was completely unexpected, it can, naively, be explained in terms of non-interacting electrons confined in a two-dimensional system, with a strong magnetic field perpendicular to the two-dimensional plane in which the electrons live. In these systems, the electronic states organize themselves in *Landau levels*, which are highly degenerate, while the states are extended. The Landau levels are separated by large gaps $\hbar\omega_c$ ($\omega_c$ is the cyclotron energy), in comparison to the other energy scales in the problem, which are the Zeeman and interaction energy.

With $p$ (an integer) Landau levels completely filled, the Hall conductance is quantized to $\sigma_{\mathrm{H}} = \frac{I}{V_{\mathrm{H}}} = p\frac{e^2}{h}$. To explain the plateau behaviour, the effects of disorder have to be taken into account. The effect of the disorder is to localize some of the extended states, while they are also shifted a bit in energy. By changing the magnetic field, one changes the filling fraction and thus the Fermi-level. If the Fermi-level is in a region where only localized states, which do not contribute to the conductance, are present, changing the magnetic field does not change the conductance, and hence we observe a plateau. The regions where the Hall conductance changes from one plateau to another correspond to magnetic fields where the Fermi-level lies in the region of the extended states, which do contribute to the conductance. To explain that, also in the presence of disorder, the Hall conductance is quantized to an integer times the fundamental conductance quantum, one can use a gauge argument [66, 54]. Due to gauge invariance, adiabatically changing the flux by one flux quantum will result in the transfer of charge from one edge to another. If, say, $n$ electrons

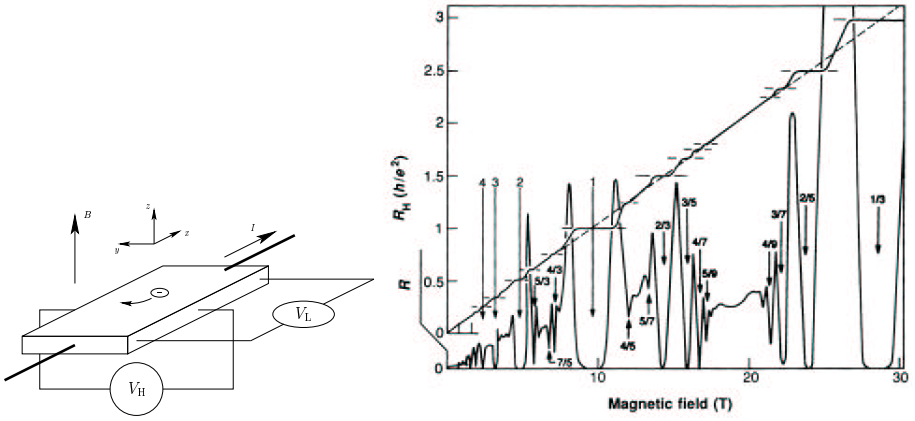


Figure 1.1: Left: schematic setup. Right: resistance measurements; figure taken from [28].

are transferred, this leads to a Hall conductance $\sigma_{\rm H} = n\frac{e^2}{h}$, also in the presence of disorder. The quantization of the Hall conductance can be so precise as found in the experiments because it is based only on gauge invariance and the presence of a mobility gap.

Remarkable as the integer quantum Hall effect is, nature showed its beauty by providing an even more astonishing state of matter. In extremely clean GaAs/$\mathrm{Al}_x\mathrm{Ga}_{1-x}$As heterostructures, D.C. Tsui, H.L. Störmer and A.C. Gossard observed a quantum Hall effect at values of the magnetic field which correspond to a partially filled Landau level [99]. The value of the Hall conductance at these plateaux was a simple *fraction* of the fundamental conductance quantum $\sigma_{\rm H} = \frac{p}{q}\frac{e^2}{h}$, where $p, q$ are small integers, while $q$ is *odd*. This effect is called the *fractional* (or anomalous) quantum Hall effect. See figure 1.1 for typical resistance measurements on clean samples. This fractional quantum Hall effect can not be explained using the 'simple' picture which was used above to explain the integer quantum Hall effect, because in that picture, no gap can arise within a Landau level. The presence of a plateau in the Hall conductance and the vanishing longitudinal voltage drop imply the presence of a (mobility) gap. The interactions between the electrons are crucial in the formation of such a gap.

The first step in explaining the fractional quantum Hall effect was made by R. Laughlin [67], who proposed a set of quantum Hall states, in the form of trial wave functions, which were shown to capture the basic features of the fractional quantum Hall states. The Laughlin wave functions are variational wave functions for the problem of two dimensional interacting electrons in the presence of a magnetic field. The hamiltonian for this system is as follows

$$\mathcal{H} = \frac{1}{2m_e}\sum_i \left(-i\hbar\nabla + eA_{\rm EM}\right)^2 + \frac{e^2}{2}\int d^2x d^2x' \frac{\rho(x)\rho(x')}{4\pi\epsilon|x-x'|}\,. \tag{1.1}$$

The first term describes the kinetic energy of the electrons (with $m_e$ the (effective) mass of the electrons and $A_{\rm EM}$ the vector potential for the magnetic field), while the second term represents the Coulomb interaction ($\rho(x)$ is the electron density in the two-dimensional

system). In this description it is assumed that all the electrons are polarized by the strong magnetic field. The Laughlin wave functions are obtained from this hamiltonian by doing a variational calculation. There are certain constraints on the wave functions, namely, they have to be antisymmetric under the exchange of any two electrons and they need to be eigenstates of the total angular momentum operator. In addition, we assume that the interactions are taken into account via a Jastrow factor, which is a two-body correlation, keeping the electrons apart. This factor has the general form $\prod \lambda(z_i - z_j)$. Taking these constraints into account leads to the following form of the variational wave functions (using the symmetric gauge in describing the magnetic field which is perpendicular to the plane of the electrons)

$$\Psi_{\mathrm{L}}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^M e^{-\sum_i \frac{|z_i|^2}{4l}} , \tag{1.2}$$

where we used complex coordinates $z_i$ to represent the position of the electrons, while the magnetic length, $l = \sqrt{\frac{\hbar}{eB}}$ is the basic length scale. The quantum Hall systems these wave functions describe have filling fraction $\nu = \frac{1}{M}$. They are not the exact ground state wave functions for the Coulomb interaction, but they were shown to have very good overlap with the numerically obtained ground state wave functions for a large class of repulsive interactions. So studying these wave functions is a good starting point to study properties of the quantum Hall systems at $\nu = \frac{1}{M}$. On should keep in mind however, that many properties, such as the behaviour of the transition from one quantum Hall state to another, can not be addressed in this way. What can be learned in this approach are properties of the excitations over these qH systems, and they turn out to be very interesting.

Before we go on to discuss the properties of the excitations over the quantum Hall systems, we will first briefly discuss the other fractional quantum Hall systems, at filling fractions $\nu = \frac{p}{q}$, with $p > 1$. From the experimental plot in figure 1.1, it can be seen that all the fractions have an *odd* denominator. The Laughlin states only describe a systems of fermions when $M$ is odd. To explain the other quantum Hall systems, Jain proposed a scheme in which *composite fermions* play a crucial role [60]. In this approach, an even number of flux quanta is bound to the electrons, to form the composite fermions. These composite fermions effectively feel a reduced magnetic field, and can form a integer quantum Hall system. The filling fraction of the original electrons becomes $\nu = \frac{p}{2pn \pm 1}$. Almost all the fractions observed can be obtained in this way. Note that the filling fractions in the composite fermion scheme all have an odd denominator, related to the fact that the quantum Hall systems are built from electrons.

## 1.1 Excitations in quantum Hall systems

Quasiholes in quantum Hall systems can be 'made' by locally increasing the magnetic flux through the sample in an adiabatic manner. These quasiholes can be shown to have a *fractional* charge, and also the statistics is fractional, in the sense that it interpolates between fermi and bose statistics.

But before we come to the point of the statistics, we will first show how the fractional charge of the quasiholes arise, in the case of the Laughlin states. So we are in a situation

where the conductance of the system is completely off-diagonal, and quantized, $\sigma_{\rm H} = \frac{1}{M}\frac{e^2}{h}$, $\sigma_{\rm L} = 0$. If we now locally increase the magnetic flux through the sample by one flux quantum $\Phi_0 = \frac{h}{e}$ (in a adiabatic fashion), this has the effect of expelling some charge form this region. The amount of charge expelled is calculated to be $q_{\rm qh} = \frac{1}{M}$, in units where the charge of the electron is $q_{\rm el} = -1$. The prediction of the existence of quasiparticles with fractional charge has been confirmed for the $\nu = \frac{1}{3}$ Laughlin state by means of shot noise experiments [80, 89]. Later on, also quasiholes with charge $\frac{1}{5}$ have been observed in a $\nu = \frac{2}{5}$ quantum Hall state [87].

Let us now turn to the statistics of these quasihole excitations. As said above, the exchange statistics of the quasiholes is *fractional*. It is only in two-dimensional systems that particles which satisfy (braid) statistics other than the familiar fermi of bose statistics can occur. In three dimensions (or more), different exchange paths of two identical particles can be continuously deformed into each other. Thus, after two successive exchanges, we come back to the original system, described by the same wave function. The phase factor corresponding to the exchange of two particles has to be $\pm 1$. In two dimensions, the situation is different, because the exchange paths can not be deformed into each other. The reason is that one would have to pull the path through the position of one of the particles, which is not allowed. Mathematically, this is explained in terms of the fundamental group of the configuration space, which is euclidian space with $N$ punctures at the positions of the particles. For a 2-dimensional system, this fundamental group is the *braid group*, while in 3 or more dimensions, it is the permutation group.

The Laughlin quasiholes transform according to a 1-dimensional representation of the braid group $\mathcal{B}_N$. The phase factor the wave function picks up when two quasiholes are exchanged can be calculated from the Aharonov-Bohm effect [1]. Due to the Aharonov-Bohm effect, particles can interact via the gauge potentials (of for instance the magnetic field), while the electro-magnetic fields themselves may vanish at the position of the other particle. For the Laughlin quasiholes, the phase factor $e^{i\pi\theta}$ corresponding to the exchange is calculated to be $\theta = \frac{1}{M}$. The fractionally charged quasiholes of the Laughlin state indeed satisfy fractional statistics. However, the statistics of these quasiholes has not been measured directly, in contrast with the fractional charge.

Another interesting consequence of the fact that particles in two dimensions have to form a representation of the braid group rather than the permutation group is that higher dimensional representations can be possible (see, for instance, [44]). In this thesis, we will see many examples of quantum Hall systems, where this is indeed the case. The study of such quantum Hall systems was prompted by the observation of a quantum Hall effect at a filling fraction with an *even* denominator, which will be discussed in the next section.

## 1.2 The $\nu = \frac{5}{2}$ quantum Hall effect

All the quantum Hall states discussed so far had a filling fraction with an odd denominator, which was explained via the hierarchy schemes starting with the Laughlin states. In 1987, a first experimental indication was found of a quantum Hall effect at a filling fraction with an *even* denominator, namely $\nu = \frac{5}{2}$. In 1999, the corresponding Hall plateau was observed [76], proving beyond any doubt that there is indeed a quantum Hall effect at filling $\nu = \frac{5}{2}$,

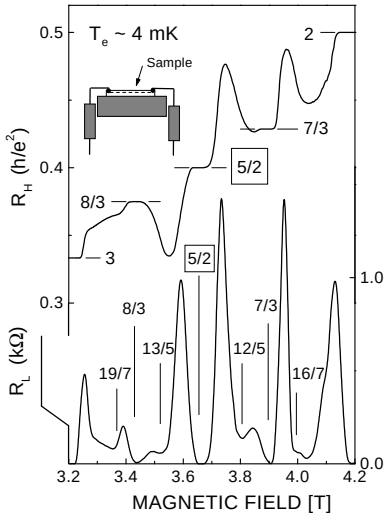


Figure 1.2: The $\nu = \frac{5}{2}$ plateau, figure taken from [76].

see figure 1.2. More recently, similar results were obtained for filling fraction $\nu = \frac{7}{2}$, see [27].

The composite fermion and hierarchy schemes can not explain these quantum Hall systems, as they only cover states with an odd denominator filling. So it was clear immediately that this quantum Hall system was different from all the others observed before. Many quantum Hall states with peculiar properties were proposed to account for this effect. Among these is the paired quantum Hall state of Moore and Read [71]. In this quantum Hall state, the electrons are spin-polarized, and form pairs, similar to the Cooper pairs of BCS superconductivity. Though it was first believed that the quantum Hall state at $\nu = \frac{5}{2}$ was not spin-polarized, nowadays, the experiments point out that this quantum Hall state is indeed spin-polarized [75]. Also, at the same filling fraction, but at high(er) temperatures, where the quantum Hall system has disappeared, Fermi-surface effects have been observed. This indicates that there might be a Cooper instability, which cases the electrons to pair. Subsequently, these pairs might condense to form the paired state of Moore and Read. For more on this, we refer to section 3.2. The experiments outlined above, together with numerous numerical evidence, has led to the consensus that the quantum Hall systems at $\nu = \frac{5}{2}$ can be described by the paired quantum Hall state of Moore and Read, which will be dealt with in great detail in section 3.2.

One of the interesting properties of the Moore-Read state is that the quasihole excitations over this state have interesting statistics properties. Due to the clustering of electrons, the system in which quasiholes are present can be formed in more than one way. For instance, if four quasiholes are present, the system can be in two different states (why this is so is explained in section 3.3). These states form a two-dimensional representation of the braid

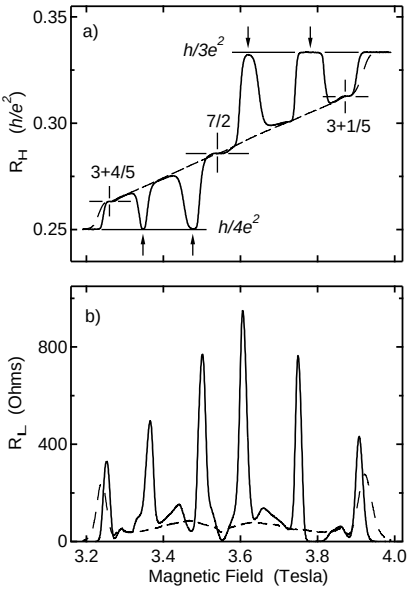


Figure 1.3: The $\nu = \frac{7}{2}$ plateau; solid curve, $T \approx 15\mathrm{mK}$; dashed curve, $T = 50\mathrm{mK}$. Figure taken from [27].

group. Exchanging the particles gives rise to phase matrices, and it can be shown that these matrices do not commute [71, 74, 95]. This form of statistics is therefore called *non-abelian* statistics.

The second Landau level is special in the sense that quantum Hall effects occurs at a filling fractions with an even denominator. Moreover, in [27], it was found that around this plateau other interesting states occur, which are shown in figure 1.3. In a sense, there is a reentrant integer quantum Hall state, disjunct from the ordinary integer quantum Hall plateaux. The nature of these states, which are also observed around filling fraction $\nu = \frac{5}{2}$ is unclear at this point.

In other half filled Landau levels, also very interesting physics is found. Let us first take a short look at the lowest half filled Landau level. This system can be described as a fluid of composite fermions, which live in the absence of magnetic field. They form a compressible Fermi liquid, in which no quantized Hall effect is observed, no plateau or vanishing longitudinal resistance.

In the higher Landau levels, another state is observed. This state shows a very anisotropic behaviour in the longitudinal resistance and is called a striped phase, see [111] for a study of the situation at $\nu = \frac{9}{2}$ (and references therein). This phase is observed at filling fractions $\nu = \frac{9}{2}, \ldots \frac{15}{2}$, and also at $\nu = \frac{5}{2}, \frac{7}{2}$, when an additional in-plane magnetic field is applied, but keeping the filling fraction, which is set by the component of the magnetic field perpendicular to sample, the same. So many electronic states are observed at half filled

Landau levels. Which state actually forms depends on the details of the energetics. We will not address this interesting problem, but instead focus on the quantum Hall states similar to the state which is now believed to form at filling $\nu = \frac{5}{2}$.

The observation of a quantum Hall effect at an even denominator filling fraction and the subsequent description by a system which exhibits non-abelian statistics has led to a great interest in quantum Hall systems with non-abelian statistics. In this thesis, we will describe various of these quantum Hall states, which can also be characterized by a clustering of the electrons.

Before we start with that discussion, we would like to point out another context in which the clustered states were shown to be relevant. This is the arena of the rotating Bose-Einstein condensates.

## 1.3 Rotating Bose-Einstein condensates

This section is the only section of this thesis in which the underlying particles are bosons instead of fermions. Recently, it has become clear that some of the clustered states described in this thesis can be relevant in the description of rotating Bose-Einstein condensates (BECs).

The hamiltonian describing $N$ weakly interacting atoms (bosons) in a rotating trap (with angular velocity $\omega\hat{z}$ and trap frequency $\omega_0$) is given by (see, for instance [24])

$$\begin{aligned}\mathcal{H} = &\sum_i \left( \frac{(\mathbf{p}_i^2 - m\omega\hat{\mathbf{z}}\times\mathbf{r})^2}{2m} + \frac{m}{2}\big((\omega_0^2-\omega^2)(x^2+y^2)+\omega_0 z^2\big)\right) \\ &+ g\sum_{i<j}\delta(\mathbf{r}_i-\mathbf{r}_j)\ . \end{aligned} \tag{1.3}$$

Here, the coupling $g = 4\pi\hbar^2 a/m$, giving rise to the correct $s$-wave scattering length $a$. In the limit of $\omega \sim \omega_0$, this hamiltonian describes a two dimensional system of particles with charge $q$ in a magnetic field $B = (2m\omega/q)\hat{\mathbf{z}}$ (see also [110]). In these systems, a filling fraction can be defined as the ratio of the number of bosons $N$ (which occupy an area $A$) and the average number of vortices $N_V = 2m\omega A/h$, thus $\nu = \frac{N}{N_V}$.

The vortices which are formed in the rotating BEC systems form a vortex lattice when the angular velocity is not to high. However, in the limit of $\omega \to \omega_0$, this vortex lattice melts, and the system becomes equivalent to the quantum Hall liquids.

The energy gap for a system with 6 vortices was studied in [24]. At various filling fractions, incompressible states have been found via cusps in the (numerically obtained) energy gaps. Among the fractions at which an incompressible state has been observed is $\nu = \frac{1}{2}$, which was interpreted as a (bosonic) Laughlin state [109]. However, incompressible systems at filling $\nu = 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, 6$ also have been found [24]. These states can not be described by simple Laughlin states, but based on overlap studies on the sphere, these states were interpreted as bosonic versions of the Read-Rezayi states [85], which are clustered analogs of the Moore-Read state [71]. Thus, the arena of the rotating Bose-Einstein condensates may also be a good place to look for bosonic quantum states with very peculiar properties, analogous to the quantum Hall states which will be studied in this thesis.

## 1.4 This thesis

In this thesis, we will concentrate on the description of quantum Hall states, using a conformal field theory approach. This will limit the questions we can ask, because we will not be able to address questions like in which way, and at what filling fractions do quantum Hall states form. What we can do is given a quantum Hall effect at a certain filling fraction, ask ourselves what are the possible states at this filling fraction? It turns out that quantum Hall states with very peculiar properties might form. There is now consensus on the nature of the $\nu = \frac{5}{2}$ state, which is believed to be a quantum state with a pairing structure. The excitations over this state are expected to show peculiar statistics properties, which go under the name of *non-abelian* statistics. This prompted the study of new classes of quantum Hall states, with similar properties. It is these quantum Hall states which will be the subject of this thesis.

To set the scene, we will have to introduce some conformal field theory methods, and point out the relation with quantum Hall systems, which will be done in chapter 2. Notably, we will explain the relation between conformal field theory, and *Chern-Simons* theory, which is used to describe quantum Hall systems.

Using the conformal field theory connection, new quantum Hall states can be defined, with similar pairing, or, in general, clustering properties as the quantum state which is related to the quantum Hall effect at $\nu = \frac{5}{2}$. One of the surprises was that there are spin full versions of the spin-polarized state at $\nu = \frac{5}{2}$, which are not only spin-singlets, but show a separation of the spin and charge degrees of freedom of the fundamental excitations over these states. This is on top of the non-abelian statistics, which the quasiholes also satisfy. The simplest of these states occurs at filling $\nu = \frac{2}{3}$, see section 3.6. The reason that the quasiholes of the clustered states can satisfy non-abelian statistics will be explained in section 3.3.

One of the main themes in this thesis is the study of the quasihole excitations in the clustered quantum Hall systems. In chapter 4, we will introduce the concept of *exclusion statistics*, which is used in the description of the excitations. Very important in this respect is the connection between the *composites* present in the electron sector, and the so-called *pseudoparticles*, which lie at the heart of the non-abelian statistics of the quasiholes. In fact, this connection comes back throughout this thesis. Another very important connection is the one between the exclusion statistics and the K-matrices which describe the topological properties of the particles in the quantum Hall states. It turns out that the exclusion statistics matrices, which are obtained in chapter 5 for the states under consideration in this thesis, also can be viewed as the K-matrices of the quantum Hall states.

One very nice application of the concepts introduced here, is the state counting of chapter 6. The quantum Hall states in this thesis can be viewed as (zero energy) eigenstates of idealized hamiltonians. These hamiltonians are constructed in such a way that the eigenstates have a clustering property, which is also present in the quantum Hall states. The hamiltonians obtained in that way can be diagonalized in a spherical geometry, in the presence of a magnetic field. The degeneracies obtained via this diagonalization study for the quantum Hall states in the presence of quasihole excitations can be explained in terms of the exclusion statistics properties of the *parafermion* fields present in the CFT description of the quantum Hall states. In a sense, the hamiltonians with the clustering property have

'knowledge' of the statistics properties of the parafermion fields!

With that result, we have come full circle. The clustering properties of the quantum Hall states are incorporated at the level of the underlying CFTs by the presence of parafermion fields. As a consequence, the quasihole excitations can have the so called non-abelian statistics (see section 3.3). This causes the states with quasiholes present to be degenerate, and, as discussed in chapter 6, these degeneracies can be understood in terms of the statistics properties of the parafermions.

# Chapter 2

# The quantum Hall - conformal field theory connection

Two dimensional systems are special in many respects, compared to higher dimensional systems. One example, which will come back frequently in this thesis, are the possible statistics properties. The group of local conformal transformations is infinite dimensional *only* in two dimensions, making *conformal field theory* a very powerful tool in the study of the critical behaviour of two dimensional systems, sometimes allowing for an exact solution of the 2-dimensional model under consideration.

Also in the study of the fractional quantum Hall effect, the use of conformal field theory (CFT) will be useful in determining, for instance, the statistics properties of the particles. However, it is very important to keep in mind that the systems we are describing in this thesis are *not* critical. There is thus a need to motivate the use of CFTs to describe quantum Hall systems. Also, one should keep in mind that the important problem of transitions between various quantum Hall states is not addressed in this thesis. We will assume throughout this thesis that a quantum Hall effect is observed at a certain filling fraction. We ask ourselves the question what kind of quantum Hall states can be formed and what are the properties of these states. Using conformal field theory, one can hope to provide a partial answer to this kind of questions.

The link between quantum Hall states and CFT can be made, in short, as follows. The effective field theories describing quantum Hall states are so called $2+1$-dimensional *Chern-Simons* theories (see, for instance,[119, 69]). A link between these topological field theories in $2+1$ dimensions and conformal field theory in $2$ (or $1+1$) dimensions was made by E. Witten in [112]. G. Moore and N. Read [71] proposed to use CFTs to describe and investigate the properties of quantum Hall systems. This gives us a powerful tool to study quantum Hall systems.

In this chapter, we will follow the line of reasoning as outlined above. We start with a short introduction to the topological field theories describing fractional quantum Hall states in section 2.1. These theories all have a common feature, the Chern-Simons (CS) term. In this context, the so-called K-matrices appear for the first time. These K-matrices play vari-

ous different roles throughout this thesis. The connection between the topological theories and conformal field theory will be given in section 2.2. In section 2.3, we will describe in which way these conformal field theories can be used to describe quantum Hall states and explain the construction using the Laughlin and Halperin states as examples. The conditions to be satisfied by the CFTs, in order to describe quantum Hall systems are given in appendix 2.A.

## 2.1 Topological theories

The connection between fractional quantum Hall states and conformal field theory is made via the presence of a topological effective field theory, the Chern-Simons theory. Therefore, we will first explain why the Chern-Simons term describes the low-energy limit of a quantum Hall system. In this section, we restrict the discussion to the ordinary, *abelian* case. The *non-abelian* case will be discussed in the following chapters.

The presence of a Chern-Simons term in the low energy effective field theory for quantum Hall states follows from the following, very general considerations (see, for instance, [107])

1. The system we are describing is $(2+1)$-dimensional system of electrons.
2. The electromagnetic current $J^{\rm em}_\mu$ is conserved: $\partial^\mu J^{\rm em}_\mu = 0$.
3. Parity and time reversal symmetry are broken by the magnetic field.
4. We want to describe the low energy part of the system with in a field theoretical setup.

In three dimensions, the conservation law $\partial^\mu J^{\rm em}_\mu = 0$ implies that $J^{\rm em}_\mu$ can be written as a curl: $J^{\rm em}_\mu = \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\partial^\nu a^\lambda$. In turn, making the change $a^\nu \to a^\nu + \partial^\nu \Lambda$ leaves the current $J^{\rm em}_\mu$ unchanged, implying that $a^\nu$ is a gauge potential. Invoking the assumption that we want an effective field theory for the long-distance, low frequency behaviour of the system, we will write down the most relevant term of gauge fields which is possible in $2+1$ dimensions which is also gauge invariant (see, for more details on gauge invariance, section 2.2). This turns out to be the Chern-Simons term (see [39] for more details)

$$\mathcal{L}_{\rm CS} = \frac{k}{4\pi}\varepsilon_{\mu\nu\lambda}a^\mu\partial^\nu a^\lambda \,. \tag{2.1}$$

Thus if the coupling $k$ is non-zero, this term will dominate the low energy behaviour. It turns out that this term indeed gives the correct physics. An important property of the Chern-Simons term is that it is a *topological* term: it does not depend on the metric. As a consequence, the corresponding hamiltonian is zero, and all eigenstates have zero energy. Naively, one might think that the problem has become trivial, or uninteresting. However, the topological properties and the degeneracy of the ground state make this problem very interesting indeed.

Above we showed that the low energy physics of an $(2+1)$ dimensional system is governed by the Chern-Simons term (2.1). Below, we consider a more general form, by allowing several gauge fields, which are coupled by a matrix, known in the literature as the

'K-matrix'. In addition, the gauge fields will be coupled to other fields, which for instance describe external fields. These terms also have the Chern-Simons form.

In the following, we will use the notation of Wen [105] in describing the *quantum Hall data*. This data characterizes the quantum Hall system; it consists of the K-matrix, a charge and spin vector, $\mathbf{t}$ and $\mathbf{s}$, respectively. These vectors have the charge and spin quantum numbers of the particles as there entries. In addition, there is an angular momentum vector $\mathbf{j}$. A few remarks with respect to the notation of spin vectors need to be made at this point. In [108, 107], the concept of a 'spin vector' was introduced. This 'spin vector' is in fact related to the angular momentum of the electrons on (for instance) the sphere and is needed to calculate the so-called shift (see equations (2.5) and (2.6) below). In our case we need to distinguish between this angular momentum vector and the vector containing the real $SU(2)$ spin of the particles. Therefore, we have denoted the angular momentum vector by $\mathbf{j}$, and the vector containing the spin quantum numbers by $\mathbf{s}$. Note that the spin quantum nubers are given in units of $h/2$ in the spin vectors throughout this thesis.

As stated above, the K-matrices play several roles in the description of the quantum Hall states which are discussed in this thesis. The entries of these matrices do not only serve as the couplings between the various gauge fields, they also can be interpreted as the (exclusion) statistics parameters of the particles, as is described in chapter 4. To be able to make contact with the 'statistics interpretation', we distinguish between the K-matrix for the 'electron part' and the quasihole part of the theory. These matrices are denoted by $\mathbb{K}_e$ and $\mathbb{K}_\phi$, respectively. The corresponding charge, spin and angular momentum vectors are $\mathbf{t}_e$, $\mathbf{t}_\phi$, $\mathbf{s}_e$, $\mathbf{s}_\phi$, $\mathbf{j}_e$, and $\mathbf{j}_\phi$ in an obvious notation. In all the cases we considered, it is possible to formulate the theory in such a way that the K-matrices for the quasihole and electron sectors are just each others inverse. This form of 'duality' will be encountered frequently in this thesis (see, in particular, [6] and [7]).

The Chern-Simons part of the lagrangian for a system on a surface of genus $g$ has the following from

$$\mathcal{L}_{\mathrm{CS}} = \frac{1}{4\pi}\varepsilon^{\mu\nu\lambda}\left(\mathbb{K}_e^{ij}a_\mu^i\partial_\nu a_\lambda^j + 2\mathbf{t}_e^i A_\mu\partial_\nu a_\lambda^i + 2\mathbf{j}_e^i\omega_\mu\partial_\nu a_\lambda^i + 2\mathbf{s}_e^i\beta_\mu\partial_\nu a_\lambda^i\right) , \qquad (2.2)$$

where the fields $a$ are the Chern-Simons gauge fields. The Greek indices run over $\{0,1,2\}$, and the Roman indices over the number of channels. The first term is the Chern-Simons term, which has the effect of changing the statistics of the matter-fields, which are coupled to the Chern-Simons (CS) gauge fields in a full theory [118]. The electromagnetic field is described by the gauge field $A_\mu$ and the coupling to the CS gauge fields involve the charge vector $\mathbf{t}_e$. This vector specifies the charge of the electronic degrees of freedom. The other terms of the lagrangian have the following interpretation. $\omega_\mu$ is the 'spin connection' and gives rise to the curvature of the space on which the quantum Hall system is defined. The last term is the ($SU(2)$) spin equivalent of the second term, and describes the spin Hall conductance. By analogy with the (electronic) filling fraction $\nu$, one can define the spin filling fraction, $\sigma$, and the spin Hall conductance.

In general, one would define the spin conductance in the same way as the charge conductance, namely as a response to a certain field. In the case of a quantum Hall system, the role of the electric field is taken over by a gradient in the Zeeman energy. The gauge field describing this is denoted by $\beta_\mu$ in eq. (2.2). The spin Hall conductance is related to the

'spin current' induced perpendicular to the direction of the gradient of the Zeeman energy.

Let us now briefly recall the results obtained from this formulation for the filling factors and the shift corresponding to a surface of genus $g$. The filling factors can be calculated by means of simple inner products[1]

$$\begin{aligned} \nu &= \mathbf{t}_e \cdot \mathbb{K}_e^{-1} \cdot \mathbf{t}_e = \mathbf{t}_\phi \cdot \mathbb{K}_\phi^{-1} \cdot \mathbf{t}_\phi \,, \\ \sigma &= \mathbf{s}_e \cdot \mathbb{K}_e^{-1} \cdot \mathbf{s}_e = \mathbf{s}_\phi \cdot \mathbb{K}_\phi^{-1} \cdot \mathbf{s}_\phi \,. \end{aligned} \tag{2.3}$$

The relation between the charge (and spin) vectors of the electron and quasihole parts are given by

$$\mathbf{t}_\phi = -\mathbb{K}_e^{-1} \cdot \mathbf{t}_e \,, \qquad\qquad \mathbf{s}_\phi = -\mathbb{K}_e^{-1} \cdot \mathbf{s}_e \,. \tag{2.4}$$

The last important property we will discuss is the so called 'shift' in the flux on surfaces of general genus $g$. The relation between the number of electrons $N_e$ and the corresponding number of flux quanta $N_\Phi$ is given by

$$N_\Phi = \frac{1}{\nu} N_e - \mathcal{S} \,, \tag{2.5}$$

where the shift $\mathcal{S}$ is given by

$$\mathcal{S} = \frac{2(1-g)}{\nu} \left( \mathbf{t}_e \cdot \mathbb{K}_e^{-1} \cdot \mathbf{j}_e \right). \tag{2.6}$$

Although $\mathbf{j}_e$ plays a somewhat different role than $\mathbf{t}_e$ and $\mathbf{s}_e$, we define $\mathbf{j}_\phi$ by analogy to (2.4)

$$\mathbf{j}_\phi = -\mathbb{K}_e^{-1} \cdot \mathbf{j}_e \,. \tag{2.7}$$

The equations (2.3)-(2.7) were derived in the context of abelian quantum Hall states [105]. In section 4.2.2, we will see that these relations also hold, under certain conditions, in the non-abelian cases as well.

We end this section by noting that the description in terms of the quantum Hall data is by no means unique. The lagrangian eq. (2.2) is invariant under $SL(n,\mathbb{Z})$ transformations on the K-matrix (denoted by the matrix $\mathbb{W}$) (for more details, see for instance [105])

$$\begin{aligned} \tilde{\mathbb{K}}_{\rm e} &= \mathbb{W} \cdot \mathbb{K}_{\rm e} \cdot \mathbb{W}^T \,, & \tilde{\mathbb{K}}_\phi &= (\mathbb{W}^{-1})^T \cdot \mathbb{K}_{\rm qh} \cdot \mathbb{W}^{-1} \,, \\ \tilde{\mathbf{v}}_{\rm e} &= \mathbb{W} \cdot \mathbf{v}_e \,, & \tilde{\mathbf{v}}_\phi &= (\mathbb{W}^{-1})^T \cdot \mathbf{v}_\phi \,, \end{aligned} \tag{2.8}$$

where $\mathbf{v}$ is one of the vectors containing the quantum numbers of the system. It is easily checked that this leaves all the physical properties of the system, such as the filling fraction, invariant. We will refer to the transformation (2.8) as a 'W-transformation'.

[1] Throughout this thesis, the transpose on the vectors in equations like (2.3) is implicitly understood in order to simplify the notation.

## 2.2 Topological field theory and CFT

In this section, the relation between topological field theories, the Chern-Simons theories of the previous section, and conformal field theory is described.

Let us start with the gauge invariance of the Chern-Simons action (2.1). Making the transformation $a^\mu \mapsto a^\mu + \partial^\mu \Lambda$ leaves the action invariant, up to a surface term

$$\delta \mathcal{L} = \varepsilon_{\mu\nu\lambda} \partial^\mu (a^\nu \partial^\lambda \Lambda) \; . \tag{2.9}$$

Of course, on a physical sample with an edge, this surface term is important, and it is cancelled by the gauge transformation of the so called edge currents. Already in 1982, B.I. Halperin realized the importance of the current-carrying edge states [54]. They occur because the confining potential lifts the energy of the Landau levels. At the edge of the sample, they cross the Fermi surface, and this leads to gapless edge excitations. The transformation properties of these edge-currents makes the complete system gauge invariant, as it of course should be.

The dynamics on the edge is described by a *chiral Luttinger liquid* ($\chi ll$) [103, 102, 104], described by the lagrangian

$$\mathcal{L}_{\chi ll} = \frac{1}{4\pi} \int \mathrm{d}t \mathrm{d}u \big( (\partial_t \varphi)^2 - v^2 (\partial_u \varphi)^2 \big) \; , \tag{2.10}$$

and the chirality constraint $\partial_t \varphi = v \partial_u \varphi$; $u$ is the space coordinate along the edge. In fact, the lagrangian describes a massless scalar field, and is a conformal field theory. See [94] for a discussion of the chiral Luttinger liquid in the context of a unified field theory for fractional quantum Hall systems.

The $\chi ll$ of eq. (2.10) has a conserved current, which is given by

$$J^{\mathrm{edge}}_\mu = \varepsilon_{\mu\nu} \partial^\nu \varphi \; , \tag{2.11}$$

where $\varepsilon_{\mu\nu}$ is the anti-symmetric symbol, and we work in units where $v = 1$.

Using canonical quantization, it can be shown that the Fourier modes of the edge currents $J^{\mathrm{edge}}_0$ satisfy the algebra

$$[j_m, j_n] = \delta_{m+n,0} \sigma_H \; , \tag{2.12}$$

which is a $\mathfrak{u}(1)$ Kac-Moody algebra. Thus, the edge degrees of freedom can be described by a conformal field theory, which is related to the topological field theory, as was pointed out above. Thus, at least for the quantum Hall states at simple filling fractions, it has become clear that the dynamics of the edge excitations are described by a simple conformal field theory. The situation for systems in which counter-flowing edge modes occur is much more complicated, and still under debate. However, for systems in which all the edge modes flow in one direction, this description is undisputed. In this thesis, we will consider this situation, but consider more complicated situations than the one described above.

So we pointed out the connection between the topological field theory in $2+1$ dimensions, and the conformal field theory (a chiral Luttinger liquid) on the $1+1$ dimensional edge of a quantum Hall sample. There is also a correspondence between the topological

field theory and a conformal field theory describing the bulk properties (such a the braid statistics of the excitations). This connection is due to Witten [112]. It states that the physical Hilbert space of a three dimensional topological Chern-Simons theory can be interpreted as the space of conformal blocks of the corresponding Wess-Zumino-Witten model in two dimensions. So the particles (sources) in the Chern-Simons theory can be interpreted as the fields in the conformal correlators. In [37], this connection is worked out in detail, giving consistency conditions for the (chiral) conformal field theory. These will be quoted in appendix 2.A. But before we come to that point, we first explain in which way CFT is used to study quantum Hall systems, concentrating on some simple examples.

## 2.3 Quantum Hall systems and conformal field theory

With the connection between topological Chern-Simons field theory and (chiral) correlators (or conformal blocks) at hand, G. Moore and N. Read proposed to use conformal field theory as a method to describe quantum Hall states. More specifically, they used the conformal blocks of chiral CFTs to obtain (representative) wave functions for known quantum Hall systems. Pushing the connection further, they proposed to other CFTs in a similar way, to define new states, which can be interpreted (under certain conditions) as quantum Hall states. Using this setting, the possible excitations of these new quantum Hall systems and their properties can be studied. We refer to [34] for a details on conformal field theories.

Throughout this thesis, we will assume that a quantum Hall system is observed at a certain filling fraction. Using CFT, one can try to construct a quantum state and study its properties. However, we will not be able to say which of the various possible states will actually occur, as the answer to this question depends on the details of the energetics.

Interpreting chiral correlators as wave functions for actual quantum systems also should be done with care. The details of the wave functions of quantum systems depend on the details of the interaction. However, as long as the real interaction between the electrons lies in the same universality class as the interaction for which the wave function is an exact ground state, the topological properties of the particles are the same in both cases. Here, we made an implicit assumption, namely that there is a, possibly idealized, interaction for which the wave function under investigation is exact. The existence of such an interaction is very important. As we will see in chapter 6, the idealized interaction can be used to study the properties of the quasihole excitations in the quantum Hall systems via numerical diagonalization studies. These results will be compared to analytic results, obtained by using the underlying conformal field theory for these systems. Knowledge about the ground state wave functions is not necessary in these studies. In fact, in many cases, the actual form of the ground state wave functions in the presence of quasiholes is not known.

In this thesis, we will mainly concentrate on states in the lowest Landau level. As the gap to the higher Landau Levels is of order $\hbar\omega_c$, which is much larger than all the other energy scales (the Zeeman and electron interaction energy) in the system, we can neglect excitations in which higher Landau levels are involved, as long as we restrict ourselves to filling fractions $\nu < 2$ (note that a Landau level completely filled with spin up and down electrons gives $\nu = 2$). As all the electrons are in the lowest Landau level, the kinetic energy is just a constant. Thus, the wave functions for quantum Hall states in the lowest Landau

level can be viewed as eigenstates of an interaction hamiltonian.

Wave functions for quantum Hall systems in the lowest Landau level can be written in the form

$$\Psi(\{z_i\}) = \widetilde{\Psi}(\{z_i\}) e^{-\sum_i \frac{|z_i|^2}{4l^2}} , \tag{2.13}$$

where the $z_i$ are the complex electron coordinates and $l = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length. $\widetilde{\Psi}(\{z_i\})$ is a polynomial in the electron coordinates and is called the 'reduced wave function'. The important point is that this polynomial only depends on $z_i$ and not on the complex conjugates $\bar{z}_i$. Note that $\Psi(\{z_i\})$ is not normalized. In this thesis, we concentrate on the reduced wave functions, but simply speak of 'wave functions' (we keep the tilde, however, to remind the reader of this).

We will now describe the way in which conformal correlators can be used to define quantum Hall systems. This will be done using two examples, namely the Laughlin and Halperin states. In the next chapter, we will define the clustered quantum Hall states and investigate their properties in subsequent chapters.

### 2.3.1 Example: the Laughlin wave function

Following the reasoning of Moore and Read, the wave functions for quantum Hall systems can be written as correlators in a chiral conformal field theory. We explain how this can be done using the Laughlin wave functions as an example, as they are the simplest fractional quantum Hall states. In the next subsection, we will treat the (somewhat) more complicated Halperin states [55].

The Laughlin wave functions,

$$\widetilde{\Psi}_{\mathrm{L}}^{M}(\{z_i\}) = \prod_{i<j} (z_i - z_j)^M , \tag{2.14}$$

can be reproduced as a correlator of vertex operators of a free boson $\varphi$, compactified on a radius $R^2 = M$. This vertex operator has the form

$$V_{\mathrm{el}}(z_i) = :e^{i\sqrt{M}\varphi}(z_i): . \tag{2.15}$$

The correlator which gives the Laughlin factor is

$$\widetilde{\Psi}(\{z_i\}) = \lim_{z_\infty \to \infty} z_\infty^{MN^2} \langle V_{\mathrm{el},1} V_{\mathrm{el},2} \cdots V_{\mathrm{el},N} : e^{-i\sqrt{M}N\varphi}(z_\infty) : \rangle . \tag{2.16}$$

The background charge is inserted in the correlator to satisfy charge neutrality. The factor $z_\infty^{MN^2}$ is inserted to obtain a non-zero result in the limit where the background charge is sent to infinity. This procedure of dealing with the background charge is different compared to [71], where a homogeneous background charge was used. Using a homogeneous background charge is more involved, but has the advantage of also reproducing the exponential factors (with the magnetic length set to $l = 1$) of the full wave function [71].

That the correlator in eq. (2.16) indeed reduces to the Laughlin wave function eq. (2.14), can be seen from the operator product expansion (OPE) for the vertex operators

$$: e^{i\alpha\varphi}(z_1) : : e^{i\beta\varphi}(z_2) : = (z_1 - z_2)^{\alpha\beta} : e^{i(\alpha+\beta)\varphi}(z_2) : , \tag{2.17}$$

and the use of Wick's theorem. One of the most important gains of using CFT to describe quantum Hall systems is the fact that also the quasiholes can be represented in this setup. Once this has been established, on can use the powerful methods of CFT to study the properties of the quasiholes. Some of the properties which can be addressed in this way are the statistics and the degeneracy of quantum Hall systems in the presence of quasiholes. This can be done, even though sometimes the correlators containing the quasiholes are not known explicitly.

This brings us to the subject of the quasihole states. As assumed in the above, these states can also be written as a correlator in the conformal field theory, by inserting quasihole operators in the chiral correlators. In the case of Laughlin quasiholes, the operators which create the quasiholes are also vertex operators

$$V_{\mathrm{qh}}(w) =: e^{i/\sqrt{M}\varphi}(w) : \ . \tag{2.18}$$

In the following, we denote the electron and quasihole coordinates by $z_i$ and $w_j$, respectively. Inserting $n$ quasihole operators and an adjusted background charge in the correlator, together with the electron operators gives an expression for the wave function of a Laughlin system with quasiholes

$$\begin{aligned}\widetilde{\Psi}^{M}_{\mathrm{L,qh}}(\{z_i, w_i\}) = \lim_{z_\infty \to \infty} z_\infty^{M(N+\frac{n}{M})^2} \langle V_{\mathrm{el},1} V_{\mathrm{el},2} \cdots V_{\mathrm{el},N} \\ \times V_{\mathrm{qh},1} V_{\mathrm{qh},2} \cdots V_{\mathrm{qh},n} : e^{-i(\sqrt{M}N + n/\sqrt{M})\varphi}(z_\infty) : \rangle \, .\end{aligned} \tag{2.19}$$

Evaluating this correlator gives the wave function for the Laughlin state in the presence of quasiholes

$$\widetilde{\Psi}^{M}_{\mathrm{L,qh}}(\{z_i, w_j\}) = \prod_{i<j} (w_i - w_j)^{\frac{1}{M}} \prod_{i,j} (z_i - w_j) \prod_{i<j} (z_i - z_j)^M \, . \tag{2.20}$$

This form corresponds to the form proposed by Laughlin in [67].

The wave functions of quantum Hall states with quasiparticles are more difficult to write down than there quasihole counterparts. This is caused by the fact that the quasiparticle wave functions involve non-analytic functions. At present, the corresponding CFT quasiparticle operators are not known. We will therefore concentrate on the quasihole excitations.

As can be seen from the wave function (2.20), the electrons are expelled from the from the positions of the quasiholes. The strength of this repulsion however, is weaker than the strength with which the electrons repel each other. This is directly related to the fact that the charge of the quasiholes is *fractionalized*; As explained in chapter 1, the charge of the Laughlin quasiholes is $q_{\mathrm{qh}} = \frac{1}{M}$, in units where the electron charge is $q_{\mathrm{e}} = -1$. Because the electrons are expelled from the regions $w_i$, there is in fact a charge deficit, and the quasiholes have opposite charge in comparison to the electrons. For quantum Hall states with spin, a similar 'inversion' of quantum numbers for the quasiholes occurs.

It is important to keep in mind that specifying a conformal field theory (in this case the $c = 1$ chiral boson compactified on a radius $R^2 = M$) is not enough to define a quantum Hall state. In addition, the form of the electron operator(s) is also needed to define the quantum Hall state.

### 2.3.2 The Halperin states

As an example of a series of states in which the spin degree of freedom is not neglected, we discuss the simplest case, namely the Halperin states. In general, these states are written in terms of two types of coordinates, which can also stand for two layers. For now, we keep the discussion general and the Halperin wave functions take the form [55]

$$\widetilde{\Psi}_{\mathrm{H}}^{(m,m',n)}(\{z_i^a, z_j^b\}) = \prod_{i<j}(z_i^a - z_j^a)^m \prod_{i<j}(z_i^b - z_j^b)^{m'} \prod_{i,j}(z_i^a - z_j^b)^n \,, \tag{2.21}$$

This state describes a double layer (or spin) system, in which the two layers (or particles with opposite spins) are coupled by via the third factor. The filling fraction can be determined by means of the K-matrix formalism. For (simple) abelian quantum Hall states, the K-matrices can easily be obtained from the braid behaviour of the electrons, as encoded in the wave functions. For the Halperin states, the K-matrix and charge vector read

$$\mathbb{K}_e = \begin{pmatrix} m & n \\ n & m' \end{pmatrix} \,, \tag{2.22}$$

$$\mathbf{t}_e = -(1,1) \,. \tag{2.23}$$

The corresponding filling fraction becomes $\nu = \frac{m+m'-2n}{mm'-n^2}$. Restricting to the case $m-1 = m'-1 = n$ the Halperin states are spin-singlets (if the coordinates are interpreted as the coordinates of spin up and down electrons). This gives rise to a series of singlet states with filling fraction $\nu = \frac{2}{2m+1}$. For $m$ even, this state is describing a fermionic spin-singlet state. The condition for the total state, including the spin part of the wave function, to be a spin-singlet translates into the *Fock cyclic* conditions of the positional part of the wave function [56]. These conditions on the orbital part of the wave function read (see also Girvin, in [81])

$$\widetilde{\Psi} \pm \sum_j e(z_i^{\uparrow}, z_j^{\downarrow})\widetilde{\Psi} = 0 \,, \tag{2.24}$$

where the $+$ $(-)$ sign applies for a bosonic (fermionic) state respectively. The operator $e(z_i^{\uparrow}, z_j^{\downarrow})$ exchanges the coordinates $z_i^{\uparrow}$ and $z_j^{\downarrow}$ in the function on which it acts. It is easily shown that the Halperin wave functions (2.21), with parameters $(m+1, m+1, m)$ satisfy the conditions (2.24) and thus are spin-singlet states. More physically, the case $m = 0$ corresponds to a completely filled Landau level (with spin up and down electrons), which is a spin-singlet. Changing $m$ will only result in the multiplication of the wave function with a completely symmetric factor, with the only effect of changing the properties of the state which correspond to the charge degrees of freedom.

Let us now turn to the description of these states in terms of conformal field theory. In this description, two chiral boson fields are needed: $\varphi_c$ and $\varphi_s$, which describe the charge and spin degrees of freedom, respectively. The electron operators for the spin up and spin down electrons are

$$V_{\mathrm{el}}^{\uparrow} = :e^{\frac{i}{\sqrt{2}}(\sqrt{2m+1}\varphi_c + \varphi_s)}(z^{\uparrow}):, \tag{2.25}$$

$$V_{\mathrm{el}}^{\downarrow} = :e^{\frac{i}{\sqrt{2}}(\sqrt{2m+1}\varphi_c - \varphi_s)}(z^{\downarrow}):\ . \tag{2.26}$$

The quasihole operators can also be written in terms of $\varphi_c$ and $\varphi_s$.

$$V_{\rm qh}^{\uparrow} = :e^{\frac{i}{\sqrt{2}}(\frac{1}{\sqrt{2m+1}}\varphi_c+\varphi_s)}(w^{\uparrow}):, \tag{2.27}$$

$$V_{\rm qh}^{\downarrow} = :e^{\frac{i}{\sqrt{2}}(\frac{1}{\sqrt{2m+1}}\varphi_c-\varphi_s)}(w^{\downarrow}): . \tag{2.28}$$

The scaling dimensions are given by $\Delta_{\rm e} = \frac{m+1}{2}$ and $\Delta_{\rm qh} = \frac{m+1}{4m+2}$. The correlator which gives the Halperin $(m+1, m+1, m)$ spin-singlet states can now be written as

$$\begin{aligned}
\widetilde{\Psi}^{m}_{\rm H,qh} &= \lim_{z_\infty\to\infty} z_\infty^a \langle V^{\uparrow}_{{\rm el},1}\cdots V^{\uparrow}_{{\rm el},N_\uparrow} V^{\downarrow}_{{\rm el},1}\cdots V^{\downarrow}_{{\rm el},N_\downarrow} V^{\uparrow}_{{\rm qh},1}\cdots V^{\uparrow}_{{\rm qh},n_\uparrow} V^{\downarrow}_{{\rm qh},1}\cdots V^{\downarrow}_{{\rm qh},n_\downarrow} \\
&\quad \times :e^{\frac{-i}{\sqrt{2}}\left((\sqrt{m+1}(N_\uparrow+n_\downarrow)+\frac{1}{\sqrt{m+1}}(n_\uparrow+n_\downarrow))\varphi_c+(N_\uparrow-N_\downarrow+n_\uparrow-n_\downarrow)\varphi_s\right)}(z_\infty):\rangle \\
&= \prod_{i<j}(z_i^{\uparrow}-z_j^{\uparrow})^m \prod_{i<j}(z_i^{\downarrow}-z_j^{\downarrow})^m \prod_{i,j}(z_i^{\uparrow}-z_j^{\downarrow})^{m-1} \\
&\quad \times \prod_{i,j}(z_i^{\uparrow}-w_j^{\uparrow}) \prod_{i,j}(z_i^{\downarrow}-w_j^{\downarrow}) \\
&\quad \times \prod_{i<j}(w_i^{\uparrow}-w_j^{\uparrow})^{\frac{m+1}{2m+1}} \prod_{i<j}(w_i^{\downarrow}-w_j^{\downarrow})^{\frac{m+1}{2m+1}} \prod_{i,j}(w_i^{\uparrow}-w_j^{\downarrow})^{\frac{-m}{2m+1}} .
\end{aligned} \tag{2.29}$$

The total numbers of electrons and quasiholes are denoted by $N = N_\uparrow + N_\downarrow$ and $n = n_\uparrow + n_\downarrow$, respectively. In eq. (2.29), we inserted the most general background charge. But as we are interested in spin-singlet states, we need to impose the constraint that the 'back ground charge' only consists of the charge boson $\varphi_c$. This leads to $N_\uparrow + n_\uparrow = N_\downarrow + n_\downarrow$, a necessary condition for the state to be a spin-singlet. Using this constraint, the parameter $a$ is calculated to be $a = \frac{2m+1}{2}(N + \frac{n}{2m+1})^2$. We used the same methods as in the previous section to work out the correlator. This state is the spin-singlet state at filling fraction $\nu = \frac{2}{2m+1}$ mentioned above.

The construction of the last two subsections can be generalized to obtain the so-called *clustered* states. This will be the subject of the next chapter. But before we come to that, we first give conditions which need to be satisfied by the conformal field theory and the operators in order to describe a quantum Hall system.

## 2.A Constraints on the CFTs for quantum Hall systems

In this subsection, we will point out the consistency conditions, which need to be satisfied by a chiral CFT and the electron operators, in order to (possibly) describe a quantum Hall state. We will follow the discussion given in [37].

First of all, we have to specify the conformal field theory itself; this is done by specifying the *chiral algebra* $\mathcal{A}$, and the set of *unitary irreducible representations* $\Lambda$. The chiral algebra determines the symmetry of the conformal field theory. For the quantum Hall states, the symmetry is associated to simple Lie algebras. $\Lambda$ must at least contain a unique vacuum, $\omega$, which has scaling dimension $\Delta_\omega = 0$. The electrical current is described by a chiral $\mathfrak{u}(1)$ current algebra. Thus the chiral algebra must at least contain one $\mathfrak{u}(1)$ current algebra.

Moreover, we will assume the chiral algebra has the form (see [37] for states based on minimal models)

$$\mathcal{A} = \mathcal{P} \otimes \mathfrak{u}(1)^{r-1} \otimes \mathfrak{u}(1)_c \,, \tag{2.A1}$$

where $r \geq 1$ and $\mathcal{P} \otimes \mathfrak{u}(1)^{r-1}$ is an electrically neutral chiral algebra. $\mathfrak{u}(1)_c$ denotes the $\mathfrak{u}(1)$ current algebra associated to the charge degree of freedom. In the cases discussed in this thesis, $\mathcal{P}$ has a parafermionic symmetry. In those cases, $\mathcal{A}$ is a affine lie algebra. Note that we will also allow a deformation of the affine Lie algebra symmetry by a modification of the 'charge direction' . This deformation can be described by a so called *shift map*, see [105, 38, 7]. The set of unitary irreducible representations of $\mathcal{P}$ has to be finite and closed under fusion.

The conditions to be satisfied by the conformal field theory and the operators corresponding to the electrons and quasiholes can now be stated. They all are motivated by general physical considerations.

1. The presence of the electron.
   As we are describing quantum Hall states, we must at least have one representation $\Lambda_\mathrm{e}$ which corresponds to the electron. This (set of) representation(s) can be used to define the set of multi-electron representations $\Lambda_m$, which are obtained by fusions of the electrons. These can be thought of the electron clusters, which will play an important role in the K-matrix formulation of the non-abelian states discussed in chapter 5.

2. Physically realized representations.
   The particles (electrons and excitations) of the qH system are labeled by the unitary representations of the chiral algebra, and because $\mathcal{A}$ has the special form (2.A1), the unitary representations can be decomposed as

   $$\Lambda \subseteq \Lambda_p \times \mathbb{R}^r \,. \tag{2.A2}$$

   Thus the labels of the excitations take the form $(\lambda_p, \mathbf{r})$, where $\lambda_p \in \Lambda_p$ and $\mathbf{r}$ is a point in $\mathbb{R}^r$.

3. Charge and statistics of *electron-like* particles
   The statistics related to a particle with label $\lambda$ can be related to the scaling dimension $\Delta_\lambda$. Upon a rotation of $2\pi$, a phase factor $e^{2\pi i \Delta_\lambda}$ is picked up. Thus if $\Delta_\lambda \in \mathbb{N}$, the excitation is a boson, if $\Delta_\lambda \in \mathbb{N} + \frac{1}{2}$, the particle obeys fermionic statistics, and finally, if $\Delta_\lambda \not\equiv 0 \pmod{\frac{1}{2}}$, the excitation obeys fractional statistics.

   It is natural to assume that the multi-electron particles obey bose statistics if they are obtained by the fusion of an even number of electrons, and fermi statistics if they resulted from the fusion of an odd number of electrons. Thus we impose a constraint on the multi-electron particles, namely a *charge - statistics* relation

   $$\begin{aligned} (\mathbf{t}_e)_m = 0 \pmod 2 &\Longrightarrow \Delta_m = 0 \pmod 1 \,, \\ (\mathbf{t}_e)_m = 1 \pmod 2 &\Longrightarrow \Delta_m = \frac{1}{2} \pmod 1 \,, \end{aligned} \tag{2.A3}$$

   where $m$ runs over all the electron-like particles.

4. Condition on the quasiparticles
   The wave functions for the quantum Hall states are analytic in the coordinates of the electron like particles. This lays constraints on the possible quasihole excitations, which have to be *relatively local* with the electron like particles. This condition reads

$$\Delta_\lambda + \Delta_m - \Delta_{\lambda'} = 0 \pmod 1 , \tag{2.A4}$$

   where $\lambda'$ is in the fusion product of $\lambda$ and $m$.

5. Charge and spin
   Just as the charge of the multi electron particles is determined by the electrons, this also holds for the spin, if it is a good quantum number in the system under consideration. The following connection between charge and spin can by found

$$\begin{aligned} (\mathbf{t}_e)_m = 0 \pmod 2 &\Longrightarrow s_m = 0 \pmod 1 , \\ (\mathbf{t}_e)_m = 1 \pmod 2 &\Longrightarrow s_m = \frac{1}{2} \pmod 1 , \end{aligned} \tag{2.A5}$$

   where $s_m$ is the $su(2)$ spin of the multi electron particle $m$.

Let us end this chapter by saying that the conformal field theories and particle operators used in the next chapter, do indeed satisfy the conditions stated in this section. This guarantees that the quantum Hall systems described in the next section do have the right physical properties.

# Chapter 3

# Clustered quantum Hall states

In this chapter, we will define sets of *clustered* quantum Hall states, and study some of their properties. G. Moore and N. Read (MR) were among the first to propose a quantum Hall state with a clustering, or better, pairing structure [71]. It was motivated by the observation of a quantum Hall effect at *even* denominator filling fraction $\nu = \frac{5}{2}$. Subsequently, this state was generalized to a series of clustered quantum Hall states by N. Read and E. Rezayi (RR) [85].

The states mentioned above are all considered to be spin-polarized states; the electron spin is absent in the construction of these states. In [10], spin was included in a natural way, and spin-singlet analogs of the states in [85] were constructed. In this chapter, we will describe in which way these clustered quantum Hall states can be constructed, and some of the properties are discussed. This chapter will mainly be concerned with the construction of the states by defining the electron and quasihole operators. We will provide explicit forms of the wave functions for states without quasiholes. The properties of these quasiholes will be studied in great detail in the following chapters.

The last set of states we will treat in this chapter are the spin-singlet states of [8]. These states have a clustering which is somewhat different with respect to the other states discussed in this chapter. Interestingly, the excitations over these states show a separation of their $SU(2)$-spin and charge degrees of freedom.

The outline of this chapter is as follows. We start in section 3.1 by defining a clustering property for qH states. In the subsequent sections, various clustered quantum Hall states will be described, on the level of the underlying CFT. Some of their properties will be addressed. Note that the description in terms of K-matrices will be given in chapter 5.

The paired state proposed by Moore and Read [71] will serve to explain the construction in general, as it is the simplest example of a clustered state (section 3.2). In section 3.3, we will explain why the states discussed in this chapter are called *non-abelian* quantum Hall states. The key point here is the structure of the quasiholes, which in turn is closely related to the clustering property of the electrons.

The clustered generalizations of the Moore-Read state are treated in section 3.4. The spin-singlet analogs of these states are discussed in section 3.5, while the states which show a separation of the spin and charge degrees of freedom can be found in section 3.6. An

overview of the properties of the clustered states, including the properties of the underlying conformal field theories, will be given in section 3.7. The details of the underlying parafermion CFTs can be found in appendix 3.A.

## 3.1 The cluster property

Quantum Hall states are said to be clustered (at order $k$), if the wave function satisfies the following property.

$$\widetilde{\Psi}_{\rm cluster}(z_i) = \Phi_{\rm bos}(z_i) \prod_{i<j} (z_i - z_j)^M \ , \tag{3.1}$$

where the fully symmetric factor $\Phi_{\rm bos}$ has the property

$$\begin{cases} \Phi_{\rm bos}(z_1 = \cdots = z_i) \neq 0 & i \leq k \\ \Phi_{\rm bos}(z_1 = \cdots = z_i) = 0 & i > k \ . \end{cases} \tag{3.2}$$

Thus, as up to any $k$ particles are brought at the same position, the factor $\Phi_{\rm bos}$ remains non-zero. However, if $k+1$ or more particles are brought together, this factor will become zero. This implies that states of the form (3.1) with $M = 0$, are zero energy ground state of the hamiltonian

$$H = V \sum_{i_1 < i_2 \cdots < i_{k+1}} \delta^2(z_{i_1} - z_{i_2}) \delta^2(z_{i_2} - z_{i_3}) \cdots \delta^2(z_{i_k} - z_{i_{k+1}}) \ , \tag{3.3}$$

where $\delta^2(z_i - z_j)$ is the 2-dimensional delta-function. Note that for $k = 2$, we speak of a pairing property, for obvious reasons.

The states described in this chapter all have the above mentioned clustering property, except for the spin-charge separated states of section 3.6. These states have a clustering among the spin up and spin down electrons separately. That this clustering property is different is closely related to the fact that the underlying affine Lie algebra is different. For the RR states and the non-abelian spin-singlet (NASS) states, the underlying affine Lie algebras are the simply laced algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ respectively, while for the spin-charge separated state, this algebra is $\mathfrak{so}(5)$, a non-simply laced Lie algebra. In effect, the consequence is that the states already have a pairing structure at level $k = 1$.

## 3.2 The Moore-Read quantum Hall state

The way in which the clustered quantum Hall states are constructed will be explained by using the MR quantum Hall state as an example, as this is the simplest paired stated. Nevertheless, it has most of the features of the general clustered states.

The construction of the MR state goes along the same lines as the construction of the Laughlin state based on conformal field theory, see section 2.3.1. The difference lies in the conformal field theory used. In the case of the MR quantum Hall state, this is a theory with $\mathfrak{su}(2)_2$ symmetry and can be described in terms of a free chiral boson and a Majorana fermion.

The pairing structure is build into the wave function via the electron operator, which contains the Majorana fermion $\psi$ (of the Ising model) in addition to a free chiral boson. This already points to the fact that the CFT describing the MR state is a $c = \frac{3}{2}$ theory. The electron operator now reads

$$V_{\mathrm{el}}^{\mathrm{mr}} = \psi \, : e^{i\sqrt{M+1}\varphi_c} : \, . \tag{3.4}$$

The MR quantum Hall state is, like the Laughlin state, constructed as a correlator of electron operators and a suitable background charge. This correlator can easily be calculated, because the Majorana fermion is also a free field. Thus by using (the fermionic form of) Wick's theorem and the correlator of two Majorana fermions

$$\langle \psi(z_1)\psi(z_2)\rangle = \frac{1}{z_1 - z_2} \, , \tag{3.5}$$

one finds the wave function

$$\begin{aligned} \widetilde{\Psi}_{\mathrm{MR}}^{M}(z_i) &= \lim_{z_\infty \to \infty} z_\infty^{(M+1)N^2} \langle V_{\mathrm{el},1}^{\mathrm{mr}} \cdots V_{\mathrm{el},N}^{\mathrm{mr}} : e^{-i\sqrt{M+1}N\varphi}(z_\infty) : \rangle \\ &= \langle \psi(z_1) \ldots \psi(z_N)\rangle \prod_{i<j} (z_i - z_j)^{M+1} \\ &= \mathrm{Pf}\Big(\frac{1}{z_i - z_j}\Big) \prod_{i<j} (z_i - z_j)^{M+1} \, . \end{aligned} \tag{3.6}$$

The second line is obtained by working out the correlator of the vertex operators of the chiral bosons. $\mathrm{Pf}(\mathcal{M})$ is the pfaffian of an anti-symmetric matrix $\mathcal{M}$ is the anti-symmetrized product

$$\mathrm{Pf}(\mathcal{M}_{i,j}) = \mathcal{A}(\mathcal{M}_{1,2}\mathcal{M}_{3,4} \ldots) \, . \tag{3.7}$$

In fact, the pfaffian of an anti-symmetric matrix can also be thought of as the square root of its determinant, $\mathrm{Pf}(\mathcal{M}_{i,j}) = \sqrt{\det \mathcal{M}}$. Because of the presence of the pfaffian factor, the state corresponding to the wave function (3.6) is also called the pfaffian quantum Hall state. Note that the number of electrons $N$ need to be even, in order to get a non-zero correlator. Below, when the quasihole wave functions are introduced, it will become clear that wave functions with an odd number of electrons are possible when quasiholes are present. The Moore-Read state (3.6) describes a quantum Hall system at filling fraction $\nu = \frac{1}{M+1}$, as can be inferred from the form of the form of the vertex operator of $\varphi_c$ in the electron operator. Equivalently, the filling fraction is determined by the Laughlin factor in the wave function. Note that for fermionic states, $M$ has to be odd, as will be the case throughout this thesis. The conformal dimensions of the electron operators are easily calculated. The conformal dimension of the Majorana fermion is $\Delta_\psi = \frac{1}{2}$. In general, the conformal dimension of the vertex operator $e^{i\vec{\alpha}\cdot\vec{\phi}}$ is given by $\Delta_{\mathrm{v.o.}} = \frac{\alpha^2}{2}$. Details on dimensions of fields and CFT in general can be found in the book [34]. Put together, the dimension of the electron operator is given by $\Delta_{\mathrm{el}} = \frac{M+2}{2}$. In fact, the electron operators are constructed in such a way that for $M = 0$ they are currents of the underlying Lie algebra CFT. For $M$ odd, the electron operator should have half integer dimension, in order to represent a fermionic object. For all

the states discussed in the thesis, the electron operators indeed have conformal dimension $\Delta_{\rm el} = \frac{M+2}{2}$.

Before we go on with the discussion on the quasiholes over the MR state, we first will comment on the relevance of this state. At the time this state was proposed as a quantum Hall state at filling $\nu = \frac{1}{2}$, it was believed that the quantum Hall effect observed at filling $\nu = \frac{5}{2}$ was due to an *unpolarized* quantum Hall state. Experiments which led to this conclusion were tilted field experiments, in which the total magnetic field is increased, while the component perpendicular to the sample is kept the same. As the filling fraction is only determined by this perpendicular component and the spin couples to the total magnetic field, these experiments can shed light on the spin of the quantum Hall state under investigation. In these tilted field experiments at $\nu = \frac{5}{2}$, it was observed that upon increasing the total magnetic field, the quantum Hall state disappeared [29].

The interpretation of these experiments was that the state at $\nu = \frac{5}{2}$ is unpolarized, and can be destroyed by increasing the in-plane magnetic field. Nevertheless, M. Greiter, X.-G. Wen and F. Wilczek proposed that the observed quantum Hall effect at $\nu = \frac{5}{2}$ could be related to the spin-polarized MR state. The electrons in the second Landau level, which is half filled, are thought to form a MR state, while the first Landau level is completely filled with spin up and down electrons. The presence of this completely filled Landau level will alter the details of the interactions between the electrons in the second Landau level. Of course, in real systems, the states will always have a certain extension in the perpendicular direction; this will in general lead to a coupling to the in-plane magnetic field, which eventually could destroy the quantum Hall state. Over the years, evidence built up that the quantum Hall effect at filling $\frac{5}{2}$ is indeed related to the MR quantum Hall state. First of all, there are extensive numerical studies which point in this direction [72, 86]. Experimentally, it has been established that the state is indeed spin-polarized [75]. Also, at 'high' temperatures, at which the quantum Hall state has disappeared, Fermi surface effects have been observed. This indicates that a Cooper instability can occur, providing a mechanism for the pairing of electrons. Applying an in-plane magnetic field will results in the formation of a striped phases, with anisotropic behaviour of the resistances. This explains the disappearing of the quantum Hall state at $\nu = \frac{5}{2}$ upon applying an in-plane magnetic field.

We will now continue with the description of the quasiholes excitations. As indicated in appendix 2.A, the quasihole operators are restricted to be relatively local with respect to the electron operators. This leads to the following, *smallest charge* quasihole operator

$$V_{\rm qh}^{\rm mr} = \sigma \ : e^{\frac{i}{2}\frac{1}{\sqrt{M+1}}\varphi_c} : , \tag{3.8}$$

where $\sigma$ is the spin field of the Ising model. It has conformal dimension $\Delta_\sigma = \frac{1}{16}$. This gives $\Delta_{\rm qh} = \frac{M+3}{16(M+1)}$ for the conformal dimension of the quasiholes. The MR wave func-

tion in the presence of quasiholes can now be written in term of a CFT correlator

$$\begin{aligned}\widetilde{\Psi}^{M}_{\mathrm{MR,qh}}(z_i,w_j) &= \lim_{z_\infty\to\infty} z_\infty^{(M+1)(N+\frac{n}{2(M+1)})^2} \\ &\times \langle V^{\mathrm{mr}}_{\mathrm{qh},1}\cdots V^{\mathrm{mr}}_{\mathrm{qh},n} V^{\mathrm{mr}}_{\mathrm{el},1}\cdots V^{\mathrm{mr}}_{\mathrm{el},N} : e^{-i\left(\sqrt{M+1}N+\frac{n}{2\sqrt{M+1}}\right)\varphi}(z_\infty) :\rangle \\ &= \langle \sigma(w_1)\cdots\sigma(w_n)\psi(z_1)\psi(z_N)\rangle \\ &\times \prod_{i,j}(w_i-w_j)^{\frac{1}{4(M+1)}} \prod_{i,j}(z_i-w_j)^{\frac{1}{2}} \prod_{i<j}(z_i-z_j)^{M+1}\,. \end{aligned} \tag{3.9}$$

This wave function is analytic in the electron coordinates, as it should be. Note that the factors $(z_i - w_j)^{\frac{1}{2}}$ are canceled by similar factors present in the correlator of Majorana and spin fields. This is guaranteed by the structure of the electron and quasihole operators. To use the correlator eq. (3.9) to obtain explicit forms of the wave functions is a difficult task. The spin field are not free fields, and therefore, the correlators containing quasihole operators can not be obtained by applying Wick's theorem. An explicit form of the MR wave function with quasiholes present can be found in [84]. In fact, this is the only case in which these explicit wave functions are known. However useful, the explicit form of these quasihole wave functions is not necessary to obtain properties of the quasiholes.

Like in the case of BCS superconductors, the flux quantum is halved due to the pairing of the electrons. This means that if the flux quantum is increased by one, two quasiholes are created. The charge of these quasiholes is given by $q_{\mathrm{qh}} = \frac{1}{2(M+1)}$. Compared to the Laughlin quasiholes, there is an additional fractionalization of the charge. Note that this structure is also embodied in the K-matrix description of the MR state, which is discussed in chapter 5.

One of the properties which caused great interest in these paired quantum Hall states is the statistics of the quasihole excitations. It is said that these quasiholes satisfy non-abelian (braid) statistics. As this is a very important prediction, which will also come back in the other clustered quantum Hall states, we will describe the situation for the quasiholes over the Moore-Read state in some detail in the next section.

## 3.3 Non-abelian statistics

In this section, we will explain that the quasiholes of the MR quantum Hall states, and the clustered states in general, satisfy what is called non-abelian statistics.

The key point is the presence of the spin field in the quasihole operator. In turn, this was possible because of the presence of the Majorana fermion in the electron operators, which cause the wave function to be paired. So in effect, the pairing (or clustering in general, as we will see in the following sections), is intimately related to the non-abelian statistics of the quasiholes. This relationship will again be found in the K-matrix structure, which is explained in chapter 5.

To explain the non-abelian statistics, we have to take a look at the fusion properties of the parafermion and spin fields, present in the electron and quasihole operators. The fusion

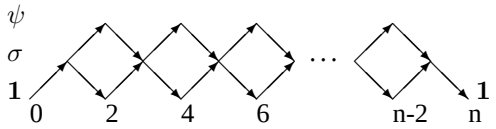


Figure 3.1: The Bratteli diagram for the Moore-Read state.

rules of the parafermion fields $\psi$ is trivial

$$\psi \times \psi = \mathbf{1} \, . \tag{3.10}$$

For the spin fields $\sigma$, the fusion rule is more complicated

$$\sigma \times \sigma = \mathbf{1} + \psi \, . \tag{3.11}$$

The consequence is that upon calculating a correlator which contains a certain number of spin field, on in general has a choice of many different *fusion paths* which fuse the (spin) fields to the identity. In the end, after fusing all the fields, one has to end with the identity, in order to obtain a non-zero correlator. The number of ways in which this can be done can be obtained from a so called *Bratteli diagram*. In such a diagram, the fusion of fields is encoded in arrows, see figure 3.1. Each arrow stands for fusing with a certain field (in this case, the spin field $\sigma$). The field which is fused with the field corresponding to the arrow is at the starting point of the arrow, while the arrow points at a position corresponding to a field in the fusion. Taking the fusion $\sigma \times \psi = \sigma$ into account, one finds the diagram 3.1.

From the diagram in figure 3.1, one easily determines that the number of spin fields in the correlator has to be even. Only after the fusion of an even number of spin fields, one can end up in either the $\mathbf{1}$ or the $\psi$-sector. In the first case, the number of electrons need to be even as well, to end up in the 'identity sector'. In the second case the number of Majorana fermions, and accordingly, the number of electrons, has to be odd. In both cases, the number of fusion paths which lead to the identity is determined to be $2^{\frac{n}{2}-1}$.

The fact that there is more than one fusion channel makes the conformal correlator in (3.9) stand for a set of wave functions, or better, a wave vector. If one now takes a MR quantum Hall state, in which four quasiholes are present, and one braids these quasiholes, this will result in a phase, which depends on a phase matrix, instead of a simple phase factor. These phase matrices have been calculated [71, 74, 95], and it was found that they do not commute. Thus the ordering of the braiding is essential, explaining the nomenclature *non-abelian* statistics.

At this point, it is useful to spend a few words on the underlying Lie algebra structure. That is, the electron operator for the MR state can be viewed as an $\mathfrak{su}(2)$ current when $M = 0$. The Majorana fermion is viewed as the simplest *parafermion* related to the $\mathfrak{su}(2)_2/\mathfrak{u}(1)$ parafermion theory, which is in fact just the Ising model. In the other clustered quantum Hall states, which we will address in the following sections, parafermion fields will be present in the electron operators. Again, as a consequence, the quasiholes with smallest possible charge will contain spin fields. These spin fields also have non-trivial fusion rules, which are generalizations of eq. (3.11). Thus one can again argue, that the quasiholes over

these states satisfy non-abelian statistics. Or in other words, these states have an *intrinsic* degeneracy in the presence of quasiholes.

One of the main themes in this thesis will be the study of the statistics of these quasiholes. In particular, the intrinsic degeneracies described in this section will come back in chapter 6, where the degeneracy of clustered quantum Hall states will be studied in a spherical geometry. There, we will compare numerical results to analytical studies, in which the intrinsic degeneracies play a crucial role. However, there is another source of degeneracies, due to the spherical geometry. Combining those two types of degeneracies will turn out to be non-trivial, and is in fact the crucial point.

Al this is very nice, but a natural question immediately rises itself: is non-abelian statistics possible in physical systems? And if so, what are the consequences of the non-abelian statistics? Before we go into this, we first want to remark that in general, it is very hard to measure the statistics of particles in condensed matter systems. Though the fractional charge of the Laughlin quasiholes has been confirmed via shot-noise experiments, the *fractional statistics* which these particles are believed to satisfy, still manages to keep out of the hands of experimentalists. But apart from that, one might wonder whether or not non-abelian statistics can be observed in principle.

One of the problems is the fact that for realistic potentials, the quasihole states which are degenerate for the ultra local tree body interaction might not be degenerate anymore. However, the *topological* statistics properties obtained from the idealized hamiltonian are expected to carry over to other hamiltonians which lie in the same universality class. Of course, it is essential that the braiding of quasiholes is done adiabatically slow.

Another worry is of course the presence of disorder in the physical systems. Also here, the fact that the properties of the quasiholes are topological, protects them against breaking down, if the exchange is done slow enough. For more details on the effect on disorder can be found in [83].

All the above have led people to propose the MR quantum Hall state as the building block for a quantum computer, see, for instance, [35]. In such a quantum computer, the quasiholes would form the q-bits. Though this is a very interesting proposal, making a quantum computer based on a quantum Hall system is very far from being realized. This topic will not be addressed in this thesis.

## 3.4 The Read-Rezayi states

The construction of quantum Hall states in terms of correlators in CFTs makes generalizations of the MR state possible. In [85], Read and Rezayi introduced clustered analogs of the Moore-Read states. In fact, they proposed to use the $\mathbb{Z}_k$ parafermions in the electron operator, in the same way as the Majorana fermion is used. The states they obtained have filling fraction

$$\nu_{k,M} = \frac{k}{kM+2} \,. \tag{3.12}$$

The $\mathbb{Z}_k$ (or $\mathfrak{su}(2)_k/\mathfrak{u}(1)$) conformal field theory contains the parafermion primary fields $\psi_l$, $l = 0, 1, \ldots, k-1$, where $\psi_0 = \mathbf{1}$. In addition, there are spin fields $\sigma_l$, with $l = 1, \ldots, k-1$.

The fusion rules for the parafermion fields $\psi_l$ read

$$\psi_l \times \psi_{l'} = \psi_{l+l'} \ , \tag{3.13}$$

where all the labels are modulo $k$. The (spin-polarized) clustered states are defined in terms of the electron operator

$$V_{\rm el}^{\rm rr} = \psi_1 \ : e^{\frac{i}{\sqrt{k}}\sqrt{kM+2}\varphi_c} : \ . \tag{3.14}$$

Note that in this definition, the most 'basic' parafermion in the $\mathbb{Z}_k$ theory is used. This reflects that the electron has the smallest charge of all 'electronic particles'.

From (3.13) it follows that the number of electrons $N$ needs to be a multiple of $k$ (in the absence of quasiholes). The wave function is easily written in terms of a correlator

$$\begin{aligned} \widetilde{\Psi}_{\rm RR}^{k,M}(z_i) &= \lim_{z_\infty \to \infty} z_\infty^{(M+\frac{2}{k})N^2} \langle V_{{\rm el},1}^{\rm rr} \cdots V_{{\rm el},N}^{\rm rr} : e^{\frac{-i}{\sqrt{k}}\sqrt{kM+2}N\varphi_c}(z_\infty) : \rangle \\ &= \langle \psi_1(z_1) \cdots \psi_1(z_N) \rangle \prod_{i<j} (z_i - z_j)^{M+\frac{2}{k}} \ . \end{aligned} \tag{3.15}$$

To actually calculate the parafermion correlator is much harder compared to the MR case, because the parafermions $\psi_1$ are not free fields. In [85], a form (to be given below) was conjectured, which was inspired by the structure of the zeroes implied by the operator product expansion of the parafermion fields. In [47], this form was proven to be equal to the correlator. A different way of characterizing the wave function can be found in [23].

We will now describe the explicit form of the Read-Rezayi wave function, in the case of $M = 0$. The wave functions for $M > 0$ are obtained by multiplying with the Laughlin factor $\prod_{i<j}(z_i - z_j)^M$. To obtain the wave functions of the RR-states for $N = pk$ electrons, ($p$ is a positive integer) first, the particles have to be divided into groups of $k$ particles. Let us consider the simplest way of doing this

$$(z_1, z_2, \ldots, z_k), (z_{k+1}, \ldots, z_{2k}), \ldots, (z_{(p-1)k+1}, \ldots, z_{pk}) \ . \tag{3.16}$$

To each pair of two clusters (say the $a^{\rm th}$ and $b^{\rm th}$ cluster), the following factor is associated

$$\begin{aligned} \chi_{a,b} = &(z_{(a-1)k+1} - z_{(b-1)k+1})(z_{(a-1)k+1} - z_{(b-1)k+2}) \\ &\times (z_{(a-1)k+2} - z_{(b-1)k+2})(z_{(a-1)k+2} - z_{(b-1)k+3}) \\ &\times \ldots \times (z_{ak} - z_{bk})(z_{ak} - z_{(b-1)k+1}) \ . \end{aligned} \tag{3.17}$$

To obtain the wave function, one has to multiply the factors (3.17) for each pair of clusters, and sum over all the possible ways of forming the $p$ clusters of $k$ particles. The last step is equivalent (up to a normalization factor) to symmetrize the product of $\chi$ factors in all the electron coordinates. The wave function thus becomes

$$\widetilde{\Psi}_{\rm RR} = \mathcal{S}[\prod_{a<b} \chi_{a,b}] \ . \tag{3.18}$$

It has been shown in [85] that this wave function indeed satisfies the cluster property eq. (3.2). That the wave functions should satisfy this property follows from the fusion rules of

the parafermion fields $\psi_l$, given in eq. (3.13). It was proven in [47] that this wave function is in fact equal to the state defined in eq. (3.15) (for $M = 0$).

Another way of obtaining an (equivalent) explicit form of the wave function was described (and proven) in [23]. The idea is to divide the electrons in $k$ groups, and giving them different 'colours' (denoted by $a, b, \ldots$) so that they become discernible. The wave function for the system consists of Laughlin factors. To obtain the clustered wave function, on has to symmetrize all the different electrons

$$\widetilde{\Psi}_{\rm RR} = \tilde{\mathcal{S}}\big[\prod_{i<j}(z_i^{(a)} - z_j^{(a)})^2 \prod_{i<j}(z_i^{(b)} - z_j^{(b)})^2 \ldots\big] \,. \tag{3.19}$$

The filling fraction of the Read-Rezayi states can be read off from the electron operators, and is given by $\nu = \frac{k}{kM+2}$. Another way of determining the filling fraction is via the maximal degree of the wave function with respect to one of the coordinates, say $z_1$ (any electron coordinate can be used). The degree of the wave function is equal the to the number of flux quanta needed to tune to a particular state on the sphere. One has the following relation

$$N_\Phi = \frac{1}{\nu} N_e - \mathcal{S} \,. \tag{3.20}$$

The shift is due to the fact that we are in fact using a spherical geometry in this determination. Using the explicit form (3.18), on indeed finds that $\nu = \frac{k}{kM+2}$. In addition, the shift on the sphere is found to be $\mathcal{S} = M + 2$.

The quasiholes over these states are also defined in complete analogy with the MR case. Thus, the operator creating the most basic quasihole (i.e. the one with the smallest charge) will contain a spin field, namely $\sigma_1$. The quasihole operator is, like the MR case, written in terms of a spin field and a vertex operator of the charge boson

$$V_{\rm qh}^{\rm rr} = \sigma_1 \ : e^{\frac{i}{\sqrt{k}} \frac{1}{\sqrt{kM+2}} \varphi_c} : \ . \tag{3.21}$$

The vertex part of these operators is determined by the condition that they have to be relatively local with respect to the electron operator (3.14). The charge of the quasiholes can be determined from the operators (3.21) to be $q_{\rm qh} = \frac{1}{kM+2}$.

The fusion rules imply a condition on the number of quasiholes which can be placed in the correlator. This condition is that after fusing the spin fields $\sigma_1$ and the parafermions $\psi_1$ of the electron operators, one has to end up with the identity operator $\mathbf{1}$. For instance, if the number of electrons is a multiple of $k$, the fusion rules imply that a multiple of $k$ quasiholes need to be placed in the correlator. Though we don't have explicit forms for the wave functions in the presence of quasiholes, formally, they can be written as a CFT correlator

$$\begin{aligned}\widetilde{\Psi}^{k,M}_{\rm RR,qh}(z_i, w_j) = &\langle \sigma_1(w_1) \cdots \sigma_1(w_n) \psi_1(z_1) \cdots \psi_1(z_N) \rangle \\ &\times \prod_{i<j} (w_i - w_j)^{\frac{1}{k(kM+2)}} \prod_{i,j} (z_i - w_j)^{\frac{1}{k}} \prod_{i<j} (z_i - z_j)^{M+\frac{2}{k}} \,.\end{aligned} \tag{3.22}$$

In general, the correlator of $n$ $\sigma_1$ fields ($n$ is the number of quasiholes) and $N$ $\psi_1$ fields is hard to calculate. Moreover, because of the non-trivial fusion rules of the spin fields, the

correlator in eq. (3.22) stands for more than one wave function. Like in the MR case, the quasiholes over the RR states satisfy non-abelian statistics.

Note that in [23], a rather explicit form of the quasihole wave functions was given. However, it is in fact an over complete set of wave functions; it can't be used for the state counting as described in chapter 6, because one has to reduce this set of states. This is in general a very difficult task, as can be seen from the MR case, were such a reduction was in fact performed [74].

## 3.5 Non-abelian spin-singlet states

In this section, we will describe a set of spin-singlet states, which have the same clustering property as the RR-states. These states can be viewed as spin-singlet analogs of the RR-states, in the same manner as the Halperin states $(m+1, m+1, m)$ are spin-singlet analogs of the spin-polarized Laughlin states. Alternatively, they can be viewed as clustered analogs of the Halperin states, in the same way as the RR-states are clustered analogs of the Laughlin states. We will follow the discussion of the defining paper [10] and the work presented in [9]. The electron and spin filling fractions for these states are given by

$$\nu_{k,M} = \frac{2k}{2kM+3}\,, \qquad\qquad \sigma_{k,M} = 2k\,. \tag{3.23}$$

The underlying structure of these states is a CFT with the symmetry of the affine Lie algebra $\mathfrak{su}(3)_k$. The states are defined in terms of the parafermions $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ and two chiral bosons, for charge and spin: $\varphi_c$ and $\varphi_s$. The parafermion fields are associated to the roots of $su(3)$. In the root diagram, a charge and a spin direction are chosen in such a way that a spin doublet is present. In this way, the spin up and spin down electrons can be identified. Note that only the case $M = 0$ has an affine $\mathfrak{su}(3)_k$ symmetry. For $M > 0$, the charge axis is deformed. As a consequence, the charged part of the vertex operator of the bosons $\varphi_c, \varphi_s$ depends on $M$.

In figure 3.2, we indicate the correspondence between the particle operators and the root diagram of $su(3)$. As can be seen form figure 3.2, the quasihole operators correspond to (fundamental) weights of $su(3)$. These operators consist of a spin field corresponding to the weight, and a vertex operator, which is also related to the $su(3)$ diagram. The spin part of the vertex operators are defined in such a way that the $z$ component of the spin of the electrons, which can be measured with $S_z = \frac{i}{\sqrt{2k}} \oint \frac{dz}{2\pi i} \partial\varphi_s$, is $\frac{1}{2}$. The charge is normalized in such a way that the electrons have charge 1. For $M \neq 0$ this implies that the charge axis of the root diagram is 'stretched', and the full symmetry is now $\mathfrak{su}(3)_{k,M}$, where the second label indicates the stretching of the charge axis. The electron and quasihole operators can be written as

$$V^{\text{nass}}_{\text{el},\uparrow} = \Psi^{\uparrow} = \psi_1 \ : e^{(\frac{i}{\sqrt{2k}}(\sqrt{2kM+3}\,\varphi_c + \varphi_s))} : , \tag{3.24}$$

$$V^{\text{nass}}_{\text{el},\downarrow} = \Psi^{\downarrow} = \psi_2 \ : e^{(\frac{i}{\sqrt{2k}}(\sqrt{2kM+3}\,\varphi_c - \varphi_s))} : , \tag{3.25}$$

$$V^{\text{nass}}_{\text{qh},\uparrow} = \phi^{\uparrow} = \sigma_{\uparrow} \ : e^{(\frac{i}{\sqrt{2k}}(\frac{1}{\sqrt{2kM+3}}\,\varphi_c + \varphi_s))} : , \tag{3.26}$$

$$V^{\text{nass}}_{\text{qh},\downarrow} = \phi^{\downarrow} = \sigma_{\downarrow} \ : e^{(\frac{i}{\sqrt{2k}}(\frac{1}{\sqrt{2kM+3}}\,\varphi_c - \varphi_s))} : , \tag{3.27}$$

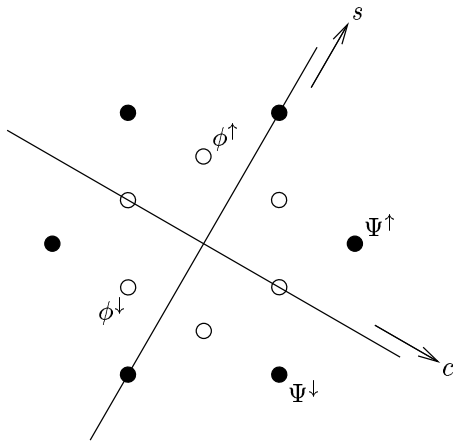


Figure 3.2: The roots ($\bullet$) and weights ($\circ$) of $su(3)$.

where we have written $\psi_{\alpha_1} = \psi_1$, $\psi_{-\alpha_2} = \psi_2$ , $\sigma_{\varpi_1} = \sigma_\uparrow$ and $\sigma_{-\varpi_2} = \sigma_\downarrow$ for simplicity (details on the notation can be found in the appendix 3.A).

The most basic spin fields $\sigma_{\uparrow,\downarrow}$ transform as a doublet of the $su(2)$ subalgebra we identify with the spin of the particles. Note that for $M = 0$, the electron operators are currents of the affine Lie algebra theory.

The wave function for the NASS state can be written as a correlator of the electron operators in eqs. (3.24) and (3.25). By evaluating the vertex operators of the chiral bosons, we arrive at the following form

$$\begin{aligned}\widetilde{\Psi}^{k,M}_{\mathrm{NASS}}(z_i^{\uparrow}; z_j^{\downarrow}) = \langle \psi_1(z_1^{\uparrow}) \dots \psi_1(z_{N/2}^{\uparrow}) \psi_2(z_1^{\downarrow}) \dots \psi_2(z_{N/2}^{\downarrow}) \rangle \\ \times \left[\widetilde{\Psi}^{(2,2,1)}_{\mathrm{H}}(z_i^{\uparrow}; z_j^{\downarrow})\right]^{1/k} \widetilde{\Psi}^{M}_{\mathrm{L}}(z_i^{\uparrow}; z_j^{\downarrow}) \,. \end{aligned} \tag{3.28}$$

Note that the explicit form of the Laughlin and Halperin wave functions can be found in sections 2.3.1 and 2.3.2 respectively. As was the case for the Read-Rezayi states, the wave function (3.28) is non-singular, and in fact a polynomial in the electron coordinates. In describing the explicit form of the wave functions for the NASS states in absence of quasi-holes, we closely follow the results presented in [9].

Because the structure of the parafermions of both the types $\psi_{l\alpha_1}$ and $\psi_{-l\alpha_2}$ closely resembles that of the $\mathbb{Z}_k$ parafermions $\psi_l$, we expect the structure of the trial wave functions (that is, of the chiral correlators (3.28)) of the NASS states to be similar to that of the RR states, and also to generalize the Halperin $(2, 2, 1)$ state. The RR wave functions were constructed by dividing the particles into clusters of $k$, writing down a product of factors for each pair of clusters, and finally symmetrizing over all ways of dividing the particles into clusters. Hence in the case with spin, we guess that we should divide the up particles into groups of $k$, the downs into groups of $k$ and then multiply together factors that connect up with up, down with down, or up with down clusters, and finally ensure that the function

is of the correct permutational symmetry type to yield a spin-singlet state (in particular, it should be symmetric in the coordinates of the up particles, and also in those of the downs). We expect that the up-up and down-down parts of this should closely resemble the RR wave functions, before the symmetrization; it was shown in ref. [85] that the functions found there vanish when $k+1$ particles come to the same point, even inside the sum over permutations that symmetrizes the final function. These considerations guided the following construction.

Due to the spin-singlet nature of the state, the wave function will be non-zero only if the number of spin up and spin down particles is the same. Furthermore, there must be an integer number of clusters, so the total number of particles $N$ must be divisible by $2k$, and will be written as $N = 2kp$, where $p \in \mathbb{N}$. One example was already given in [10], namely the wave function for the case $k = 2, M = 0$ with the number of particles equal to 4 (i.e., $p = 1$),

$$\widetilde{\Psi}^{k=2,M=0}_{\mathrm{NASS}}(z^{\uparrow}_1, z^{\uparrow}_2; z^{\downarrow}_1, z^{\downarrow}_2) = (z^{\uparrow}_1 - z^{\downarrow}_1)(z^{\uparrow}_2 - z^{\downarrow}_2) + (z^{\uparrow}_1 - z^{\downarrow}_2)(z^{\uparrow}_2 - z^{\downarrow}_1)\,. \tag{3.29}$$

This is part of the two-dimensional irreducible representation of the permutation group on 4 objects, $S_4$, as can easily be seen. This is the correct symmetry type to obtain a spin-singlet state, as we discuss further below.

We will now describe the different factors that enter the NASS wave functions. Because the only effect of $M$ being non-zero is to give an overall Laughlin factor, we will assume at first that $M = 0$. First we give the factors that involve particles of the same spin, say spin up. They are the same as in RR [85]. We will divide the particles into clusters of $k$ in the simplest way,

$$(z^{\uparrow}_1, \ldots, z^{\uparrow}_k), (z^{\uparrow}_{k+1}, \ldots, z^{\uparrow}_{2k}), \ldots, (z^{\uparrow}_{(p-1)k+1}, \ldots, z^{\uparrow}_{pk})\,, \tag{3.30}$$

and the same for the $z^{\downarrow}$'s. (In a more precise treatment, we would say that the first $N/2$ particles are spin up, the remainder spin down.) We write down factors that connect the $a^{\mathrm{th}}$ with the $b^{\mathrm{th}}$ cluster:

$$\begin{aligned}\chi^{z^{\uparrow}}_{a,b} = {} & (z^{\uparrow}_{(a-1)k+1} - z^{\uparrow}_{(b-1)k+1})(z^{\uparrow}_{(a-1)k+1} - z^{\uparrow}_{(b-1)k+2}) \\ & \times (z^{\uparrow}_{(a-1)k+2} - z^{\uparrow}_{(b-1)k+2})(z^{\uparrow}_{(a-1)k+2} - z^{\uparrow}_{(b-1)k+3}) \\ & \times \ldots \times (z^{\uparrow}_{ak} - z^{\uparrow}_{bk})(z^{\uparrow}_{ak} - z^{\uparrow}_{(b-1)k+1})\,.\end{aligned} \tag{3.31}$$

For $k = 1$, we would write $\chi^{z^{\uparrow}}_{a,b} = (z^{\uparrow}_a - z^{\uparrow}_b)^2$. The factors that connect up with down spins are simpler:

$$\chi^{z^{\uparrow}, z^{\downarrow}}_{a,b} = (z^{\uparrow}_{(a-1)k+1} - z^{\downarrow}_{(b-1)k+1})(z^{\uparrow}_{(a-1)k+2} - z^{\downarrow}_{(b-1)k+2}) \ldots (z^{\uparrow}_{ak} - z^{\downarrow}_{bk})\,. \tag{3.32}$$

For $k = 1$, the factor would be $\chi^{z^{\uparrow}, z^{\downarrow}}_{a,b} = (z^{\uparrow}_a - z^{\downarrow}_b)$. We multiply all these factors for all pairs of clusters, up-up, down-down, or up-down:

$$\prod_{a<b}^{p} \chi^{z^{\uparrow}}_{a,b} \prod_{c,d}^{p} \chi^{z^{\uparrow}, z^{\downarrow}}_{c,d} \prod_{e<f}^{p} \chi^{z^{\downarrow}}_{e,f}\,. \tag{3.33}$$

Notice that for $k = 1$, we do obtain the Halperin (2,2,1) wave function.

To obtain a spin-singlet state when the spatial function is combined with the spin state (which lies in the tensor product of $N$ spins $1/2$), some symmetry properties must be satisfied. For the $M = 0$ case, the particles are bosons, hence the full wave function must be invariant under permutations of spins and coordinates of any two particles. This can be used to obtain the correct form of the function from that component in which, say the first $N/2$ are spin up, the rest spin down, as above, so knowledge of that component is sufficient. The requirement that the full wave function be a spin-singlet can be shown to reduce to the Fock conditions: the component just defined must be symmetric under permutations of the coordinates of the up particles, and also of the down particles, and must also obey the Fock cyclic condition, as given in ref. [56] (modified in an obvious way for the boson case). These three conditions can be shown to imply that the spatial wave function is of a definite permutational symmetry type (belongs to a certain irreducible representation of the permutation group), that corresponds to the Young diagram with two rows of $N/2$ boxes each. In general, given a function of arbitrary symmetry, a Young operator can be constructed that projects it onto a member of the correct representation (though the result may vanish); this construction generalizes the familiar symmetrization and anti-symmetrization operations. For the present case, the Young operator is the following operation, equivalent to summing over the function with various permutations of its arguments, and some sign changes: First, anti-symmetrize in $z_1$, $z_{N/2+1}$; then in $z_2$, $z_{N/2+2}$; ..., $z_{N/2}$, $z_N$; then symmetrize in $z_1$, ..., $z_{N/2}$; then finally symmetrize in $z_{N/2+1}$, ..., $z_N$. This clearly satisfies the first two requirements of Fock, and can be proved to satisfy also the cyclic condition. It remains to check that it is nonzero, we believe it is. Incidentally, the application of the Young operator is the analog of symmetrizing over the down spins in the spatial wave function of the permanent state (see e.g. ref. [84]), to which it reduces for the case of BCS paired wave functions of spin 1/2 bosons (there are similar statements in the more familiar case of spin-singlet pairing of spin 1/2 fermions). However, based on the example of the Halperin ($k = 1$) case, we also considered the function defined as in eq. (3.33), and then simply symmetrized over all the ups and over all the downs. For the Halperin function [which in fact is already symmetric in eq. (3.33)], this satisfies the cyclic condition, as can be seen using the fact that the (1,1,0) state is a Landau level filled with both spins, plus the Pauli exclusion principle for fermions. For $k = 2$, 3, we verified the cyclic condition numerically for several moderate sizes. Hence, we expect that this simpler form actually works for all $k$ (as well as for all $N$ divisible by $k$). Apparently, this procedure and the application of the Young operator give the same function in the end (up to a normalization).

For $M = 0$, our wave function is then:

$$\widetilde{\Psi}^{k,0}_{\mathrm{NASS}} = \mathrm{Sym} \prod_{a<b}^{p} \chi^{z^\uparrow}_{a,b} \prod_{c,d}^{p} \chi^{z^\uparrow, z^\downarrow}_{c,d} \prod_{e<f}^{p} \chi^{z^\downarrow}_{e,f} \,, \tag{3.34}$$

where $\mathrm{Sym}$ stands for the symmetrization over the ups and also over the downs. This function is nonzero, as may be seen by letting the up coordinates coincide in clusters of $k$ each, and also the downs, all clusters at different locations, and making use of the result in RR [85] that only one term in the symmetrization is nonzero in the limit. This term is the Halperin $(2k, 2k, k)$ function for $2p$ particles. To obtain the wave function for general $M$,

we multiply by an overall Laughlin factor, $\widetilde{\Psi}_{\rm L}^{M}$.

We can give a simple proof that our wave function (for $M = 0$) vanishes if any $k + 1$ particles, each of either spin, come to the same point. This works also for the RR wave functions, and is simpler, though less informative, than the proof in RR [85]. It works term by term, inside the sum over permutations in the symmetrizer. Thus, without loss of generality, we may use the simple clustering considered above. We note that on the clock face formed by the labels $1, \ldots, k$ within each cluster, there is always a factor connecting any two particles at the same position, regardless of their spin. This factor vanishes when the particles coincide. Since there are only $k$ distinct positions, when $k + 1$ particles come to the same point, the clock positions must coincide in at least two cases, so that the wave function vanishes, which completes the proof.

We do not have a direct general proof of the equality of these explicit wave functions and the formal expressions eq. (3.28), but we have performed a number of consistency checks. First, the wave functions are polynomials of the correct degree. From eq. (3.28)), we can infer what the total degree should be. The parafermions of the correlator contribute with (see [43]) $-1 \cdot 2kp \cdot (1 - \frac{1}{k})$. The factors of the 2,2,1 part are $2 \cdot \frac{2}{k} \cdot \frac{1}{2} kp(kp-1)$ and $1 \cdot \frac{1}{k} \cdot (kp)^2$. Adding these gives, for $M = 0$, $pk(3p - 2)$. We need to check whether eq. (3.33) gives the same degree. For the $i$th up particle, the degree of $z_i^{\uparrow}$ in the product of up-up factors is $2(p - 1)$, and in the up-down factors is $p$. Thus the net degree in $z_i^{\uparrow}$ is $N_\phi = 3p - 2 = 3N/2k - 2$, or for general $M$, $N_\phi = 3p + M(N - 1) - 2 = (M + 3/2k)N - 2 - M$. This gives the filling factor $\nu = 2k/(2kM + 3)$ [10], which reduces to that for the Halperin states for $k = 1$, and also the shift, defined as $N_\phi = N/\nu - \mathcal{S}$, which here is $\mathcal{S} = M + 2$ on the sphere (for more on the shift, see ref. [105]). Finally, the total degree is $N/2$ times that in $z_i^{\uparrow}$, namely $kp(3p - 2)$ for $M = 0$, the same as for the correlator. Also, the numerical work described in section 6.1 below confirms that the ground state of the appropriate Hamiltonian on the sphere for $k = 2$, $M = 1$ at the given number of flux does have a unique spin zero ground state at zero energy, so that the correlator and the wave function constructed above must coincide. This also implies that the wave functions above must be spin singlet. As was the case for the Read-Rezayi states, an alternative expression for the wave function of the NASS states is possible; this form is discussed in [92].

Though we do not have an explicit form of the wave functions with quasiholes, it can be characterized by the correlator of quasihole and electron operators. Working out the chiral boson part results in the form

$$
\begin{aligned}
&\widetilde{\Psi}^{k,M}_{\mathrm{NASS,qh}}(z_i^{\uparrow}; z_j^{\downarrow}; w_i^{\uparrow}; w_j^{\downarrow}) = \\
&\langle \sigma_{\uparrow}(w_1^{\uparrow}) \ldots \sigma_{\uparrow}(w_{n_\uparrow}^{\uparrow}) \sigma_{\downarrow}(w_1^{\downarrow}) \ldots \sigma_{\downarrow}(w_{n_\downarrow}^{\downarrow}) \psi_1(z_1^{\uparrow}) \ldots \psi_1(z_{N_\uparrow}^{\uparrow}) \psi_2(z_1^{\downarrow}) \ldots \psi_2(z_{N_\downarrow}^{\downarrow}) \rangle \\
&\times \left[ \widetilde{\Psi}_{\rm H}^{(2,2,1)}(z_i^{\uparrow}; z_j^{\downarrow}) \right]^{1/k} \widetilde{\Psi}_{\rm L}^{M}(z_i^{\uparrow}; z_j^{\downarrow}) \prod_{i,j} (z_i^{\uparrow} - w_j^{\uparrow})^{\frac{1}{k}} \prod_{i,j} (z_i^{\downarrow} - w_j^{\downarrow})^{\frac{1}{k}} \\
&\times \prod_{i<j} (w_i^{\uparrow} - w_j^{\uparrow})^{\frac{1}{2kM+3}(\frac{2}{k}+M)} \prod_{i<j} (w_i^{\downarrow} - w_j^{\downarrow})^{\frac{1}{2kM+3}(\frac{2}{k}+M)} \\
&\times \prod_{i,j} (w_i^{\uparrow} - w_j^{\downarrow})^{\frac{-1}{2kM+3}(\frac{1}{k}+M)} \, .
\end{aligned}
\tag{3.35}
$$

As in the previous cases where spin fields were present in the correlator, the expression (3.35) stands for a set of wave functions. The number can be deduced from the spin fields of the $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ CFT. We will come back in detail on the subject of state counting in chapter 6.

## 3.6 The spin-charge separated states

The construction of the previous section, namely constructing spin-singlet analogs of the MR quantum Hall state by using $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ parafermions, can be generalized, by making use of other rank 2 affine Lie algebras. There are only two other rank two Lie algebras, namely $so(5)$ and $G_2$. We will concentrate on the $so(5)$ case, as the quasiholes over those states have an interesting property on top of the non-abelian statistics, namely, a separation of the spin and charge degrees of freedom. This section is based on the article [8], which deals with the case $k = 1$. Here, we will be more general, and in most cases present results for general $k$, with the exception of the explicit wave functions. Note that one can also use Lie algebras with rank $r > 2$, as long as one can assign a proper $(SU(2))$-spin direction in the root diagram; the electrons should transform as a doublet under spin rotation. In this section, we will concentrate on the $so(5)$ case.

The underlying structure of the spin-charge separated states is the affine Lie algebra $\mathfrak{so}(5)_k$. The roots and weights, and the operators assigned to them are shown in figure 3.3.

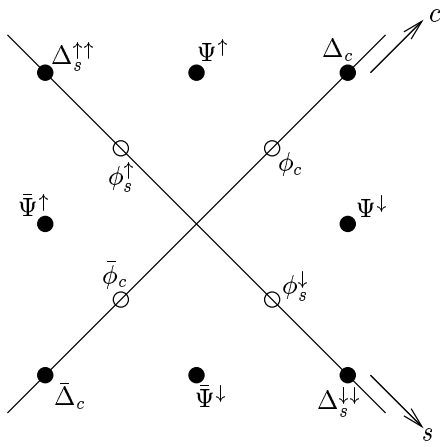


Figure 3.3: The roots ($\bullet$) and weights ($\circ$) of $so(5)$.

At level $k = 1$, the only parafermion field in the parafermionic coset $\mathfrak{so}(5)_1/\mathfrak{u}(1)^2$ is the Majorana fermion, because this coset *is* the Ising model. It is thus expected that the structure of the wave function at $k = 1$ closely resembles the Moore-Read state. This is indeed the case as can be seen from the explicit wave function (the electron operators are

given in eq. (3.37))

$$\widetilde{\Psi}^{(M)}_{\mathrm{SCsep}}(z_i^{\uparrow}, z_j^{\downarrow}) = \mathrm{Pf}\Big(\frac{1}{x_i - x_j}\Big)\widetilde{\Psi}^{(M+1,M+1,M)}_{\mathrm{H}}(z_i^{\uparrow}; z_j^{\downarrow})\,, \tag{3.36}$$

where $x_i$ can be either a spin up or a spin down electron. The pairing property is somewhat different from the previous spin-singlet states. We will discuss the pairing property using the $k = 1, M = 0$ wave function in equation (3.36). Note that for $M = 0$, this wave function has poles when a spin up electron is at the same position as a spin down electron (this problem does not occur for the physical situation $M = 1$).

In the case $k = 1$, up to two particles of the same spin can be brought to the same location while the wave function is still non-zero. So even for two spin up and two spin down particles at the same location, the wave function doesn't vanish (in fact, has a pole). Thus, the clustering holds for the spin up and spin down particles separately, while for the spin-singlet clustered states (3.34), putting any $k + 1$ electrons at the same position make the wave function vanish.

In general, the electron operators for the $\mathfrak{so}(5)_k$ states are given by

$$V^{\mathrm{sc}}_{\mathrm{el},\uparrow} = \Psi^{\uparrow} = \psi_{\uparrow} \,:e^{\frac{i}{\sqrt{2k}}(\sqrt{2kM+1}\,\varphi_c+\varphi_s)}:\,, \tag{3.37}$$

$$V^{\mathrm{sc}}_{\mathrm{el},\downarrow} = \Psi^{\downarrow} = \psi_{\downarrow} \,:e^{\frac{i}{\sqrt{2k}}(\sqrt{2kM+1}\,\varphi_c-\varphi_s)}:\,, \tag{3.38}$$

where now the parafermion fields are in fact the fields $\psi_{\uparrow} = \psi_{\alpha_1+\alpha_2}$ and $\psi_{\downarrow} = \psi_{\alpha_1}$. Here, $\alpha_1$ is a short root, and $\alpha_2$ a long root. Note that in the case $k = 1$, they are just a Majorana fermion $\psi_{\uparrow} = \psi_{\downarrow} = \psi$ (because of a field identification in the parafermion CFT, see appendix 3.A).

The operators $\Delta$ indicated in figure 3.3 are characterized as

$$\Delta_c = :e^{i\sqrt{\frac{4kM+2}{k}}\varphi_c}:\,, \tag{3.39}$$

$$\Delta_s^A = :e^{\pm i\sqrt{\frac{2}{k}}\varphi_s}:\,, \tag{3.40}$$

where $A = \uparrow\uparrow, \downarrow\downarrow$ refers to the spin eigenvalues $s_z = \pm 1$.

The operators creating the quasiholes in this quantum Hall state are given by

$$V^{\mathrm{sc}}_{\mathrm{qh,c}} = \phi_c = \sigma_{\varpi_1} \,:e^{\frac{i}{\sqrt{2k}}(\frac{1}{\sqrt{2kM+1}}\varphi_c)}:\,, \tag{3.41}$$

$$V^{\mathrm{sc}}_{\mathrm{qh},\uparrow,\downarrow} = \phi_s^{\alpha} = \sigma_{\pm(\varpi_1-\varpi_2)} \,:e^{\frac{\pm i}{\sqrt{2k}}\varphi_s}:\,. \tag{3.42}$$

$\varpi_1$ and $\varpi_2$ are fundamental weights of the Lie algebra $so(5)$. Again, one has to bear in mind that for $k = 1$, both the spin fields appearing in the quasihole operators are equal to the spin field of the Ising model. Apart from the non-abelian statistics, the quasihole excitations over these quantum Hall states show a separation of their spin and charge degrees of freedom. The charge of the holon is given by $q_{\mathrm{qh}} = \frac{1}{2kM+1}$, while it has no spin. The spinon just have spin up or spin down (namely $s_z = \pm\frac{1}{2}$) and no charge. This structure of spin-charge separation comes back in the K-matrix description of these states (discussed for level $k = 1$ in [8]; results for general $k$ are presented in section 5.4 of this thesis). A more general account on K-matrices for conformal field theories can be found in [7].

Also for this case, one can study the ground state degeneracy of states with quasiholes present. For $k = 1$, the results will be similar as those for the MR state. However, we didn't study this case yet. It will be more involved than the cases discussed in chapter 6, because the Lie algebra $so(5)$ is non-simply laced, which complicates matters quite a bit.

## 3.7 Overview of properties

In this section, we give an overview of the properties of the clustered quantum Hall states discussed in this chapter. We will concentrate on the properties of the states without quasihole excitations, because the properties of these excitations will be studied in the next chapters.

For now, let us start by recalling the filling fractions of the various quantum Hall states. The (electronic) filling fraction is determined by the coefficients of the charged chiral bosons in the electron operators. We refer to table 3.1 for the values of the filling fractions, or the various sections in this chapter. For the spin-singlet states, one can define a spin Hall conductance, similar to the (electronic) Hall conductance (see section 2.1). For both types of spin-singlet quantum Hall states of sections 3.5 and 3.6, the spin Hall filling is given by $\sigma_{\mathrm{H}} = 2k$, independent of $M$ (which only affects the 'charge' properties of the quantum Hall states, see also section 4.2.3).

From the definitions of the electron and quasihole operators, the scaling dimension can be obtained by the standard CFT techniques (see, for instance, [34]). The details of the underlying parafermion CFTs will be discussed in appendix 3.A. These scaling dimensions (also tabulated in table 3.1) are important in the description of the various tunneling experiments one can, in principle, do. However, these experiments are very delicate, and at this point doing such experiments on samples which show the famous plateau at $\frac{5}{2}$ filling is still out of reach. Nevertheless, the tunneling characteristics of the clustered states might provide an experimental check which can discern the various quantum Hall states at the same filling fraction.

To illustrate this, we will take a closer look at the spin-charge separated state at filling fraction $\nu = \frac{2}{3}$ (i.e. $k = 1, M = 1$). At this filling fraction, another spin-singlet quantum Hall state has been proposed. This is a state of the so called Jain-series [60], with anti-parallel flux attachment [113]. In short, the idea behind the construction of Jain is that an even number of flux quanta gets bound to the electrons. In effect, the composite particles, which are fermions again, feel a reduced magnetic field. If these *composite fermions* fill an integer number of (effective) Landau levels, a quantum Hall effect can occur. The filling of the electrons is a fraction, so this construction gives rise to fractional quantum Hall states. The (electron) filling fraction in the case of $n$ filled Landau levels of composite fermions with $2p$ units of flux attached, is given by $\nu = \frac{n}{2pn \pm 1}$. The $+$ $(-)$ sign applies in the case where the attached flux has the same (opposite) direction as the applied magnetic field.

So at certain filling fractions, there may be various proposals for quantum Hall states. Which state is formed depends heavily on the details of the energetics, which is very hard to calculate analytically. Numerical analysis might indicate which state is the most relevant one, as was done in the case of the $\nu = \frac{5}{2}$ qH state (see, for instance, [72]). On the experimental side, one can look for properties which differ for the various proposals. Often, the

tunneling characteristics of the quantum Hall states indeed differ among the different proposals. As indicated above, doing these tunneling experiments might be hard; however, we think they provide an interesting check to see which states occur under various conditions.

The relevant electron and quasihole operators needed to calculate the tunneling characteristics of the Jain states were identified in [70]. For the spin-charge separated states, they can be found in table 3.1.

Here, in describing the tunneling behaviour, we will follow the discussion presented in [70]. We will concentrate on the process of tunneling electrons from a Fermi liquid into the edge of the quantum Hall system. Moreover, we will only address the scaling behaviour, rather than the amplitudes. The tunneling current $I$ has the scaling behaviour $I \propto V^{\alpha}$. The exponent $\alpha$ is determined by the scaling dimension of the tunneling operator, which in turn is determined by the electron and quasihole operators. So because in general the scaling dimensions of the electron operators differ between the various quantum Hall states, also the I-V characteristics is different. For the composite fermion state at filling $\nu = \frac{2}{3}$, the scaling dimensions of the electron and quasiholes are calculated in [70] to be $g_{\rm e} = 2$ and $q_{\rm qh} = \frac{2}{3}$ respectively. The $I-V$ characteristics for tunneling electrons into the edge is given by $I \sim V^{g_{\rm e}}$, so the composite fermion state gives rise to a quadratic $I - V$. For the spin-charge separated states, the scaling dimensions are given by $g_{\rm e} = M+2$, $g_{\rm hol} = \frac{2M+5}{16M+8}$ and $g_{\rm sp} = \frac{5}{8}$ for the electron, holon and spinon, respectively (see also [8]). Thus, for the spin-charge separated state at filling $\nu = \frac{2}{3}$, we predict a cubic $I - V$. Though experimentally measuring the scaling behaviour of the $I - V$ of the tunneling processes might be very hard, it is a probe which distinguishes (some of) the various quantum Hall states which are proposed at the various filling fractions.

We end this section by providing a summary in the form of table 3.1 of the properties of the various clustered quantum Hall states discussed in this section. The main properties of the quantum Hall states are indicated, as well as some of the properties of the underlying CFTs.

# 3.A General parafermion CFTs

Following the work of Gepner [43] throughout this appendix, we will state how the general parafermion fields can be described and indicate which fields are used in the construction of the clustered quantum Hall states. Here, we only give the minimal information necessary, for details we refer to [43].

The fields of the $\mathfrak{g}_k/\mathfrak{u}(1)^r$ parafermion CFT (where $\mathfrak{g}_k$ is a simple affine Lie algebra) are written as $\Phi^{\Lambda}_{\lambda}$. Both $\Lambda$ and $\lambda$ are weights of the simple Lie algebra $g$. $\lambda$ is considered to be a *charge* and is defined modulo $k\mathcal{M}_L$, where $\mathcal{M}_L$ is the long root lattice of $g$. The action of the (proper) external automorphisms the affine Lie algebra (unfortunately also denoted by $\sigma$) imposes field identifications among the fields

$$\Phi^{\Lambda}_{\lambda} \equiv \Phi^{\sigma(\Lambda)}_{\lambda+\sigma(0)} \,. \tag{3.A1}$$

Details on the external automorphisms $\sigma$ can be found in [34]. Another constraint on the labels of the fields $\Phi$ is that the weight $\lambda$ needs to be 'accessible' from $\Lambda$ by the subtraction

| State | Read-Rezayi $\widetilde{\Psi}^{k,M}_{\mathrm{RR}}$ | Non-abelian spin-singlet $\widetilde{\Psi}^{k,M}_{\mathrm{NASS}}$ | Spin-charge separated $\widetilde{\Psi}^{k,M}_{\mathrm{SCsep}}$ |
|---|---|---|---|
| $\nu$ | $\frac{k}{kM+2}$ | $\frac{2k}{2kM+3}$ | $\frac{2k}{2kM+1}$ |
| $q_{\mathrm{qh}}$ | $\frac{1}{kM+2}$ | $\frac{1}{2kM+3}$ | $c:\ \frac{1}{2kM+1}$<br>$s:\ 0$ |
| electron operators | $\psi_\alpha e^{\frac{i}{\sqrt{k}}(\sqrt{kM+2}\varphi_c)}$ | $\psi_{\alpha_1} e^{\frac{i}{\sqrt{2k}}(\sqrt{2kM+3}\varphi_c+\varphi_s)}$<br>$\psi_{-\alpha_2} e^{\frac{i}{\sqrt{2k}}(\sqrt{2kM+3}\varphi_c-\varphi_s)}$ | $\psi_{\alpha_1+\alpha_2} e^{\frac{i}{\sqrt{2k}}(\sqrt{2kM+1}\varphi_c+\varphi_s)}$<br>$\psi_{\alpha_1} e^{\frac{i}{\sqrt{2k}}(\sqrt{2kM+1}\varphi_c-\varphi_s)}$ |
| quasihole operators | $\sigma_\varpi e^{\frac{i}{\sqrt{k}}(\frac{1}{\sqrt{kM+2}}\varphi_c)}$ | $\sigma_{\varpi_1} e^{\frac{i}{\sqrt{2k}}(\frac{1}{\sqrt{2kM+3}}\varphi_c+\varphi_s)}$<br>$\sigma_{-\varpi_2} e^{\frac{i}{\sqrt{2k}}(\frac{1}{\sqrt{2kM+3}}\varphi_c-\varphi_s)}$ | $\sigma_{\varpi_1} e^{\frac{i}{\sqrt{2k}}(\frac{1}{\sqrt{2kM+1}}\varphi_c)}$<br>$\sigma_{\varpi_1-\varpi_2} e^{\frac{-i}{\sqrt{2k}}\varphi_s}$ |
| $\Delta_{\mathrm{e}}$ | $\frac{M+2}{2}$ | $\frac{M+2}{2}$ | $\frac{M+2}{2}$ |
| $\Delta_{\mathrm{qh}}$ | $\frac{(k-1)M+3}{2(k+2)(kM+2)}$ | $\frac{(5k-1)M+8}{2(k+3)(2kM+3)}$ | $c:\ \frac{2(4k-3)M+5}{4(k+3)(2kM+1)}$<br>$s:\ \frac{5}{4(k+3)}$ |
| $\Delta_\psi$ | $1-\frac{1}{k}$ | $1-\frac{1}{k}$ | $1-\frac{1}{2k}$ |
| $\Delta_\sigma$ | $\frac{k-1}{2k(k+2)}$ | $\frac{k-1}{k(k+3)}$ | $\frac{4k-3}{4k(k+3)}$ |
| based on | $\mathfrak{su}(2)_k$ | $\mathfrak{su}(3)_k$ | $\mathfrak{so}(5)_k$ |
| $c_{\mathrm{CFT}}$ | $\frac{3k}{k+2}$ | $\frac{8k}{k+3}$ | $\frac{10k}{k+3}$ |
| $k=1$ | $\widetilde{\Psi}^{M+2}_{\mathrm{L}}$ | $\widetilde{\Psi}^{(M+2,M+2,M+1)}_{\mathrm{H}}$ | $\mathrm{Pf}(\frac{1}{x_i-x_j})\widetilde{\Psi}^{(M+1,M+1,M)}_{\mathrm{H}}$ |

Table 3.1: Properties of clustered quantum Hall states and the underlying CFTs.

of roots (including $\alpha_0$). The minimal number of times the root $\alpha_0$ needs to be subtracted will be denoted by $n_\lambda^\Lambda$. Thus if $\lambda \in \Lambda$ (now at the level of the simple Lie algebra $g$), then $n_\lambda^\Lambda = 0$.

The conformal dimensions of the fields are given by

$$\Delta_\lambda^\Lambda = \frac{\Lambda \cdot (\Lambda + 2\rho)}{2(k+g)} - \frac{\lambda \cdot \lambda}{2k} + n_\lambda^\Lambda. \tag{3.A2}$$

In this equation, the inner products are defined with respect to the quadratic form matrix. $2\rho$ is the sum of all the positive roots of the corresponding Lie algebra.

The central charge of the parafermionic CFT is given by $c_{\rm pf} = c_{\rm aLa} - r$, where $c_{\rm aLa}$ is the central charge of the CFT with affine Lie algebra symmetry $\mathfrak{g}_k$ and $r$ is the rank of the Lie algebra $g$. The central charge of the CFT with affine Lie algebra symmetry is given by $c_{\rm aLa} = \frac{dk}{k+g}$, where $d$ is the dimension of the Lie algebra and $g$ the dual Coxeter number (no confusion should arise in using the same symbol for the Lie algebra and its dual Coxeter number).

The remainder of this section is devoted to the (parafermion) field we used in the description of the paired quantum Hall states discussed in this chapter. Some of the properties of the various fields used are given in table 3.1.

The parafermion fields $\psi_\alpha$, used in the definition of the electron operators, are in fact the fields $\Phi_\alpha^{\mathbf{1}}$. In this notation, $\mathbf{1}$ denotes the vacuum representation $\mathbf{1} = (0, \ldots, 0)$ and $\alpha$ is a root. The operator product expansion (OPE) of the parafermion fields have the following form

$$\psi_\alpha(z_i)\psi_\beta(z_j) = (z_i - z_j)^{\Delta_{\alpha+\beta} - \Delta_\alpha - \Delta_\beta} \psi_{\alpha+\beta}(z_j)\ . \tag{3.A3}$$

The scaling dimensions of the parafermion fields $\psi_\alpha$ can be obtained using eq. (3.A2). In the case of the parafermions used in the electron operator, $\psi_\alpha = \Phi_\alpha^{\mathbf{1}}$, where $\alpha$ is in the adjoint representation, one finds that $n_\alpha^{\mathbf{1}} = 1$, implying $\Delta_\psi = 1 - \frac{1}{bk}$, where $b = 1$ in the case of simply-laced Lie algebra. $b = 3$ for the Lie algebra $G_2$ and $b = 2$ for all the other non simply-laced Lie algebras.

The form $\Phi_\alpha^{\mathbf{1}}$ for the parafermion fields $\psi_\alpha$ implies that the fusion of two parafermion fields is always trivial

$$\psi_\alpha \times \psi_\beta = \psi_{\alpha+\beta}\ , \tag{3.A4}$$

where the labels are modulo $k$ times the long root lattice $\mathcal{M}_L$. Together with $r$ boson fields, these parafermions can be used to make the currents $J^\alpha$ of the corresponding affine Lie algebra CFT. In general, the spin fields $\sigma_\varpi$ are the fields $\Phi_{\varpi'}^{\varpi}$, where $\varpi'$ lies in the fundamental representation $\varpi$.

To obtain the operators for creating the electrons and quasiholes, the parafermion and spin fields must be combined with vertex operators of chiral boson fields. For $M = 0$, this is done in such a way that the electron operators become currents of the underlying affine algebra. For the rank 2 Lie algebras, we choose a spin and charge axis in the root diagram consistent with the spin of the electron. The conformal dimension of a vertex operator $e^{i\vec{\alpha}\cdot\vec{\varphi}}$ is given by $\Delta_{\rm v.o.} = \frac{\vec{\alpha}^2}{2}$. Thus the form of the vertex operator part of the electron operator is fixed. Together with the assignment of charge and spin to the root diagram, this also fixes the form of the quasihole operators. This procedure automatically takes care of the fact

that the quasiholes have to be local with respect to the electron operators. The results for the various states are given in the preceding sections; they are also collected in table 3.1 in section 3.7.

An important property of quantum Hall states, at least from a theoretical point of view, is the degeneracy of the state on a torus. This degeneracy can also be characterized by the number of fields in the chiral algebra of the CFT for the quantum Hall state (see, for instance, [71]). In the case of the abelian quantum Hall states, this torus degeneracy can be calculated from the K-matrix of the electron sector, as was shown by Wen (see, for instance, [105]). The result is simply the determinant of the K-matrix for the electron sector. For the non-abelian quantum Hall states, this result does not hold anymore, because of the parafermionic CFTs (for the abelian qH states, only chiral boson fields are present). Of course, one would like to have a way of obtaining the torus degeneracy directly from the K-matrices for the non-abelian quantum Hall states, as described in chapter 5. At this point, we do not have such a formula. However, we can calculate the number of primary fields in the parafermion theories, and combine this result to the degeneracy 'caused' by the chiral boson fields present in the electron operator.

Using the constraints of the beginning of this appendix, we find the following numbers of (parafermion) primary fields for the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ states, namely $\frac{1}{2}k(k+1)$ and $\frac{1}{6}k^2(k+1)(k+2)$, respectively. For the $\mathfrak{so}(5)$ parafermions we find $\frac{1}{2}k^2(k+1)(k+2)$ primary fields.

These numbers need to be combined with the degeneracy caused by the chiral boson fields in the electron operators. For the RR, NASS and SCsep states, these are given by $M+\frac{2}{k}$, $\frac{2kM+3}{k^2}$ and $\frac{2kM+1}{k^2}$ respectively.

Combined, the degeneracies of the various quantum Hall states on the torus are given by $\frac{1}{2}(k+1)(kM+2)$ and $\frac{1}{6}(k+1)(k+2)(2kM+3)$ for the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ states, while the $\mathfrak{so}(5)$ states have degeneracy $\frac{1}{2}(k+1)(k+2)(2kM+1)$.

# Chapter 4

# Statistics properties

In this chapter, we will introduce the concepts needed to discuss the statistics properties of the electrons and quasiholes of the paired quantum Hall states. The language we use is that of the so called *exclusion statistics* principle introduced by Haldane in the early 90-ties [51]. The idea behind this principle is to generalize the Pauli exclusion principle, which states that no two electrons (or fermions in general) can occupy the same quantum state. Thus the presence of an electron diminishes the allowed states for the other electrons by one. Haldane introduced the concept of *fractional* exclusion statistics. This concept will be very useful in the state counting, which is discussed in chapter 6. It turns out that the matrix containing the statistics parameters of the particles of the quantum Hall states has an direct relation with the K-matrix description of the quantum Hall states. These K-matrices of the clustered quantum Hall states will be discussed in the following chapter. This chapter deals with the formalism of describing non-abelian (exclusion) statistics which is needed in the next chapter. This chapter is based on the sections 4 and 5 of [7].

## 4.1 Abelian exclusion statistics

To explain the concept of *exclusion statistics*, we will follow the way in which Haldane originally defined this concept, but focus on the concepts needed in this thesis (such as the state counting). As an example, we will take the spinons which form in the Haldane-Shastry spin chain [50, 93]. This spin chain is a $S = \frac{1}{2}$ Heisenberg chain with *inverse-square* exchange.

The idea behind exclusion statistics [51] is that, in finite systems, the number of states which are available for a particle, depends on the number of particles already present. More precisely, the addition of a particle diminishes the number of available states for particles which are added afterwards. For fermionic particles, the number of available states would diminish by one if a particle is added to the system. In the other familiar case, namely bosons, the number of states would stay the same. Haldane proposed the following interpolation between fermions and bosons. Let $d_i$ be the number of available states for species $i$.

By adding a particle of species $j$ (i.e. $\Delta N_j = 1$), the number $d_i$ changes according

$$\Delta d_i = -\sum_j g_{ij} \Delta N_j \ . \tag{4.1}$$

The elements $g_{ij}$ are called the statistical interaction parameters. For bosons, we have $g_{ij} = 0$, while fermions obey $g_{ij} = \delta_{ij}$, which is the Pauli exclusion principle. The idea of Haldane is to consider more general forms of statistics, such as fractional statistics. The $g_{ij}$ have to be rational, in order to have a well defined thermodynamic limit. Crucial for this definition to work is the assumption that we want to describe *finite* systems, and that the particles are added while the boundary conditions are fixed. That systems with fractional exclusion parameters can occur will be demonstrated for *spinons* systems, see below.

The exclusion statistics can also be defined via the consequences it has for the state counting. The number of states for $N$ identical bosons or fermions in $G$ orbitals is well known, and given by

$$W_{\rm b} = \frac{(G+N-1)!}{N!(G-1)!} \ , \qquad\qquad W_{\rm f} = \frac{G!}{N!(G-N)!} \ . \tag{4.2}$$

The state counting for particles obeying general exclusion statistics is given by [114]

$$W = \prod_i \frac{\big(G_i + N_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})\big)!}{(N_i)!\big(G_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})\big)!} \ . \tag{4.3}$$

Using these multiplicities and the grand canonical partition function for a system of particles obeying fractional statistics, the equations (4.4) (see below) are obtained. From these equations, which describe the 1-particle partition functions, the other thermodynamic properties can be derived.

In this thesis, we will use the concept of fractional exclusion statistics mainly to describe the fractional quantum Hall states. We find an interesting relation between the exclusion statistics matrices and the K-matrices describing the topological properties of the quantum Hall states. In chapter 6, we will use the exclusion statistics of the parafermions to obtain the ground state degeneracies of the clustered quantum Hall states on the sphere, in the presence of quasihole excitations. Before we come to that point, we will first take a closer look at a system, which is best described as an 'ideal gas of fractional statistics particles'.

An important consequence of the concept of an 'ideal gas of fractional statistics particles' is the notion of 1-particle distribution functions which generalize the familiar Fermi-Dirac and Bose-Einstein distributions. These distributions can be derived from '1-particle grand canonical partition functions'. These quantities, which we denote by $\lambda_i$, satisfy the following set of equations, which were independently derived by Isakov, Dasnières de Veigy-Ouvry and Wu (IOW) [59, 26, 114]

$$\left(\frac{\lambda_i - 1}{\lambda_i}\right) \prod_j \lambda_j^{\mathbb{K}^{\rm st}_{ij}} = z_i \ , \tag{4.4}$$

where $\lambda_i = \lambda_i(z_1, \ldots, z_n)$, with $z_i = e^{\beta(\mu_i - \epsilon)}$, is the generalized fugacity of species $i$. Note that the energy $\epsilon$ may also include contributions from the coupling of the charge and

spin of the quasiparticles to external electric and magnetic fields. Hence the information about charge and spin of the quasiparticles is also encoded in these generalized fugacities. The fugacities of the particles will be important for the distinction between abelian and non-abelian statistics, as we will point out later. The matrix $\mathbb{K}^{\mathrm{st}}$ is the so-called 'statistics matrix' and describes, at least in the original situation in which Haldane introduced his new notion of statistics, the statistical interaction of particles of different species.

From the solutions $\lambda_i$ of the IOW-equations (4.4) the one-particle distribution functions $n_i(\epsilon)$ are obtained as

$$n_i(\epsilon) = z_i \frac{\partial}{\partial z_i} \log \prod_j \lambda_j \Big|_{z_i = e^{\beta(\mu_i - \epsilon)}} = \sum_j z_j \frac{\partial}{\partial z_j} \log \lambda_i \Big|_{z_i = e^{\beta(\mu_i - \epsilon)}} , \tag{4.5}$$

where we have assumed that the matrix $\mathbb{K}^{\mathrm{st}}$ is symmetric.

Before we go on to describe the in which way the statistics matrices are used in a quantum Hall situation, we will first consider a basic example of the application of exclusion statistics, namely the of the spinons related to $su(2)_1$.

### 4.1.1 Spinons

In this section, we will use *spinons* to explain the concept of exclusion statistics. In general (including arbitrary dimension) spinon excitations can occur in the background of a antiferromagnetic resonating valance bond (RVB) state; they are unpaired spins. Consider a system of $N$ spins, and $N_{\mathrm{sp}}$ spinon excitations, leaving an integer $(N - N_{\mathrm{sp}})/2$ unbroken bonds. Because of the non-orthogonality of the states, the dimension of the Hilbert space is given by $1 + \frac{N - N_{\mathrm{sp}}}{2}$, independent of the spinon type [52, 51]. To clarify this statement, we take a look at a three site system. Assume that the spins on two of the sites form a bond, while the third site contains a spinon (of either spin). Moving the spinon to one of the sites of the bond only leads to one other independent state; thus, for each bond, there is only one extra spinon state. Of course, summing over all possible states of a system of $N$ sites leads to a total number of states of $2^N$, as expected.

Thus, we come to the conclusion that adding $2n$ spinons to the system reduces the dimension of the Hilbert space for the next added spinon by $n$: the spinons obey a *semionic* exclusion statistics, interpolating between the fermion and boson statistics. The statistics parameters are given by

$$\mathbb{K}_{\mathrm{sp}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} . \tag{4.6}$$

One place where these spinons occur is the Haldane-Shastry spin chain. In [52], Haldane derived the exact spectrum and the thermodynamics of this system. In this analysis, the spinons played a crucial role. Without going into all the details of the system, we give the hamiltonian for the model on a ring, with periodic boundary conditions

$$H = J \sum_{n \leq n'} \big(d(n - n')\big)^2 \mathbf{S}_n \cdot \mathbf{S}_{n'} , \tag{4.7}$$

where $d(n) = \frac{N}{\pi}\sin(\frac{\pi|n|}{N})$ is the *chord distance*. Thus the model (4.7) is an example of a 1-dimensional system in which particles with fractional (exclusion) statistics are present.

On the level of conformal field theory, the properties of the spinons were derived in [90]. The 1-particle distribution functions obtained there exactly correspond to the distributions obtained form the exclusion statistics matrix (4.6). The method used in [90] is that of 'truncated chiral partition functions'. This method will be used (and explained) in this thesis in chapter 6.

### 4.1.2 Abelian quantum Hall fluids

The relation between, on the one hand, the K-matrix of an abelian quantum Hall fluid and, on the other hand, the exclusion statistics of its charged edge excitations, can be described as follows. The charged edge excitations are described by a specific conformal field theory, also known as a chiral Luttinger liquid. Following a procedure first proposed in [90], one may associate a notion of fractional exclusion statistics to a set of fundamental excitations in this CFT. Selecting a particular set of negatively charged 'electron type' excitations together with a 'dual' set of positively charged quasihole excitations, one precisely finds fractional exclusion statistics in the sense of Haldane, with statistics matrix $\mathbb{K}^{\rm st}$ given by

$$\mathbb{K}^{\rm st} = \mathbb{K}_e \oplus \mathbb{K}_\phi \ , \tag{4.8}$$

with $\mathbb{K}_e$ and $\mathbb{K}_\phi$ the K-matrices for the abelian quantum Hall state. For the principal Laughlin series at filling fraction $\nu = 1/M$, this result was obtained in [30], in its general form it first appeared in [6]. The relation of the identification (4.8) with character identities involving so called universal chiral partition functions will be discussed in section 4.3

In [40], a slightly different identification between the K-matrix and a statistics matrix, amounting to $\mathbb{K} = \mathbb{K}_e$, was proposed. The two proposals can be reconciled by realizing that we, in our analysis of edge excitations, restrict ourselves to quanta of positive energy only. From the duality relations that we discuss below, one learns that, in a precise sense, quasihole quanta of positive energy can be traded for holes in a 'Fermi sea' of electron-type quanta at negative energy, and in this way one arrives at a complete description in terms of the matrix $\mathbb{K}_e$ alone.

In chapter 5, we will identify the statistics matrices for excitations over non-abelian quantum Hall states, following [6]. Extending the identification (4.8) to the non-abelian case, we shall propose K-matrices for the non-abelian quantum Hall states. We would like to stress that, although many of the formulas from the well known abelian K-matrix description still hold for the generalized K-matrices we find here, the description for the non-abelian states is on an entirely different footing. The abelian K-matrices were introduced to describe quantum Hall states in the 'most general' way, i.e. by trying to implement the hierarchical schemes in a general way. In the non-abelian case, we need the K-matrix structure to keep track of the non-abelian statistics. So although we use a matrix structure, we are not describing a hierarchical situation.

We continue this section with a discussion of the fundamental 'particle-hole' duality between the electron and the quasihole sectors of the theory. To show how this duality works, we assume that we have $n$ quasiholes $\phi$ and $n$ electron-like particles $\Psi$ described

by the matrices $\mathbb{K}_\phi$ and $\mathbb{K}_e$, respectively. We assume that (i) $\mathbb{K}_\phi = \mathbb{K}_e^{-1}$, and (ii) there is no mutual exclusion statistics between the two sectors (meaning that the statistics matrix is given by the direct sum (4.8)). These two conditions in fact constitute what we mean by duality in this context. In the context of low-energy effective actions for abelian fqH systems, a similar notion of duality has been considered (see, e.g. [107] and references therein).

With the matrices $\mathbb{K}_\phi$ and $\mathbb{K}_e$, two independent systems of IOW-equations can be written down, and these systems are related by the duality (for clarity, we will denote the single level partition function for the quasiholes and electron-like particles by $\lambda_i$ and $\mu_i$ respectively; the corresponding fugacities will be denoted by $x_i$ and $y_i$)

$$\lambda_i = \frac{\mu_i}{\mu_i - 1} \ , \qquad\qquad x_i = \prod_j y_j^{-(\mathbb{K}_e)^{-1}_{ij}} \ , \tag{4.9}$$

as can be verified easily.

As an illustration of the duality, we calculate the central charge of the conformal field theory that describes the edge excitations. We focus on the abelian case. In the non-abelian case, which we discuss in the next section, there will be a subtraction term due to the presence of pseudoparticles.

In general, for abelian quantum Hall states, the central charge $c_{\rm CFT}$ is given by

$$c_{\rm CFT} = \frac{6}{\pi^2} \int_0^1 \frac{dz}{z} \log \lambda_{\rm tot}(z) \ , \tag{4.10}$$

where $\lambda_{\rm tot}(z)$ denotes the product $\prod_j \lambda_j$ evaluated at $z_j = z$ for all $j$. It has been shown (see [22, 17] and references therein) that, upon using the IOW-equations (4.4), this can be rewritten in the following form

$$c_{\rm CFT} = \frac{6}{\pi^2} \sum_i L(\xi_i) \ , \tag{4.11}$$

where $L(z)$ is Rogers' dilogarithm

$$L(z) = -\frac{1}{2} \int_0^z dy \left( \frac{\log y}{1-y} + \frac{\log(1-y)}{y} \right) \ . \tag{4.12}$$

In [63], many interesting identities involving the dilogarithm can be found. The quantities $\xi_i$ which appear in eq. (4.11) are solutions to the central charge equations

$$\xi_i = \prod_j (1 - \xi_j)^{\mathbb{K}_{ij}} \ . \tag{4.13}$$

For the abelian quantum Hall case, we have two matrices $\mathbb{K}_\phi$ and $\mathbb{K}_e$ and we need the solutions $\xi_i$ and $\eta_i$ of the equations

$$\xi_i = \prod_{j=1}^n (1 - \xi_j)^{(\mathbb{K}_\phi)_{ij}} \ , \qquad\qquad \eta_i = \prod_{j=1}^n (1 - \eta_j)^{(\mathbb{K}_e)_{ij}} \ . \tag{4.14}$$

By virtue of the duality, these solutions are related by a simple equation: $\eta_i = 1 - \xi_i$. This leads to

$$\sum_i L(\xi_i) + \sum_i L(\eta_i) = \sum_i \big(L(\xi_i) + L(1-\xi_i)\big) = nL(1) = n\,\frac{\pi^2}{6}\,. \tag{4.15}$$

So in the abelian case, we correctly find that the central charge is just given by the number of species in the theory, $c_{\rm CFT} = n$.

## 4.2 Non-abelian exclusion statistics

In this section, we focus on K-matrices and statistics matrices for non-abelian quantum Hall states. We shall first introduce new types of particles, pseudoparticles and composite particles, and explain the role they play in the non-abelian case. We also extend the notion of duality to the non-abelian case. After that we discuss various aspects (filling factors and shift map) of the quantum Hall data $\mathbb{K}$, $\mathbf{t}$, $\mathbf{s}$ and $\mathbf{j}$ in the non-abelian case.

Among the new particles that appear in non-abelian theories are so called 'composite' particles in the electron sector. These will show up as particles which have multiple electron charges. We introduce an integer label $l_i$ for an order-$l_i$ composite particle of charge $(\mathbf{t}_e)_i = -l_i$.

In the quasihole sector, we encounter so called pseudoparticles, which do not carry any energy, but rather act as a book-keeping device that keep track of 'internal degrees of freedom' of the physical quasiholes. The notion of a 'pseudoparticle' can be traced back to so-called string solutions to the Bethe equations for quantum integrable systems in one dimension, such as the Heisenberg $XXX$ chain (see [96], where the contribution to the thermodynamics of the string solutions for the $XXX$ chain is computed). Pseudoparticles were used (and received their name) in the TBA analysis of integrable systems with non-diagonal particle scattering (see, e.g. [116]). In the context of exclusion statistics they have been discussed in [40, 48, 17, 6]. We assign the label $l_i = 0$ to all pseudoparticles.

An important observation, first made in [6], is that the duality between the electron and quasihole sectors naturally links the presence of composite particles in one sector to the presence of pseudoparticles in the other. Physically, this is a link between the pairing physics of the non-abelian quantum Hall states and the non-abelian statistics of their fundamental excitations.

### 4.2.1 Composites, pseudoparticles and null-particles

The presence of pseudoparticles and composite particles calls for a slight generalization of the discussion of the previous section. When focusing on the dependence of the $\lambda_i$ on the energy $\epsilon$, the natural specialization of the generalized fugacities $z_i$ is given by $z_i = z^{l_i}$, with $z = e^{-\beta\epsilon}$. In the presence of $l_i \neq 1$, the 1-particle distribution functions take the form [note that a composite particle labeled by $l_i$ carries energy $l_i\epsilon$]

$$n_i(\epsilon) = z_i \frac{\partial}{\partial z_i} \log \prod_j [\lambda_j]^{l_j}\big|_{z_i = e^{\beta(\mu_i - l_i\epsilon)}} = \sum_j l_j z_j \frac{\partial}{\partial z_j} \log \lambda_i\big|_{z_i = e^{\beta(\mu_i - l_i\epsilon)}}\,. \tag{4.16}$$

With the following definition of $\lambda_{\rm tot}(z)$

$$\lambda_{\rm tot}(z) = \prod_i [\lambda_i(z_j = z^{l_j})]^{l_i} \ , \tag{4.17}$$

the central charge $c_{\rm CFT}$ is again given by the expression (4.10). We note that in the specialized IOW equations, with $z_i = z^{l_i}$, the right hand side of the equations for pseudoparticles is equal to 1. When focusing on quantum numbers other than energy, such as spin, we will consider slightly more general versions of the quantity $\lambda_{\rm tot}$.

In all examples (abelian and non-abelian) that are explicitly discussed in this thesis, we assume a choice of particle basis such that $\mathbf{l}_e = -\mathbf{t}_e$. For the abelian quantum Hall states we further assume that $(\mathbf{t}_e)_i = -1$ for all $i$. In the quasihole sector we specify $(\mathbf{l}_\phi)_i = \frac{1}{q_{\rm qh}}(\mathbb{K}_\phi)_{ij}(\mathbf{l}_e)_j = \frac{1}{q_{\rm qh}}\mathbf{t}_\phi$, where $q_{\rm qh}$ is the smallest (elementary) charge in the quasihole sector. [This implies that, even in the abelian case, we may treat some of the quasiholes as composites of the most fundamental ones, thereby generalizing the discussion of the previous section.]

Under these assumptions, we find that under duality $\lambda_{\rm tot}(x)$ and $\mu_{\rm tot}(y)$ are related in the following way

$$\lambda_{\rm tot}(x) = x^\gamma \mu^\alpha_{\rm tot}(y) \ , \qquad\qquad y = x^{-\beta} \ , \tag{4.18}$$

with

$$\alpha = \beta = \frac{1}{q_{\rm qp}} \ , \qquad\qquad \gamma = \frac{\nu}{q^2_{\rm qp}} \ . \tag{4.19}$$

A clear sign of non-abelian statistics is found in the way the quantity $\lambda_i$ for physical particles depends on the fugacity $z_i$. Putting $z_l = 1$ for all pseudoparticles, and focusing on the small $z$ behaviour of $\lambda_i$, one finds

$$\lambda_i = 1 + \alpha_i z_i + o(z^2) \ . \tag{4.20}$$

In the abelian case, $\alpha_i = 1$, whereas in the non-abelian case $\alpha_i > 1$. The factors $\alpha_i$ lead to multiplicative factors in the Boltzmann tails of the one-particle distribution functions for physical particles. The quantities $\alpha_i$ are in fact the largest eigenvalues of the fusion matrix [22], i.e., the quantum dimensions (see, for instance, [34]) of the conformal field theory associated to the quantum Hall state, and can easily be calculated for the cases we deal with (see sections 5.1 and 5.2).

In [6], we presented a generalized K-matrix structure for some recently proposed quantum Hall states. The proposed K-matrices were identified via their role as statistics matrices for the fundamental charged edge excitations. In the quasihole sector, the non-abelian statistics leads to a specific set of pseudoparticles and an associated statistics matrix $\mathbb{K}_\phi$ [48, 17]. The matrix $\mathbb{K}_e$, related to $\mathbb{K}_\phi$ by the duality $\mathbb{K}_e = \mathbb{K}_\phi^{-1}$, refers to particles which are identified as composites of the fundamental electron-like excitation. From the point of view of the wave functions for the non-abelian quantum Hall states (see chapter 3 and [71, 85, 10]), the presence of composite excitations is very natural. This is because the non-abelian states show a clustering property, as described in section 3.1. In [46, 45, 6] it was argued that the

wave functions which show pairing (at $k = 2$), are related (in the non-magnetic limit, i.e. in the limit of $\nu \to \infty$) to BCS superconductivity.

Composite particles are identified as particles whose generalized fugacities are specific combinations of the generalized fugacities of other particles, i.e., all quantum numbers of composite particles are completely determined in terms of the quantum numbers of their constituents. It has been shown in [17] that particular kinds of composite particles, so-called null-particles, accounting for the null-states in the quasiparticle Fock spaces, are often needed to interpret the system in terms of Haldane's exclusion statistics or, equivalently, to write the partition function in UCPF form (see also section 4.3.2).

We now turn to the computation of the central charge $c_{\mathrm{CFT}}$ the non-abelian case. It was shown in [17], that the presence of pseudoparticles leads to a simple correction term that is subtracted from the abelian result $c_{\mathrm{CFT}} = n$. For the pseudoparticles, a system of equations like eq. (4.13) can be written down

$$\xi_i' = \prod_j{}' (1 - \xi_j')^{\mathbb{K}_{ij}} \ , \tag{4.21}$$

where the prime indicates that the product is restricted to pseudoparticles. The correction term is given by a sum over the dilogarithm of the solutions of (4.21), leading to

$$c_{\mathrm{CFT}} = n - \frac{6}{\pi^2} \sum_i{}' \ L(\xi_i') \ . \tag{4.22}$$

### 4.2.2 On filling factors

Up to now, we merely asserted that the statistics matrices $\mathbb{K}$ can also serve as (generalized) K-matrices for non-abelian quantum Hall states. To make this statement more clear, we will now investigate how some of the 'K-matrix results' for abelian quantum Hall states generalize to the non-abelian case. In this derivation, we make the assumption that the pseudoparticles do not carry charge or spin. In all cases that are explicitly considered in chapter 5 this assumption holds in the simplest formulation. If pseudoparticles do carry spin or charge, the formulas we obtain below need to be modified.

Let us start with the filling factor corresponding to state which is described by the IOW-equations, for a statistics matrix $\mathbb{K}_e$, charge vector $\mathbf{t}_e$, and labels $\mathbf{l}_e = -\mathbf{t}_e$. We couple the system to an electric field by taking $y_i = y^{-(\mathbf{t}_e)_i}$. [This is when the orientation of the electric field is such that the response is carried by the negatively charged excitations.] The large $y$ (i.e. low temperature) behaviour of the IOW-equations (4.4) is then given by the following set of relations

$$\prod_j \mu_j^{(\mathbb{K}_e)_{ij}} \sim y^{-(\mathbf{t}_e)_i} \ , \tag{4.23}$$

which imply, when $\mathbb{K}$ is symmetric (which is assumed throughout this thesis) and invertible

$$\mu_{\text{tot}} = \prod_i \mu_i^{-(\mathbf{t}_e)_i} \sim y^{\mathbf{t}_e \cdot \mathbb{K}_e^{-1} \cdot \mathbf{t}_e} \ . \tag{4.24}$$

Because the left hand side of Eq. (4.24) in the $T \to 0$ limit determines the filling factor $\nu$ through $\mu_{\text{tot}} \sim y^{\nu}$, we find the well-known formula

$$\nu = \mathbf{t}_e \cdot \mathbb{K}_e^{-1} \cdot \mathbf{t}_e \ . \tag{4.25}$$

For the opposite orientation of the electric field, a similar expression is obtained by starting from the K-matrix for the (positively charged) quasiholes

$$\nu = \mathbf{t}_\phi \cdot \mathbb{K}_\phi^{-1} \cdot \mathbf{t}_\phi \ . \tag{4.26}$$

This result could also have been obtained by using eq. (4.25) and the transformation properties of $\mathbb{K}_e$ and $\mathbf{t}_e$ under duality. We remark that the above derivations explicitly assume that only the physical particles respond to the electric field, i.e., that all pseudoparticles are neutral.

Let us now turn to the spin Hall conductance, and the corresponding spin filling factor. The derivation of the corresponding spin filling factor

$$\sigma = \mathbf{s}_e \cdot \mathbb{K}_e^{-1} \cdot \mathbf{s}_e \ , \tag{4.27}$$

goes along the same lines as the derivation of the electron filling factor. As an extra step, one needs to relate the fugacities of the spin up and down particles by $y_\uparrow = 1/y_\downarrow = z$. This results in

$$\prod_i \mu_i^{(\mathbf{s}_e)_i} \sim z^{\mathbf{s}_e \cdot \mathbb{K}_e^{-1} \cdot \mathbf{s}_e} \ , \tag{4.28}$$

leading to Eq. (4.27). It is important to note that this formula only holds in the cases where the pseudoparticles in the $\phi$-sector do not carry spin. As a check on this formula, one would like to have a procedure to obtain the spin filling factor directly from the wave functions, as is possible for the electron filling factor. To do this, one has to count the zeros of the wave function with respect to one reference particle (of a given spin, say, up). The total number of zeros gives the total flux needed on the sphere as a linear function of the total number of electrons $N_e$. By using the relation between $N_e$ and $N_\Phi$ given in (2.5) one obtains the electron filling factor and the shift. To obtain the spin filling factor, one has to keep track of two different types of zeros, namely those with respect to a particle of the same spin, and the ones with respect to particles of the other spin. We will denote the number of these zeros by $N_\Phi^\uparrow$ and $N_\Phi^\downarrow$ respectively. The electron and spin filling factors are obtained from

$$\begin{aligned} N_\Phi = N_\Phi^\uparrow + N_\Phi^\downarrow &= \frac{1}{\nu} N_e - \mathcal{S} \ , \\ N_\Phi^\uparrow - N_\Phi^\downarrow &= \frac{1}{\sigma} N_e - \mathcal{S} \ . \end{aligned} \tag{4.29}$$

We applied this procedure to the non-abelian spin-singlet states of [10] (the explicit form of the wave functions are given in section 3.5), and indeed found the same results for the electron and spin filling factor as obtained from the K-matrix formalism, eq. (3.23). Also the electron filling factor for the Read-Rezayi states is reproduced correctly, see eq. (3.12). In addition, for both types of states we found that the shift on the sphere is in agreement with (2.6) for $g = 0$.

Summarizing, we have presented evidence that duality relations

$$\mathbb{K}_\phi = \mathbb{K}_e^{-1} \ , \qquad \mathbf{t}_\phi = -\mathbb{K}_e^{-1} \cdot \mathbf{t}_e \ , \qquad \mathbf{s}_\phi = -\mathbb{K}_e^{-1} \cdot \mathbf{s}_e \ , \qquad \mathbf{j}_\phi = -\mathbb{K}_e^{-1} \cdot \mathbf{j}_e \ . \tag{4.30}$$

are applicable to both abelian and non-abelian quantum Hall states, and that the expressions (2.3) for the filling factors $\nu$ and $\sigma$ apply to the non-abelian case, in a formulation where pseudoparticles do not carry spin or charge.

### 4.2.3 Shift map

Suppose we have a fractional quantum Hall system which is characterized by the data $(\mathbb{K}_e, \mathbf{t}_e, \mathbf{s}_e, \mathbf{j}_e)$. We can then construct a family of fractional quantum Hall systems, parameterized by $M \in \mathbb{Z}_+$, by applying the 'shift map' $\mathcal{S}_M$ introduced in [36]. In the cases we consider, $M$ odd (even) corresponds to a fermionic (bosonic) state respectively. At the level of trial wave functions $\Psi(z)$, $\mathcal{S}_M$ simply acts as a multiplicative Laughlin factor $\prod_{i<j}(z_i - z_j)^M$. Thus, $\mathcal{S}_M$ increases the number of flux quanta by

$$N_\Phi \mapsto N_\Phi + M(N_e - 1) = (\frac{1}{\nu} + M)N_e - (\mathcal{S} + M) \ , \tag{4.31}$$

i.e.,

$$\nu^{-1} \mapsto \nu^{-1} + M \ , \qquad \sigma \mapsto \sigma \ , \qquad \mathcal{S} \mapsto \mathcal{S} + M \ . \tag{4.32}$$

In fact, $\mathcal{S}_M$ acts on the fqH data $(\mathbb{K}_e, \mathbf{t}_e, \mathbf{s}_e, \mathbf{j}_e)$ as

$$\begin{aligned} \mathcal{S}_M \mathbb{K}_e &= \mathbb{K}_e + M \mathbf{t}_e \mathbf{t}_e \ , \\ \mathcal{S}_M \mathbf{t}_e &= \mathbf{t}_e \ , \\ \mathcal{S}_M \mathbf{s}_e &= \mathbf{s}_e \ , \\ \mathcal{S}_M \mathbf{j}_e &= \mathbf{j}_e + \frac{M}{2} \mathbf{t}_e \ . \end{aligned} \tag{4.33}$$

One easily checks that (4.33), together with (4.25), leads to the shift in $\nu^{-1}$ as given in (4.32). By duality (4.30) one obtains

$$\begin{aligned} \mathcal{S}_M \mathbb{K}_\phi &= \mathbb{K}_\phi - \frac{M}{1 + \nu M} \mathbf{t}_\phi \mathbf{t}_\phi \ , \\ \mathcal{S}_M \mathbf{t}_\phi &= \frac{1}{1 + \nu M} \mathbf{t}_\phi \ , \\ \mathcal{S}_M \mathbf{s}_\phi &= \mathbf{s}_\phi \ , \\ \mathcal{S}_M \mathbf{j}_\phi &= \mathbf{j}_\phi - \frac{M}{2} \left( \frac{\nu \mathcal{S} - 1}{1 + \nu M} \right) \mathbf{t}_\phi \ . \end{aligned} \tag{4.34}$$

A few remarks should be made. By using the duality (4.30), one actually finds for the action of the shift map on $\mathbf{s}_\phi$: $\mathcal{S}_M \mathbf{s}_\phi = \mathbf{s}_\phi + \frac{M(\mathbf{t}_\phi \cdot \mathbf{s}_e)}{1 + \nu M} \mathbf{t}_\phi$. However, the shift map is only

supposed to act on the charge component of the particles, thus we would like to demand that $\mathcal{S}_M \mathbf{s}_\phi = \mathbf{s}_\phi$. Therefore, for consistency, we *require*

$$\mathbf{t}_\phi \cdot \mathbf{s}_e = -\mathbf{t}_e \cdot \mathbb{K}_e^{-1} \cdot \mathbf{s}_e = 0 \,, \tag{4.35}$$

leading to (4.34). Of course, relation (4.35) is just the statement that for spin-singlet states there should be a $\mathbb{Z}_2$ symmetry $(\mathbf{t}_e, \mathbf{s}_e) \mapsto (\mathbf{t}_e, -\mathbf{s}_e)$. Equation (4.35) is fulfilled for all our examples (if we take $\mathbf{s}_e = 0$ for the spin-polarized states). Although, in general, $\mathbf{j}_e$ has to be treated as an independent variable, for the states discussed in sections 3.4 to 3.6 all formulas are consistent with the relation $\mathbf{j}_e = \mathbf{s}_e + \frac{\mathcal{S}}{2(1-g)}\mathbf{t}_e$.

In this thesis we will be mainly concerned with fractional quantum Hall systems corresponding to conformal field theories $\mathfrak{g}_{k,M}$ which are deformations of the conformal field theory based on the affine Lie algebra $\mathfrak{g}_k$ at level $k$. The $\mathfrak{g}$ symmetry greatly simplifies the determination of the fqH data $(\mathbb{K}_e, \mathbf{t}_e, \mathbf{s}_e, \mathbf{j}_e)$ for $\mathfrak{g}_k$. The fqH data for $\mathfrak{g}_{k,M}$ are then simply obtained by applying the shift operator $\mathcal{S}_M$ as in (4.33). The action of the shift map can be visualized as follows. Charge is usually identified with a particular direction in the weight lattice of $\mathfrak{g}$. The degrees of freedom associated to this direction can be represented by a chiral boson compactified on a circle of some radius $R$. The shift map $\mathcal{S}_M$ has the effect of rescaling the radius $R$ while keeping all other directions in the weight diagram fixed.

### 4.2.4 Composites

The description of a physical system in terms of a set of $n$ quasiparticles with mutual exclusion statistics given by a matrix $(\mathbb{K}_{ij})_{1\le i,j\le n}$ is not unique. In particular one may extend the number of quasiparticles by introducing composites as we will now explain.

Consider the IOW-equations (4.4) with

$$\mathbb{K} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} , \qquad \mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} . \tag{4.36}$$

If we define the operation $\mathcal{C}_{ij}$, corresponding to adding a composite of the quasiparticles $i$

and $j$ to the system, by

$$\mathcal{C}_{ij}\mathbb{K} = \begin{pmatrix} a_{11} & \dots & a_{1n} & \vdots & a_{1i}+a_{1j} \\ \vdots & & \vdots & \vdots & \vdots \\ & a_{ij}+1 & & \vdots & \\ & & & \vdots & \\ a_{ji}+1 & & & \vdots & \\ & & & \vdots & \\ a_{n1} & \dots & a_{nn} & \vdots & a_{ni}+a_{nj} \\ \dots & \dots & \dots & \vdots & \dots \\ a_{i1}+a_{j1} & \dots & a_{in}+a_{jn} & \vdots & a_{ii}+2a_{ij}+a_{jj} \end{pmatrix} , \tag{4.37}$$

and

$$\mathcal{C}_{ij}\mathbf{z} = (\mathbf{z}_1, \dots, \mathbf{z}_n; \mathbf{z}_i\mathbf{z}_j) , \tag{4.38}$$

such that, in particular,

$$\begin{aligned} \mathcal{C}_{ij}\mathbf{t} &= (\mathbf{t}_1, \dots, \mathbf{t}_n; \mathbf{t}_i + \mathbf{t}_j) , \\ \mathcal{C}_{ij}\mathbf{s} &= (\mathbf{s}_1, \dots, \mathbf{s}_n; \mathbf{s}_i + \mathbf{s}_j) , \end{aligned} \tag{4.39}$$

then the two systems are equivalent, at least at the level of thermodynamics. The solutions $\{\lambda_i\}$ to the IOW-equations defined by $(\mathbb{K}, \mathbf{z})$ and $\{\lambda'_i\}$ defined by $(\mathbb{K}', \mathbf{z}') = (\mathcal{C}_{ij}\mathbb{K}, \mathcal{C}_{ij}\mathbf{z})$ are simply related by

$$\begin{aligned} \lambda'_i &= \frac{\lambda_i + \lambda_j - 1}{\lambda_j} , & \lambda'_j &= \frac{\lambda_i + \lambda_j - 1}{\lambda_i} , \\ \lambda'_{n+1} &= \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j - 1} , & \lambda'_k &= \lambda_k , \qquad (k \neq i, j, n+1) . \end{aligned} \tag{4.40}$$

Note that, in particular, it follows $\lambda_i = \lambda'_i\lambda'_{n+1}$ and $\lambda_j = \lambda'_j\lambda'_{n+1}$ such that $\lambda_{\text{tot}} = \lambda'_{\text{tot}}$. Also, from $\lambda_i = \lambda'_i\lambda'_{n+1}$ and $\lambda_j = \lambda'_j\lambda'_{n+1}$ one sees that the original one-particle partition functions for $i$ and $j$, receive contributions from the new particles $i$ and $j$, respectively, as well as from the composite particle $n+1$. The operation $\mathcal{C}_{ij}$ has the effect that states in the spectrum containing both particles $i$ and $j$ get less dense (their mutual exclusion statistics is bumped up by 1), while the resulting 'gaps' are now filled by the new composite particle.

A consistency check on the equivalence of the systems described by $(\mathbb{K}, \mathbf{z})$ and $(\mathbb{K}', \mathbf{z}')$ is the fact that both lead to the same central charge as a consequence of the five-term identity for Rogers' dilogarithm (see [17]).

Finally, note that the shift map $\mathcal{S}_M$ of eq. (4.33) and composite operation $\mathcal{C}_{ij}$ of eqs. (4.37) and (4.39) commute, i.e.

$$\mathcal{S}_M\, \mathcal{C}_{ij} = \mathcal{C}_{ij}\, \mathcal{S}_M , \tag{4.41}$$

as one would expect.

## 4.3 The UCPF and exclusion statistics

In this section, we will comment upon the relation between the *universal chiral partition function* (UCPF) and exclusion statistics.

### 4.3.1 Quasiparticle basis and truncated partition function

Quasiparticles in two dimensional conformal field theories are represented by so-called chiral vertex operators $\phi^{(i)}(z)$ that intertwine between the irreducible representations of the chiral algebra. Given a set of quasiparticles $\phi^{(i)}(z)$, $i = 1, \ldots, n$, one has to determine a basis for the Fock space created by the modes $\phi^{(i)}_{-s}$, i.e., a maximal, linearly independent set of vectors

$$\phi^{(i_N)}_{-s_N} \ldots \phi^{(i_2)}_{-s_2} \phi^{(i_1)}_{-s_1} |\omega\rangle \ , \tag{4.42}$$

with suitable restrictions on the mode sequences $(s_1, \ldots, s_N)$ (which may depend on the 'fusion paths' $(i_1, \ldots, i_N)$), as well as a set of vacua $|\omega\rangle$ (see [22, 17] for more details). The partition function $Z(\mathbf{z}; q)$ is then defined by

$$Z(\mathbf{z}; q) = \mathrm{Tr}\Big( (\prod_i z_i^{N_i}) q^{L_0} \Big) \ , \tag{4.43}$$

where the trace is taken over the basis (4.42) and $N_i$ denotes the number operator for quasiparticles of type $i$ while $L_0 = \sum_i s_i$ for a state of type (4.42). During this discussion on the UCPF we use the following, in the literature standard, notation $q = e^{-\beta\epsilon_0}$, where $\epsilon_0$ is some fixed energy scale, and $z_i = e^{\beta\mu_i}$.

Exclusion statistics in conformal field theory can be studied by means of recursion relations for truncated partition functions [90]. Truncated partition functions $P_{\mathbf{L}}(\mathbf{z}; q)$, for $\mathbf{L} = (L_1, \ldots, L_n)$, are defined by taking the partition function of those states (4.42) where all the modes $s$ for quasiparticles of species $i$ satisfy $s \leq L_i$. By definition, for large $\mathbf{L}$, we will have (see [22, 17] for more details)

$$P_{\mathbf{L}+\mathbf{e}_i}(z; q)/P_{\mathbf{L}}(z; q) \sim \lambda_i(z_i q^{L_i}) \ , \tag{4.44}$$

where $\mathbf{e}_i$ denotes the unit vector in the $i$-direction. In particular, if the generalized fugacities $z_i$ are given by $z_i = z^{l_i}$, for some fixed $z$, and the quasiparticle modes are truncated by $L_i = l_i L$, then we find, using (4.17)

$$P_{L+1}(z; q)/P_L(z; q) \sim \lambda_{\text{tot}}(z q^L) \ , \tag{4.45}$$

where $P_L(z; q) = P_{l_1 L, l_2 L, \ldots, l_n L}(z_i = z^{l_i}; q)$. Thus, given a set of recursion relations for the truncated partition functions $P_{\mathbf{L}}(z; q)$, one derives algebraic equations for the one-particle partition functions $\lambda_i(z)$ by taking the large $\mathbf{L}$ limit. In particular one can find an equation for $\lambda_{\text{tot}}(z)$ from $P_L(z; q)$ by using (4.45). For all conformal field theories that have been studied this way it turns out that one finds agreement between these $\lambda$-equations and the IOW-equations (4.4) corresponding to a specific statistics matrix $\mathbb{K}$ (see, in particular, [22]).

### 4.3.2 The universal chiral partition function

Based on many examples, it has become clear that the characters of the representations of all conformal field theories can be written in the form of, what is now known as, a universal chiral partition function (UCPF) (see in particular, ref. [13] and references therein)

$$Z(\mathbb{K};\mathbf{Q},\mathbf{u}|\mathbf{z};q)=\sum_{\mathbf{m}}{}'\Big(\prod_i z_i^{m_i}\Big)q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m}+\mathbf{Q}\cdot\mathbf{m}}\prod_i\begin{bmatrix}\big((\mathbb{I}-\mathbb{K})\cdot\mathbf{m}+\mathbf{u}\big)_i\\ m_i\end{bmatrix}\,, \tag{4.46}$$

where $\mathbb{K}$ is a (rational) $n\times n$ matrix, $\mathbb{I}$ is the identity matrix, $\mathbf{Q}$ and $\mathbf{u}$ are certain $n$-vectors and the sum over $m_1,\ldots,m_n$, is over the nonnegative integers subject to some restrictions (which, throughout this thesis, are taken to be such that the coefficients in the $q$-binomials are integer). The $q$-binomial (Gaussian polynomial) is defined by

$$\begin{bmatrix}M\\ m\end{bmatrix}=\frac{(q)_M}{(q)_m(q)_{M-m}}\,,\qquad\qquad (q)_m=\prod_{k=1}^{m}(1-q^k)\,. \tag{4.47}$$

The vectors $\mathbf{Q}$ and $\mathbf{u}$ as well as the restrictions on the summation variables, will in general depend on the particular representation of the conformal field theory, while $\mathbb{K}$ is independent of the representation. To write the conformal characters in the form (4.46) may require introducing null-quasiparticles which account for null-states in the quasiparticle Fock space [17]. The null-quasiparticles are certain composites, hence their fugacities $z_i$ in (4.46) are specific combinations of the fugacities of their constituents.

It has been conjectured that the UCPF (4.4) is precisely the partition function (4.43) of a set of quasiparticles with exclusion statistics given by the same matrix $\mathbb{K}$, where $u_i=\infty$ corresponds to a physical quasiparticle and $u_i<\infty$ to a pseudoparticle [48, 17]. This conjecture has been verified in numerous examples (see [48, 17] for references). A convincing piece of evidence in support of this conjecture is the fact that the asymptotics of the character (4.46) (in the thermodynamic limit $q\to 1^-$) is given by exactly the same formula as the one for the IOW-equations [17] (see also [88, 73, 63, 25] for $z_i=1$). In the next section we establish the correspondence in a more direct way.

For future convenience let us introduce the limiting form of the UCPF (4.46) when all $u_i\to\infty$, i.e. the case that all quasiparticles are physical and the exclusion statistics is abelian

$$Z_\infty(\mathbb{K};\mathbf{Q})=\sum_{\mathbf{m}}{}'\Big(\prod_i z_i^{m_i}\Big)\frac{q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m}+\mathbf{Q}\cdot\mathbf{m}}}{\prod_i(q)_{m_i}}\,. \tag{4.48}$$

Note that the limiting UCPFs (4.48) are not all independent, but satisfy (see [16])

$$Z_\infty(\mathbb{K};\mathbf{Q})=Z_\infty(\mathbb{K};\mathbf{Q}+\mathbf{e}_i)+z_iq^{\frac{1}{2}\mathbb{K}_{ii}+\mathbf{Q}_i}Z_\infty(\mathbb{K};\mathbf{Q}+\mathbb{K}\cdot\mathbf{e}_i)\,, \tag{4.49}$$

as a consequence of

$$\frac{1}{(q)_m}=\frac{q^m}{(q)_m}+\frac{1}{(q)_{m-1}}\,. \tag{4.50}$$

### 4.3.3 Relation to exclusion statistics

The relation between the UCPF and exclusion statistics can be made more explicit as follows. Suppose the truncated partition functions $P_{\mathbf{L}}(\mathbf{z};q)$ are given by 'finitized UCPFs' of the form

$$P_{\mathbf{L}}(\mathbf{z};q) = \sum_{\mathbf{m}}{}' \Big(\prod_i z_i^{m_i}\Big) q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m}+\mathbf{Q}\cdot\mathbf{m}} \prod_i \begin{bmatrix} \big(\mathbf{L}+(\mathbb{I}-\mathbb{K})\cdot\mathbf{m}+\mathbf{u}\big)_i \\ m_i \end{bmatrix} , \tag{4.51}$$

for some vectors $(\mathbf{Q},\mathbf{u})$. Of course, the number of parameters in this expression is overdetermined. Usually we think of $\mathbf{u}$ as being fixed while the meaning of the parameters $\mathbf{L}$ are determined by the cut-off scale. We can of course absorb the $\mathbf{u}$ by shifts in $\mathbf{L}$ (in fact, in practice we often make shifts in the definition of $\mathbf{L}$ to simplify the recursion relations). We also remark that we have introduced finitization parameters $L_i$ also for the pseudoparticles in (4.51) to facilitate deriving recursion relations. In making the identification with the truncated partition functions these parameters are kept at a fixed (usually 'small' or even zero) value.

Using

$$\begin{bmatrix} M \\ m \end{bmatrix} = \begin{bmatrix} M-1 \\ m \end{bmatrix} + q^{M-m} \begin{bmatrix} M-1 \\ m-1 \end{bmatrix} , \tag{4.52}$$

we find that $P_{\mathbf{L}}(\mathbf{z};q)$ satisfies the system of recursion relations

$$P_{\mathbf{L}}(\mathbf{z};q) = P_{\mathbf{L}-\mathbf{e}_i}(\mathbf{z};q) + z_i q^{-\frac{1}{2}\mathbb{K}_{ii}+\mathbf{Q}_i+\mathbf{u}_i+\mathbf{L}_i} P_{\mathbf{L}-\mathbb{K}\cdot\mathbf{e}_i}(\mathbf{z};q) . \tag{4.53}$$

Upon dividing by $P_{\mathbf{L}}(\mathbf{z};q)$, setting $q=1$, taking the large $\mathbf{L}$ limit, and using (4.44), we obtain

$$1 = \lambda_i^{-1} + z_i \prod_j \lambda_j^{-\mathbb{K}_{ji}} , \tag{4.54}$$

which are equivalent to the IOW-equations (4.4) with statistics matrix $\mathbb{K}$.

Moreover, for any polynomial $P_{\mathbf{L}}(\mathbf{z};q)$ satisfying the recursion relation (4.53), the polynomial

$$Q_{\mathbf{L}}(\mathbf{z};q) = \Big(\prod_i z_i^{-L_i}\Big)\, q^{\frac{1}{2}\mathbf{L}\cdot\mathbb{K}\cdot\mathbf{L}+(\mathbf{Q}+\mathbf{u})\cdot\mathbf{L}} P_{\mathbb{K}\cdot\mathbf{L}}(\mathbf{z};q^{-1}) , \tag{4.55}$$

satisfies the recursion relations (4.53) with dual data $(\mathbb{K}';\mathbf{Q}',\mathbf{u}',\mathbf{z}')$, given by (cf. (4.9))

$$\mathbb{K}' = \mathbb{K}^{-1} , \qquad \mathbf{Q}'+\mathbf{u}' = \mathbb{K}^{-1}\cdot(\mathbf{Q}+\mathbf{u}) , \qquad z_i' = \prod_j z_j^{-\mathbb{K}^{-1}_{ij}} . \tag{4.56}$$

Thus, under the assumption that the set of finitized UCPFs (4.51), for fixed $\mathbf{Q}+\mathbf{u}$, form a complete set of solutions to (4.53), the dual polynomial $Q_{\mathbf{L}}(\mathbf{z}',q)$ of (4.55) can again be written as a (finite) linear sum of finitized UCPFs with dual data (4.56). Moreover, by taking the large $\mathbf{L}$ limit of (4.55), using eqs. (4.44) and (4.54), one recovers the duality relations (4.9) and (4.18).

The above calculation shows that, for quasiparticles whose truncated partition function is given by an expression of the form (4.51), the thermodynamics of these quasiparticles

is described by Haldane's exclusion statistics with statistics matrix $\mathbb{K}$. Even though many truncated characters are indeed of the form (4.51) (we will encounter various examples in the remainder of this thesis) this is not the general situation. However, in examples it turns out that for all recursion relations for truncated characters there is an associated recursion relation, leading to the *same* $\lambda$-equation, which does admit a solution of the form (4.51). The true solution to this recursion relation will in general differ from (4.51) by terms of order $q^L$. In a sense we can talk about the *universality class* of recursion relations as those recursion relations that give rise to the same $\lambda$-equations and hence the same exclusion statistics.

### 4.3.4 Composites, revisited

In section 4.2.4 we have seen, at the level of thermodynamics (i.e. the IOW-equations), how to introduce composite particles into the system in such a way that the resulting system is equivalent to the original system. Due to the intimate relation of exclusion statistics with the UCPF, explained in section 4.3.3, one would expect that a similar construction is possible at the level of the UCPF. Indeed, upon substituting the following polynomial $q$-identity (see appendix A of [7] for a proof)

$$\begin{bmatrix} M_1 \\ m_1 \end{bmatrix} \begin{bmatrix} M_2 \\ m_2 \end{bmatrix} = \sum_{m \geq 0} q^{(m_1-m)(m_2-m)} \begin{bmatrix} M_1 - m_2 \\ m_1 - m \end{bmatrix} \begin{bmatrix} M_2 - m_1 \\ m_2 - m \end{bmatrix} \begin{bmatrix} M_1 + M_2 - (m_1 + m_2) + m \\ m \end{bmatrix}, \quad (4.57)$$

into the UCPF (4.46) at the $(i, j)$-th entry, and subsequently shifting the summation variables $m_i \mapsto m_i + m$, $m_j \mapsto m_j + m$, yields an equivalent UCPF, based on $n + 1$ quasiparticles with data $(\mathcal{C}_{ij}\mathbb{K}; \mathcal{C}_{ij}\mathbf{Q}, \mathcal{C}_{ij}\mathbf{u})$ and $\mathcal{C}_{ij}\mathbf{z}$, where

$$\begin{aligned} \mathcal{C}_{ij}\mathbf{Q} &= (\mathbf{Q}_1, \ldots, \mathbf{Q}_n; \mathbf{Q}_i + \mathbf{Q}_j) \,, \\ \mathcal{C}_{ij}\mathbf{u} &= (\mathbf{u}_1, \ldots, \mathbf{u}_n; \mathbf{u}_i + \mathbf{u}_j) \,, \end{aligned} \quad (4.58)$$

while $\mathcal{C}_{ij}\mathbb{K}$ and $\mathcal{C}_{ij}\mathbf{z}$ are defined in eqs. (4.37) and (4.38), respectively. Various limiting forms of (4.57), relevant to introducing a composite of two physical particles or one physical particle and one pseudoparticle, are given in [7] as well.

# Chapter 5

# K-matrices for clustered quantum Hall states

In this chapter, we will determine the K-matrices for the clustered quantum Hall states discussed in this thesis. We first deal with the K-matrices for the clustered spin-polarized states of section 3.4, followed by the spin-singlet states of section 3.5 in sections 5.1 and 5.2 respectively. These sections are based on the sections 6 and 7 of the paper [7]. In section 5.3, we give an alternative construction of the K-matrices above, inspired by the observation that the wave functions of the clustered quantum Hall states can be obtained from a related abelian *cover* state by a symmetrization procedure, described in section 3.4. This new way of constructing K-matrices for non-abelian quantum Hall states is used to obtain the K-matrices of the spin-charge separated states (see eq. (3.36) for the $k = 1$ states) in section 5.4.

Though the K-matrices for quantum Hall systems are interesting by themselves, as they characterize the topological properties of the quantum Hall systems, they also appear in many other contexts. They can be viewed as the statistics matrices appearing in the *universal chiral partition function* for the corresponding Wess-Zumino-Witten (WZW) models with the same affine symmetry, see, in particular, [7].

We will use the K-matrices for the quantum Hall systems as a starting point to obtain the statistics matrices for the parafermion CFTs in section 5.5. These statistics matrices form the basis of the fermionic character formulas for the parafermion theories. While these characters are known for the $\mathbb{Z}_k$ and $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ parafermions, the form we propose for the parafermions related to $\mathfrak{so}(5)_k/\mathfrak{u}(1)^2$ appears to be new.

The results for $\mathfrak{so}(5)_k$ described in this chapter are based on [5]. K-matrices for more general WZW models with affine Lie algebra symmetry and the related parafermions will also be presented there.

## 5.1 Spin-polarized clustered states: $\mathfrak{su}(2)_{k,M}$

In this section we discuss a family of non-abelian spin-polarized fractional quantum Hall systems with underlying conformal field theory $\mathfrak{su}(2)_{k,M}$ and filling factor

$$\nu_{k,M} = \frac{k}{kM+2} \,. \tag{5.1}$$

For $k = 2$ we are describing the Moore-Read states, which were introduced in [71] while the generalizations to $k > 2$ were introduced in [85]. The system contains a single quasihole $\phi$, with charge $1/(kM+2)$ and an electron operator $\Psi$ with charge $-1$. At the $\mathfrak{su}(2)_k$-point (i.e. $M = 0$) the quasihole operator $\phi$ has $su(2)$-weight $\alpha/2$, where $\alpha$ is the (positive) root of $su(2)$ and corresponds to one component of the chiral vertex operator transforming in the spin $1/2$ representation ('spinon', see [52, 15, 19, 20]), while the electron operator has weight $-\alpha$ and corresponds to the current $J_{-\alpha}$. For general $M$ the charge lattice has to be stretched.

The fqH data $(\mathbb{K}_e, \mathbf{t}_e)$ and their duals $(\mathbb{K}_\phi, \mathbf{t}_\phi)$ for $k = 1$ (corresponding to the abelian spin-polarized Laughlin states with $\nu = 1/(M+2)$ [67]) were discussed in [30] and for $k = 2$ in [6]. Here we discuss the generalization (see also [48]) to arbitrary $k$, corresponding to the Read-Rezayi states [85]. But we will first discuss the Moore-Read states corresponding to $k = 2$.

### 5.1.1 The Moore-Read case

As indicated before, we analyze the conformal field theory $\mathfrak{su}(2)_{k,M}$ by first analyzing the affine Lie algebra point $M = 0$ and subsequently applying the shift map to obtain the result for general $M$. Also, we will study the quasihole sector for the Moore-Read state first, as this is the simplest state with non-abelian statistics, while it shows all the basic features related to non-abelian statistics. This case was first studied in [91]. As indicated above, the quasihole transforms in the spinon representation of $\mathfrak{su}(2)_2$ (in the case of the Moore-Read state). Also, it is well known that the spinons corresponding to $\mathfrak{su}(2)_2$ have the same fusion rules as the spin field present in the quasihole operator. It is thus to be expected that the K-matrix describing the quasihole sector contains the same pseudoparticles which take care of the non-abelian statistics. Anticipating on the fact that we want to split the K-matrix in a quasihole and electron sector, we only take the positively charged quasihole and arrive at the following K-matrix and charge vector for the quasihole sector of the Moore-Read state with $M = 0$

$$\mathbb{K}_\phi^{\mathrm{MR},M=0} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \,, \qquad\qquad \mathbf{t}_\phi = (0, \frac{1}{2}) \,. \tag{5.2}$$

The first particle, which does not carry charge, is interpreted as a pseudoparticle. It is the presence of this pseudoparticle which takes care of the non-abelian statistics of the quasiholes over the Moore-Read state. To find the quantum Hall data for general $M$, we can simply apply the shift map (4.34) to find

$$\mathbb{K}_\phi^{\mathrm{MR}} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{M+2}{4(M+1)} \end{pmatrix} \,, \qquad\qquad \mathbf{t}_\phi = (0, \frac{1}{2(M+1)}) \,. \tag{5.3}$$

In the case of the MR state, we can rather easy obtain the quantum Hall data for the electron sector, because the parafermion field present in the electron operator is the Majorana fermion. The exclusion statistics corresponding to this field is equivalent to the Pauli exclusion principle, which in the language of exclusion statistics matrices corresponds to 1. The pairing structure in the Moore-Read state is taken into account in the K-matrix formalism by using both the electron and the composite 'consisting of two electrons' in the electron sector. The corresponding operators are given by $\psi : e^{i\sqrt{M+1}\varphi_c}$ : and : $e^{i2\sqrt{M+1}\varphi_c}$. Because of this structure, the K-matrix can be obtained as is the case for abelian states, with the addition of 1 for the electron, due to the Majorana fermion. The corresponding quantum Hall data is given thus given by

$$\mathbb{K}_e^{\mathrm{MR}} = \begin{pmatrix} M+2 & 2M+2 \\ 2M+2 & 4M+4 \end{pmatrix} , \qquad \mathbf{t}_e = -(1,2) . \tag{5.4}$$

It is easily verified that the data for the quasihole and electron sector satisfy the duality equation (4.30). Also, the filling fraction ($\nu = \frac{1}{M+1}$) is correctly reproduced by the equations (4.25) and (4.26). The central charge corresponding to this system is also easily obtained by the application of equation (4.22), and the fact that the pseudoparticle gives rise to a reduction of the central charge of $\frac{1}{2}$. Thus we correctly obtain the central charge $\frac{3}{2}$ of the $\mathfrak{su}(2)_2$ affine Lie algebra CFT.

As indicated before, the presence of the pairing structure in the Moore-Read state gives rise to the possibility of quasiholes with non-abelian statistics. This structure is also present in the K-matrix description described above, because the presence of the composite with charge $-2$ gives rise to a neutral particle in the quasihole sector. It is this neutral particle which is interpreted as pseudoparticle, which via the coupling to the quasihole makes the latter non-abelian.

Other properties of the K-matrix description can be obtained from the K-matrices for the Read-Rezayi states by setting $k = 2$ in the remainder of this section, where we treat the Read-Rezayi states.

### 5.1.2 The Read-Rezayi states

The exclusion statistics and UCPF for the doublet of spinon operators in $\mathfrak{su}(2)_k$ were studied in [20, 33, 48, 17]. It turns out that in this case we need $k-1$ additional charge- and spin neutral pseudoparticles. As was the case for the spin field corresponding to the Majorana fermion, the fusion rules of the spin fields corresponding to the $\mathbb{Z}_k$ parafermions are identical to those of the spinons mentioned above. Thus the quasihole in the K-matrix couples to

$k-1$ pseudoparticles in the same way as the spinons (see, in particular, [48, 17])

$$\mathbb{K}_\phi = \begin{pmatrix} 1 & -\frac{1}{2} & & & & \vdots & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \vdots & \\ & \ddots & \ddots & \ddots & & \vdots & \\ & & -\frac{1}{2} & 1 & -\frac{1}{2} & \vdots & \\ & & & -\frac{1}{2} & 1 & \vdots & -\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & -\frac{1}{2} & \vdots & \frac{1}{2} \end{pmatrix} , \qquad \mathbf{t}_\phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{pmatrix} , \tag{5.5}$$

leading, with (4.26), to a filling factor of $\nu = k/2$ in accordance with (5.1).

The data for arbitrary $M$ now follow by applying the shift map $\mathcal{S}_M$ of (4.34), i.e.

$$\mathbb{K}_\phi^M = \mathcal{S}_M \mathbb{K}_\phi = \begin{pmatrix} & & \vdots & \\ & \frac{1}{2}\mathbb{A}_{k-1} & \vdots & \\ & & \vdots & -\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots \\ & & -\frac{1}{2} \ \vdots & \frac{(k-1)M+2}{2(kM+2)} \end{pmatrix} , \quad \mathbf{t}_\phi^M = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{kM+2} \end{pmatrix} , \tag{5.6}$$

where, in order to simplify the notation, we have introduced the Cartan matrix $\mathbb{A}_{k-1}$ of $su(k)$ (cf. (6.36)). One verifies that (4.26) is satisfied. The IOW-equations, determining the exclusion statistics of the quasiholes, can now be written down explicitly. E.g., for the MR state ($k = 2$) the following equation for $\lambda_{\text{tot}}$ easily follows from (4.4), in agreement with [91]

$$(\lambda_{\text{tot}} - 1)(\lambda_{\text{tot}}^{\frac{1}{2}} - 1) = x^2 \lambda_{\text{tot}}^{\frac{3M+2}{2(M+1)}} . \tag{5.7}$$

The small $x$ behaviour of $\lambda_{\text{tot}}$ for general $k$ was obtained from the IOW-equations in [22], with the result

$$\lambda_{\text{tot}}(x) = 1 + \alpha_k x + o(x^2) , \qquad \alpha_k = 2\cos\left(\frac{\pi}{k+2}\right) . \tag{5.8}$$

It was argued that the factors $\alpha$ can also be obtained as quantum dimension of the appropriate CFT. It is easily checked that the small $x$ behaviour of $\lambda_{\text{tot}}$ in (5.7) indeed satisfies (5.8) for $k = 2$. Similar equations for $\lambda_{\text{tot}}$ with $k = 3, 4$ were given in [22].

To determine the fqH data $(\mathbb{K}_e, \mathbf{t}_e)$ in the electron sector we observe that the electron operator $\Psi(z)$ is identified with $J_{-\alpha}(z)$. By acting with the negative modes of $J_{-\alpha}(z)$ on the lowest weight vector in the lowest energy sector of some integrable highest weight module $L(\Lambda)$ at level $k$, one obtains what is known as the principal subspace $W(\Lambda)$ of $L(\Lambda)$ (or, rather, the reflected principal subspace). It is known that the character of the principal subspace can be written in the UCPF form [31, 32, 42] (see appendix B of [6] for a brief

summary of the results for $\mathfrak{su}(n)_k$). For $\mathfrak{su}(2)_k$ this requires, besides the electron operator $\Psi$ itself, clusters of up to $k$ electron operators. The corresponding K-matrix is given by the $k \times k$ matrix $\mathbb{K}_e = 2\,\mathbb{B}_k$ where $(\mathbb{B}_k)_{ij} = \min(i,j)$

$$\mathbb{B}_k = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & & 2 \\ 1 & 2 & 3 & & 3 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & k \end{pmatrix} , \tag{5.9}$$

while $\mathbf{t}_e = -(1,2,\ldots,k)$. Applying the shift map (4.33) thus gives

$$\mathbb{K}_e^M = \begin{pmatrix} M+2 & 2M+2 & \ldots & kM+2 \\ 2M+2 & 2(2M+2) & \ldots & 2(kM+2) \\ \vdots & \vdots & \ddots & \vdots \\ kM+2 & 2(kM+2) & \ldots & k(kM+2) \end{pmatrix} , \qquad \mathbf{t}_e^M = -\begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix} . \tag{5.10}$$

One easily verifies that the data $(\mathbb{K}_\phi, \mathbf{t}_\phi)$ and $(\mathbb{K}_e, \mathbf{t}_e)$ are indeed related by the duality relations (4.30), and that equations (4.25) and (4.26) are satisfied.

Moreover, the resulting IOW-equations for $\mu_{\text{tot}} = \mu_1\mu_2^2$ in case of the MR state are given by

$$(\mu_{\text{tot}}^{2(M+1)} - y^2)(\mu_{\text{tot}}^{M+1} - y) = \mu_{\text{tot}}^{3M+2} , \tag{5.11}$$

which are indeed related to (5.7) by the duality relations (4.18). Explicitly,

$$\lambda_{\text{tot}}(x) = y^{-2}\mu_{\text{tot}}^{2(M+1)}(y) , \qquad\qquad y = x^{-2(M+1)} . \tag{5.12}$$

Finally, in order to show that the quasihole-electron system based on $\mathbb{K} = \mathbb{K}_\phi^M \oplus \mathbb{K}_e^M$ gives a complete description of the $\mathfrak{su}(2)_{k,M}$ conformal field theory, we have to show that the chiral character of the latter can be written in terms of a (finite) combination of UCPF characters based on $\mathbb{K}_\phi^M \oplus \mathbb{K}_e^M$. This is indeed possible and discussed in appendix C of [6]. Here we suffice to remark that the central charge, related to the asymptotic behaviour of the characters, works out correctly. Indeed, using standard dilogarithm identities one finds with (4.22)

$$c_\phi + c_e = \frac{3k}{k+2} , \tag{5.13}$$

which equals the central charge of $\mathfrak{su}(2)_{k,M}$.

The above description of the Read-Rezayi states has an interesting application, namely the identification of a particle which acts as a supercurrent in the non-magnetic limit. This identification was made in [6], to which we refer for a more detailed discussion. We use the variable $q = 1/\nu = M + k/2$, in terms of which the non-magnetic limit corresponds to $q \to 0$. In this limit, all the statistics parameters of the largest composite (with charge $-k$), go to zero, while the statistics parameters of the quasihole diverge. This is easily seen when one writes the statistic matrices (5.10) and (5.6) in terms of $q$. For these quantum Hall states the fundamental flux quantum is $h/ke$, because of the order-$k$ clustering. Upon piercing

a quantum Hall state with this amount of flux, a quasihole with charge $e/kq$ is formed. This follows from the fact that the conductance is $e^2/qh$ in physical units. For $q \geq 1/k$ this is the lowest charge possible and the electron like excitations correspond to multiple insertions of the flux quantum. This situation changes when we take the limit $q \to 0$. Following [6], we take $q = 1/N$, with $N$ a large integer. The largest composite is formed by inserting an amount of flux $-qkh/e = -kh/Ne$, thus a fraction of the flux quantum. The maximal occupation with this particle (in absence of other particles) is $n_{\max} = 1/k^2 q = N/k^2$. Thus the maximal amount of flux that can be screened by this type of composites is $(-kh/Ne)(N/k^2) = -h/ke$, which is precisely the flux quantum. In conclusion we find that in the non-magnetic limit, the largest composite has bosonic statistics, and can screen an amount of flux up to the flux quantum. This clearly resembles the behaviour of the supercurrent in BCS superconductors.

## 5.2 Spin-singlet clustered states: $\mathfrak{su}(3)_{k,M}$

In [10] a family of non-abelian spin-singlet (NASS) states $\Psi_{k,M}$ trial wave functions with filling factors

$$\nu_{k,M} = \frac{2k}{2kM+3}\,, \qquad\qquad \sigma_{k,M} = 2k\,, \tag{5.14}$$

was constructed. The system has two quasihole excitations $\{\phi_\uparrow, \phi_\downarrow\}$ with one unit of up/down spin and charge $1/(2kM+3)$, while the electron operators $\{\Psi_\uparrow, \Psi_\downarrow\}$ have charge $-1$. The underlying conformal field theory is $\mathfrak{su}(3)_{k,M}$. In terms of $su(3)$ weights the spin and charge assignment in the $M = 0$ case is as follows. Denote the positive simple roots of $su(3)$ by $\alpha_i$, $i = 1, 2$ and the remaining positive non-simple root by $\alpha_3 = \alpha_1 + \alpha_2$. Let $\epsilon_i$, $i = 1, 2, 3$, denote the weights of the fundamental three dimensional irreducible representation **3** of $su(3)$ such that $\epsilon_i \cdot \epsilon_j = \delta_{ij} - 1/3$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, 2$, then $\{\phi_\uparrow, \phi_\downarrow\} = \{\phi^{\epsilon_1}, \phi^{\epsilon_2}\}$ while $\{\Psi_\uparrow, \Psi_\downarrow\} = \{J_{-\alpha_2}, J_{-\alpha_3}\}$ (see figure 5.1). The charge and spin direction are identified in the $su(3)$ weight diagram as indicated in the figure. For other $M$ the analogous picture is obtained by 'stretching' the charge axis. Unfortunately, the notation used in [10, 9] is different from the one used in the mathematics literature. In effect, the difference between the two notations is a rotation of the root diagram of $su(3)$ by $\frac{\pi}{3}$, which leaves this diagram and hence all the results invariant.

In the following sections we will analyze the fqH data for the conformal field theory $\mathfrak{su}(3)_{k,M}$. We first discuss the case $k = 1$ (which corresponds to the abelian spin-singlet Halperin state with parameters $(M+2, M+2, M+1)$ [55]) in some detail and then generalize to the non-abelian case $k > 1$.

### 5.2.1 $\mathfrak{su}(3)_{k=1,M}$

The exclusion statistics and UCPF character for the $\mathfrak{su}(3)_{k=1,M=0}$ conformal field theory, in terms of the quasiparticles $\{\phi^{\epsilon_1}, \phi^{\epsilon_2}, \phi^{\epsilon_3}\}$, were worked out in [21, 90, 22, 17]. Specializing

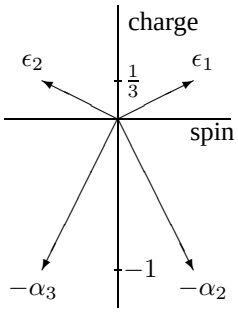


Figure 5.1: The weight diagram of $\mathfrak{su}(3)$.

to the subset $\{\phi_\uparrow, \phi_\downarrow\} = \{\phi^{\epsilon_1}, \phi^{\epsilon_2}\}$ we have

$$\mathbb{K}_\phi = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \mathbf{t}_\phi = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \qquad \mathbf{s}_\phi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.15}$$

With (4.26) this leads to $\nu = 2/3$ in agreement with (5.14). Applying the shift map (4.34), the fqH data for $\mathfrak{su}(3)_{k=1,M}$ are thus given by

$$\mathbb{K}_\phi^M = \mathcal{S}_M \mathbb{K}_\phi = \frac{1}{2M+3}\begin{pmatrix} M+2 & -(M+1) \\ -(M+1) & M+2 \end{pmatrix}, \tag{5.16}$$

while

$$\mathbf{t}_\phi^M = \begin{pmatrix} \frac{1}{2M+3} \\ \frac{1}{2M+3} \end{pmatrix}, \qquad \mathbf{s}_\phi^M = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.17}$$

The IOW-equation for the total one-particle partition function $\lambda_{\text{tot}} = \lambda_\uparrow \lambda_\downarrow$, resulting from (5.16), is given by

$$\lambda_{\text{tot}} - x_\uparrow x_\downarrow \lambda_{\text{tot}}^{\frac{2M+2}{2M+3}} - (x_\uparrow + x_\downarrow)\lambda_{\text{tot}}^{\frac{M+1}{2M+3}} - 1 = 0\,. \tag{5.18}$$

The K-matrix in the electron sector is determined as follows. First of all, the principal subspace of the $\mathfrak{su}(3)_{k=1,M=0}$ integrable highest weight modules is generated by $\{J_{-\alpha_1}, J_{-\alpha_2}\}$ and has a K-matrix given by (see appendix B of [6])

$$\mathbb{K} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{5.19}$$

The electron operators $\{\Psi_\uparrow, \Psi_\downarrow\}$, however, are identified with $\{J_{-\alpha_2}, J_{-\alpha_3}\}$. Interpreting $J_{-\alpha_3}$ as the composite $(J_{-\alpha_1} J_{-\alpha_2})$, we can apply the construction of section 4.2.4 and find an equivalent K-matrix for the combined $\{J_{-\alpha_1}, J_{-\alpha_2}, J_{-\alpha_3}\}$ system

$$\mathbb{K}' = \mathcal{C}_{12}\mathbb{K} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \tag{5.20}$$

Thus, we conclude that the electron fqH data are given by

$$\mathbb{K}_e = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} , \qquad \mathbf{t}_e = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} , \qquad \mathbf{s}_e = \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{5.21}$$

And thus, by applying the shift map

$$\mathbb{K}_e^M = \mathcal{S}_M \mathbb{K}_e = \begin{pmatrix} M+2 & M+1 \\ M+1 & M+2 \end{pmatrix} , \qquad \mathbf{t}_e^M = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{5.22}$$

Note again that the fqH data in the electron and quasihole sectors, given in eqs. (5.16), (5.17) and (5.22), are related by the duality (4.30).

The IOW-equation for $\mu_{\text{tot}} = \mu_\uparrow \mu_\downarrow$, resulting from (5.22), is given by

$$\mu_{\text{tot}}^{2M+3} - \mu_{\text{tot}}^{2M+2} - (y_\uparrow + y_\downarrow)\mu_{\text{tot}}^{M+1} - y_\uparrow y_\downarrow = 0 , \tag{5.23}$$

and is dual to (5.18) in the sense of (4.18). Explicitly,

$$\lambda_{\text{tot}}(x_\uparrow, x_\downarrow) = (y_\uparrow y_\downarrow)^{-1} \mu_{\text{tot}}(y_\uparrow, y_\downarrow)^{2M+3} , \tag{5.24}$$

where

$$y_\uparrow = x_\uparrow^{-(M+2)} x_\downarrow^{-(M+1)} , \qquad y_\downarrow = x_\uparrow^{-(M+1)} x_\downarrow^{-(M+2)} . \tag{5.25}$$

It remains to show that the fqH data $(\mathbb{K}_\phi, \mathbf{t}_\phi, \mathbf{s}_\phi)$ and their duals $(\mathbb{K}_e, \mathbf{t}_e, \mathbf{s}_e)$ give a complete description of the chiral spectrum of the $\mathfrak{su}(3)_{k=1,M}$ conformal field theory by constructing the $\mathfrak{su}(3)_{k=1,M}$ characters in terms of (finite) linear combinations of UCPFs based on $\mathbb{K}_e \oplus \mathbb{K}_\phi$. This is shown in appendix D of [6]. Here we only observe that, since there are no pseudoparticles, eq. (4.15) immediately gives $c_e + c_\phi = 2$ which is the correct value of the central charge for $\mathfrak{su}(3)_{k=1,M}$. Note also that $c_\phi$ and $c_e$ separately depend on $M$ and are, in general, not simple rational numbers, e.g., for $M = 0$ we have numerically $c_e = 0.6887$ and $c_\phi = 1.3113$ while for $M \to \infty$ all the central charge is concentrated in the $\phi$-sector.

Upon generalizing to higher levels $k > 1$, it turns out we need an equivalent description of the system described above in terms of three quasihole operators, namely by adding a quasihole operator $\phi^{-\epsilon_3}$ of $su(3)$ weight $-\epsilon_3$, i.e., of charge $2/3$ (for $M = 0$) and spinless. The K-matrix for this system can be obtained as a submatrix of the K-matrix describing quasiparticles in the $\mathbf{3} \oplus \mathbf{3}^*$ of $su(3)$ [17] or, equivalently, by using that $\phi^{-\epsilon_3}$ is the composite $(\phi^{-\epsilon_1}\phi^{-\epsilon_2})$ [21] and using eq. (4.37). We find

$$\mathbb{K}_\phi^{\prime M} = \mathcal{C}_{12} \mathbb{K}_\phi^M = \frac{1}{2M+3} \begin{pmatrix} M+2 & M+2 & 1 \\ M+2 & M+2 & 1 \\ 1 & 1 & 2 \end{pmatrix} , \quad \mathbf{t}_\phi^{\prime M} = \begin{pmatrix} \frac{1}{2M+3} \\ \frac{1}{2M+3} \\ \frac{2}{2M+3} \end{pmatrix} . \tag{5.26}$$

In the electron sector we can similarly introduce the composite $(J_{-\alpha_2} J_{-\alpha_3})$ and obtain

$$\mathbb{K}_e^{\prime M} = \mathcal{C}_{12} \mathbb{K}_e^M = \begin{pmatrix} M+2 & M+2 & 2M+3 \\ M+2 & M+2 & 2M+3 \\ 2M+3 & 2M+3 & 4M+6 \end{pmatrix} , \qquad \mathbf{t}_e' = -\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} . \tag{5.27}$$

Now we observe a curiosity; while obviously the fqH data (5.26) and (5.27) are dual, since they are equivalent to the dual systems given in (5.16) and (5.22), they are not related by the duality transformation (4.30) because both $\mathbb{K}_\phi$ and $\mathbb{K}_e$ are not invertible. The equivalence can also be observed at the level of the resulting IOW-equations which are now given by

$$\begin{aligned}(\lambda_{\rm tot}^{\frac{1}{2M+3}} - x_\uparrow x_\downarrow)(\lambda_{\rm tot} - x_\uparrow x_\downarrow \lambda_{\rm tot}^{\frac{2M+2}{2M+3}} - (x_\uparrow + x_\downarrow)\lambda_{\rm tot}^{\frac{M+1}{2M+3}} - 1) &= 0\,, \\ (\mu_{\rm tot}^{2M+3} - y_\uparrow y_\downarrow)(\mu_{\rm tot}^{2M+3} - \mu_{\rm tot}^{2M+2} - (y_\uparrow + y_\downarrow)\mu_{\rm tot}^{M+1} - y_\uparrow y_\downarrow) &= 0\,. \end{aligned} \tag{5.28}$$

Because of the first factor the equations (5.28) do not transform into each other under (5.24). However, the physical solutions, which are determined by the second factor, do! Summarizing, we conclude that it is obvious that the notion of duality should have an extension that incorporates non-invertible K-matrices. We leave this for future investigation.

### 5.2.2 $\mathfrak{su}(3)_{k,M}$

As argued in [18, 17], the generalization of the results of the previous section to levels $k > 1$ requires the addition of $2(k-1)$ pseudoparticles incorporating the non-abelian statistics of the quasihole operators $\{\phi_\uparrow, \phi_\downarrow\}$. Since these pseudoparticles couple differently to $\{\phi_\uparrow, \phi_\downarrow\}$ than to the composite particle $\phi_{\uparrow\downarrow} = (\phi_\uparrow \phi_\downarrow)$ (i.e., different than the naive coupling given by the composite construction), it appears that the first construction in Sect. 7.1 does not generalize to higher levels.

It is known that for $\mathfrak{su}(n)_{k,M=0}$ the pseudoparticles couple to the physical particles by means of the matrix $\mathbb{A}_{n-1}^{-1} \otimes \mathbb{A}_{k-1}$. Here we have used the result for the restricted Kostka polynomials as given in, e.g., [14, 25, 63, 57] (see the discussion in [18] for details). Then, by applying the shift map (4.34), we obtain

$$\mathbb{K}_\phi'^M = \begin{pmatrix} & & \vdots & & & \\ \mathbb{A}_2^{-1} \otimes \mathbb{A}_{k-1} & & \vdots & -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ & & \vdots & -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{2}{3} & -\frac{1}{3} & \vdots & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{(2k-2)M+3}{3(2kM+3)} \\ -\frac{2}{3} & -\frac{1}{3} & \vdots & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{(2k-2)M+3}{3(2kM+3)} \\ -\frac{1}{3} & -\frac{2}{3} & \vdots & \frac{(2k-2)M+3}{3(2kM+3)} & \frac{(2k-2)M+3}{3(2kM+3)} & \frac{(4k-4)M+6}{3(2kM+3)} \end{pmatrix}, \tag{5.29}$$

where the components of $\mathbb{A}_2^{-1}$ refer to the quasiholes in the $\mathbf{3}$ and $\mathbf{3}^*$, respectively, and

$$\mathbf{t}_\phi = (\underbrace{0, 0, \ldots, 0}_{2(k-1)} | \frac{1}{2kM+3}, \frac{1}{2kM+3}, \frac{2}{2kM+3})\,. \tag{5.30}$$

For instance, for level $k = 2$ we have

$$\mathbb{K}_\phi'^M = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} & \vdots & -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{4}{3} & \vdots & -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{2}{3} & -\frac{1}{3} & \vdots & \frac{7M+6}{12M+9} & \frac{7M+6}{12M+9} & \frac{2M+3}{12M+9} \\ -\frac{2}{3} & -\frac{1}{3} & \vdots & \frac{7M+6}{12M+9} & \frac{7M+6}{12M+9} & \frac{2M+3}{12M+9} \\ -\frac{1}{3} & -\frac{2}{3} & \vdots & \frac{2M+3}{12M+9} & \frac{2M+3}{12M+9} & \frac{4M+6}{12M+9} \end{pmatrix} . \tag{5.31}$$

Note that the matrix $\mathbb{K}_\phi'^M$ of (5.29) is not invertible, as was observed for $k = 1$ in section 5.2.1. Thus, we cannot simply identify the dual sector by performing the transformation (4.30).

To obtain the dual sector we proceed as in section 5.2.1. We start with the K-matrix of the principal subspace spanned by $\{J_{-\alpha_1}, J_{-\alpha_2}\}$. As discussed in appendix B of [6], for $\mathfrak{su}(3)_k$, this K-matrix is given by $\mathbb{K} = \mathbb{A}_2 \otimes \mathbb{B}_k$ and requires, besides the currents $\{J_{-\alpha_1}, J_{-\alpha_2}\}$ a set of $2(k-1)$ composites

$$(\underbrace{J_{-\alpha_i} \dots J_{-\alpha_i}}_{l}) , \qquad 2 \le l \le k \, , i = 1, 2 \, . \tag{5.32}$$

Starting with this matrix we introduce additional composites according to the procedure of section 4.2.4, beginning with the electron operator $\Psi_\downarrow = (J_{-\alpha_1} J_{-\alpha_2})$ (recall that $\Psi_\uparrow = J_{-\alpha_2}$), and continuing until all composites

$$(\underbrace{\Psi_\uparrow \dots \Psi_\uparrow}_{n_\uparrow} \underbrace{\Psi_\downarrow \dots \Psi_\downarrow}_{n_\downarrow}) , \qquad n_\uparrow + n_\downarrow \le k \, , \tag{5.33}$$

have been introduced. Note that the set of composites (5.33), for fixed $n_\uparrow + n_\downarrow$, span a $(n_\uparrow + n_\downarrow + 1)$-dimensional irreducible representation of spin $SU(2)$. The electron K-matrix is then the $\frac{1}{2}k(k+3) \times \frac{1}{2}k(k+3)$ submatrix of the resulting $\mathbb{K}$ obtained by omitting the composites which cannot be written in terms of electron operators only. Let us be illustrate this procedure the case of $k = 2$. Starting with the principal subspace K-matrix

$$\mathbb{K} = \begin{pmatrix} 2 & -1 & \vdots & 2 & -1 \\ -1 & 2 & \vdots & -1 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & -1 & \vdots & 4 & -2 \\ -1 & 2 & \vdots & -2 & 4 \end{pmatrix} , \tag{5.34}$$

we introduce the composites $\Psi_\downarrow = (J_{-\alpha_1} J_{-\alpha_2})$, $(J_{-\alpha_2}(J_{-\alpha_1} J_{-\alpha_1}))$, $(J_{-\alpha_2}(J_{-\alpha_1} J_{-\alpha_2}))$, and $(J_{-\alpha_2}((J_{-\alpha_2}(J_{-\alpha_1} J_{-\alpha_1}))))$, respectively. Then, after removing the rows and columns

corresponding to $J_{-\alpha_1}$, $(J_{-\alpha_1}J_{-\alpha_1})$ and $(J_{-\alpha_2}(J_{-\alpha_1}J_{-\alpha_1}))$, we obtain

$$\mathbb{K}'_e = \begin{pmatrix} 2 & 1 & \vdots & 2 & 2 & 1 \\ 1 & 2 & \vdots & 1 & 2 & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 1 & \vdots & 4 & 3 & 2 \\ 2 & 2 & \vdots & 3 & 4 & 3 \\ 1 & 2 & \vdots & 2 & 3 & 4 \end{pmatrix}, \qquad \mathbf{t}_e = -\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \qquad \mathbf{s}_e = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ -2 \end{pmatrix}. \tag{5.35}$$

Similarly, one obtains the electron K-matrix for $\mathfrak{su}(3)_{k,M=0}$ at higher levels, and the generalization to arbitrary $M$ follows, as before, by applying the shift map (4.33). Unfortunately, the procedure described above is ambiguous. The resulting K-matrix depends on the order in which the composites are taken as well as the precise identification of the clusters (5.33) with the original clusters (5.32), e.g., should we identify $(\Psi_\downarrow\Psi_\downarrow)$ with $(J_{-\alpha_1}(J_{-\alpha_1}(J_{-\alpha_2}J_{-\alpha_2})))$ or $((J_{-\alpha_1}J_{-\alpha_1})(J_{-\alpha_2}J_{-\alpha_2}))$? Ultimately, the 'correct' matrix $\mathbb{K}_e$ is selected by the requirement that the complete spectrum can be build out of the quasihole and electron operators or, more concretely, that the characters of $\mathfrak{su}(3)_{k,M}$ can be written as a linear combination of UCPFs based on $\mathbb{K}_\phi \oplus \mathbb{K}_e$. A nontrivial (and highly selective) check is whether the central charge, given by (4.22), works out correctly, i.e., whether $c_\phi + c_e = 8k/(k+3)$, for the K-matrices (5.29) and the 'appropriate' generalization of (5.35) to higher levels and arbitrary $M$. We have checked this numerically for low values of $k$ and $M$ as well as exactly, for all $k$, in the $M \to \infty$ limit, in which case the central charge is entirely concentrated in the $\phi$-sector. We refrain from giving the explicit matrices $\mathbb{K}_e$ until we have performed an additional simplifying reduction.

First observe that, for $k = 2$, the matrix $\mathbb{K}'_e$ of eq. (5.35) is invertible, in contrast to the matrix $\mathbb{K}'^M_\phi$ of (5.31). One could therefore simply have started with $\mathbb{K}'_e$ and have obtained the dual sector by the duality transformations (4.30). This would result in a $\phi$-sector, different from the one discussed above, with two physical quasiholes and three pseudoparticles. Unfortunately, this procedure breaks down, in general, for higher $k$ as the matrices $\mathbb{K}_e$, constructed according to the procedure outlined above, are no longer invertible. However, note that the matrix (5.35) can be reduced to an equivalent $4 \times 4$ matrix by inverting the composite procedure – in this case by removing $(\Psi_\uparrow\Psi_\downarrow)$ in the fourth column, since this column can be created by applying $\mathcal{C}_{12}$. This procedure works for general $k > 1$ and leads to a $2k \times 2k$ electron K-matrix, for the composites (5.32) with either $n_\downarrow = 0$ or $n_\uparrow = 0$ (i.e.

we lose the SU(2) multiplet structure), given by

$$\mathbb{K}_e = \begin{pmatrix} 2 & 0 & 2 & 0 & \cdots & 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 & \cdots & 0 & 2 & 1 & 2 \\ 2 & 0 & 4 & 0 & & 4 & 1 & 4 & 2 \\ 0 & 2 & 0 & 4 & & 1 & 4 & 2 & 4 \\ \vdots & \vdots & & & & & & \vdots & \vdots \\ 2 & 0 & 4 & 1 & & 2(k-1) & k-2 & 2(k-1) & k-1 \\ 0 & 2 & 1 & 4 & & k-2 & 2(k-1) & k-1 & 2(k-1) \\ 2 & 1 & 4 & 2 & \cdots & 2(k-1) & k-1 & 2k & k \\ 1 & 2 & 2 & 4 & \cdots & k-1 & 2(k-1) & k & 2k \end{pmatrix} , \quad (5.36)$$

and

$$\begin{aligned} \mathbf{t}_e &= -(1,1;2,2;\ldots;k,k) \, , \\ \mathbf{s}_e &= (1,-1;2,-2;\ldots;k,-k) \, . \end{aligned} \quad (5.37)$$

The generalization $\mathbb{K}_e^M$ to arbitrary $M$ follows by applying the shift map, in this case by adding the matrix $M(\mathbb{J}_2 \otimes \mathbb{P})$ where $\mathbb{J}_2$ is the $2 \times 2$ matrix with all entries equal to 1, and $(\mathbb{P})_{ij} = ij$ $(1 \leq i, j \leq k)$ (see [6] for an explicit expression in the case $k = 2$). This matrix is invertible, so we simply define $\mathbb{K}_\phi^M = (\mathbb{K}_e^M)^{-1}$. A convenient permutation of rows and columns of $\mathbb{K}_\phi^M$ leads to the following matrix

$$(\mathbb{K}_\phi^M)^{\text{perm}} = \begin{pmatrix} & & & & & & & \vdots & 0 & -\frac{1}{3} \\ & & & & & & & \vdots & 0 & -\frac{2}{3} \\ & & & & & & & \vdots & 0 & 0 \\ & & & \mathbb{A}_2^{-1} \otimes \mathbb{A}_{k-1} & & & & \vdots & \vdots & \vdots \\ & & & & & & & \vdots & 0 & 0 \\ & & & & & & & \vdots & -\frac{2}{3} & 0 \\ & & & & & & & \vdots & -\frac{1}{3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & -\frac{2}{3} & -\frac{1}{3} & \vdots & \frac{(4k-1)M+6}{3(2kM+3)} & \frac{-M}{3(2kM+3)} \\ -\frac{1}{3} & -\frac{2}{3} & 0 & \cdots & 0 & 0 & 0 & \vdots & \frac{-M}{3(2kM+3)} & \frac{(4k-1)M+6}{3(2kM+3)} \end{pmatrix} , \quad (5.38)$$

containing two physical particles and $2(k-1)$ pseudoparticles. Also,

$$\begin{aligned} \mathbf{t}_\phi &= (0,0,\ldots,0;\frac{1}{2kM+3},\frac{1}{2kM+3}) \, , \\ \mathbf{s}_\phi &= (0,0,\ldots,0;-1,1) \, , \end{aligned} \quad (5.39)$$

as one would expect. For $k = 2$, the couplings between the pseudoparticles and the real particles is not precisely of the form as given in eq. (5.38), so we give this matrix explicitly (the first 2 particles are pseudoparticles)

$$\mathbb{K}_{\phi}^{k=2,M} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{7M+6}{12M+9} & \frac{-M}{12M+9} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{-M}{12M+9} & \frac{7M+6}{12M+9} \end{pmatrix} . \tag{5.40}$$

We have checked that the total central charge $c_e + c_\phi$ for eqs. (5.36) and (5.38) works out correctly, namely $c_e + c_\phi = 8k/(k+3)$. Moreover, we have checked for low values of $k$ that the equation for $\lambda_{\text{tot}}$, resulting from the IOW equations based on (5.38), are identical to those based on (5.29). Furthermore, in all formulations, the equations (4.25) and (4.26) are consistent with (5.14).

For $k = 2, 3$, we checked the small $x$ behaviour for $\lambda_{\text{tot}}$, eq. (4.20). We again expect the constants $\alpha$ to be the quantum dimensions of the associated conformal field theory. Using some results in [34], these quantum dimensions are given by

$$\alpha_k = 1 + 2\cos\left(\frac{2\pi}{k+3}\right) . \tag{5.41}$$

For $k = 2$, the equation for $\lambda_{\text{tot}}$ reads (upon taking $x_\uparrow = x_\downarrow = x$)

$$(\lambda_{\text{tot}}^{\frac{1}{2}} - 1)^2 = x^2 \lambda_{\text{tot}}^{\frac{8M+5}{8M+6}} + x\lambda_{\text{tot}}^{\frac{6M+4}{8M+6}} - x\lambda_{\text{tot}}^{\frac{2M+1}{8M+6}} , \tag{5.42}$$

which leads to the following small $x$ behaviour

$$\lambda_{\text{tot}} = 1 + 2\left(\frac{1+\sqrt{5}}{2}\right) x + o(x^2) , \tag{5.43}$$

in agreement with $\alpha_2 = (1+\sqrt{5})/2$ from (5.41); the extra factor 2 comes from the sum over the two physical particles, see eq. (4.20). For $k = 3$ we find

$$(\lambda_{\text{tot}}^{\frac{1}{2}} - 1) = x\lambda_{\text{tot}}^{\frac{8M+3}{6(6M+3)}} (\lambda_{\text{tot}}^{\frac{1}{6}} + 1)^{\frac{1}{3}} (\lambda_{\text{tot}}^{\frac{1}{3}} + 1)^{\frac{2}{3}} , \tag{5.44}$$

which gives $\alpha_3 = 2$, consistent with (5.41). Note that for the abelian case $k = 1$, we find for the small $x_{\uparrow,\downarrow}$-behaviour, using (5.18),

$$\lambda_{\text{tot}} = 1 + (x_\uparrow + x_\downarrow) + o(x^2) , \tag{5.45}$$

in agreement with (5.41) and the fact that for $k = 1$ we have an abelian state.

As was the case for the spin-polarized states of section 5.1, also for the non-abelian spin-singlet states a particle behaving as a supercurrent can be identified in the non-magnetic limit. The situation here is slightly more complicated than in the case of the spin-polarized states discussed in section 5.1. This is because in the formulation above, there is no candidate particle with the property that all the statistics parameters go to zero in the limit $q \to 0$

(with $q = 1/\nu = M + 3/2k$). However, if one acts with $\mathcal{C}_{2k-1,2k}$ on $\mathcal{S}_M\mathbb{K}_e$, with $\mathbb{K}_e$ given by eq. (5.36), one introduces a composite with charge $-2k$ and spin $0$, which has the desired properties. In the $\phi$-sector, the particle content is changed to one quasihole and $2k$ pseudoparticles, of which a few carry spin.

The possibility to introduce a composite with the right properties enables one to repeat the discussion of section 5.1, with the only difference that the flux quantum in this case equals $h/2ke$. So, also in this case, we can identify a supercurrent in the non-magnetic limit.

## 5.3 Alternative construction

In this section, we will show an alternative way of obtaining the K-matrices for the non-abelian quantum Hall states which were discussed in sections 5.1 and 5.2. The wave functions of these non-abelian states could be obtained from abelian *cover* states, via a projection. The idea is to implement this projection on the level of the K-matrices. The K-matrices for the abelian cover states are easily obtained. They just are the direct sum of $k$ copies of the $k = 1, M = 0$ K-matrices. We will take $M = 0$ in this construction, and use the shift map eq. (4.33) in the end. Applying the shift map directly on the direct sum of $k = 1, M = 0$ K-matrices leads to the same result.

The property that discerns the non-abelian K-matrices form the abelian ones is the presence of composites in the electron sector. These composites are accompanied by pseudoparticles in the $\phi$-sector. To obtain the non-abelian K-matrices from the abelian cover states, one has to introduce composites. One can use the composite construction from section 4.2.4. However, this construction is based on a character identity, eq. (4.57), and will lead to an equivalent description. In this description, the electron sector is augmented with a composite, while in the $\phi$-sector, a real quasiparticle is exchanged for two pseudoparticles. Actually, this action of the composite construction on the $\phi$-sector is also related to a character identity, as will be described in the paper [5]. Applying this character identity on the $\phi$-sectors of the UCPF, one finds another UCPF, based on the transformed K-matrix $\mathcal{D}_{ij}\mathbb{K}_\phi$. This matrix can be obtained from the composite construction (4.37)

$$\mathcal{D}_{ij}\mathbb{K} = (\mathcal{C}_{ij}\mathbb{K}^{-1})^{-1} \,, \tag{5.46}$$

where, in addition, the particles $i, j$ in the transformed formulation are pseudoparticles. Performing both composite constructions yields an equivalent description, as these constructions are based on character identities.

A description of the non-abelian system can be obtained by performing a projection on the K-matrices of the abelian states. This projection is done by 'deleting' a particle in both the electron and $\phi$-sector. In the end, this description can be obtained from the abelian cover by simply applying a W-transformation on the electron sector (and the equivalent W-transform on the $\phi$-sector). In addition, some of the particles in the $\phi$-sector have to be interpreted as pseudoparticles, in accordance with the construction outlined above. In all the cases described in this thesis, we will 'add' particles in the electron sector which have the same quantum numbers. In the $\phi$-sector, this will result in particles with all quantum numbers trivial. These particles are interpreted as pseudoparticles.

The W-transformations which are equivalent to the composite construction followed by the reduction of the number of particles all have a similar form. In effect, they 'add' one particle to another, see eq. (5.48). Loosely speaking, they correspond to identity matrices, in which some of the off-diagonal elements have been changed from $0$ to $1$.

Let us clarify the construction by doing a simple example, that is, obtaining the K-matrix description for the Moore-Read state. The abelian cover state has a simple K-matrix for the electron sector

$$\mathbb{K}_e^{\rm cover} = \begin{pmatrix} 2+M & M \\ M & 2+M \end{pmatrix} . \tag{5.47}$$

The W-transformation (see eq. (2.8)) needed to obtain the K-matrix for the MR state is characterized by

$$\mathbb{W} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} . \tag{5.48}$$

Its effect is to make the second particle a composite of the two original particles; the corresponding matrix is

$$\mathbb{K}_e^{\rm mr} = \begin{pmatrix} 2+M & 2+2M \\ 2+2M & 4+4M \end{pmatrix} . \tag{5.49}$$

This is indeed the K-matrix for the MR state (compare eq. (5.10) with $k = 2$). In the $\phi$-sector, one has one pseudoparticle, in accordance with eq. (5.5). Note that this construction does not change the filling fraction.

The construction outlined above also works for the general clustered states of sections 3.4 and 3.5. For the spin-polarized states, the K-matrix of the abelian cover state is simply given by (see also [23])

$$\mathbb{K}_e^{\rm cover} = \begin{pmatrix} M+2 & M & \cdots & M \\ M & M+2 & & \vdots \\ \vdots & & \ddots & M \\ M & \cdots & M & M+2 \end{pmatrix} , \tag{5.50}$$

which is a $k \times k$ dimensional matrix. All the particles have charge $\mathbf{t}_j = -1$ and the filling fraction corresponding to this state indeed is $\nu = \frac{k}{kM+2}$. To obtain the correct K-matrices for the non-abelian states, we have to introduce composites which have 'sizes' ranging from $1$ up to $k$. This is achieved by the W-transformation

$$\mathbb{W} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix} . \tag{5.51}$$

It is easily verified that this transformation indeed reproduced the K-matrix (5.10). Also the $\phi$-sector is obtained correctly, if one interprets the pseudoparticles correctly.

The K-matrices for the spin-singlet analogs can be obtained in a similar way, however, the construction is somewhat more complicated, because the $k = 1$ description already has

two components. This gives more possibilities for the W-transformation, of which only one gives the correct central charge corresponding to the underlying CFT. The other W-transformations correspond to other projections. Whether or not these give rise to quantum Hall states is not clear at the moment. Also, to which CFTs the resulting K-matrices correspond is not clear (if they correspond to a CFT at all). For now, we will use the W-transformations which lead to K-matrices with rational central charge; it turns out to be the case that only one W-transformation leads to a rational central charge, which also corresponds to the central charge of the underlying CFT.

The K-matrix for the abelian cover state (see also [92]), is given by

$$\mathbb{K}_e^{\text{cover}} = \begin{pmatrix} M+2 & M+1 & M & M & \cdots & \cdots & M & M \\ M+1 & M+2 & M & M & & & M & M \\ M & M & M+2 & M+1 & & & & \vdots \\ M & M & M+1 & M+2 & & & & \vdots \\ \vdots & & & & \ddots & & M & M \\ \vdots & & & & & \ddots & M & M \\ M & M & & & M & M & M+2 & M+1 \\ M & M & \cdots & \cdots & M & M & M+1 & M+2 \end{pmatrix}. \tag{5.52}$$

The W-transformation which is needed in the projection, is given by

$$\mathbb{W} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & & & 0 & 1 \\ 1 & 0 & 1 & 0 & \ddots & & & \vdots \\ 0 & 0 & 0 & 1 & & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & & 0 & 0 \\ \vdots & & & \ddots & & \ddots & 0 & 1 \\ 1 & 0 & & & 1 & 0 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.53}$$

Applying this W-transformation on the matrix (5.52), indeed gives the matrix $\mathbb{K}_e$ for the spin-singlet paired states (5.36) for general $M$ (note, that a suitable permutation on the order of the particles is necessary). Also, in the $\phi$-sector, there are $2(k-1)$ pseudoparticles.

Above, we showed that the K-matrices for the non-abelian quantum Hall states could be obtained from a suitably chosen, abelian cover matrix. Though we do not have a proof for this, we will assume that such a construction is also possible for the spin-charge separated states. In the next section, we will obtain a set of matrices, which we believe, are the correct matrices to describe the $\mathfrak{so}(5)_k$ affine Lie algebra theory. We will provide a few non-trivial checks on these matrices to show that they are indeed the correct ones. More on K-matrices of general affine Lie algebra CFTs can be found in the forthcoming article [5].

## 5.4 Spin-charge separated states: $\mathfrak{so}(5)_{k,M}$

In this section, we will report some results on the K-matrix structure of the spin-charge separated states. The results for $k = 1$ are also described in [8]. For the spin-charge separated states, the underlying structure is the $\mathfrak{so}(5)_k$ affine Lie algebra. This is a *non simply-laced* algebra, meaning that the roots come in two different lengths. This makes the analysis of this algebra much harder than the *simply-laced* algebra's. Indeed, we can not rely so much on results in the mathematical literature. However, we will attempt to obtain a K-matrix description, without having the intention to be mathematically rigorous. We will use the observation from the previous section that the K-matrices for the non-abelian states can be obtained from abelian *cover* states. We will merely assume that the results will carry over to the $\mathfrak{so}(5)_k$ case, and justify the procedure afterwards by doing consistency checks.

For both the spin-polarized and spin-singlet states of sections 3.4 and 3.5, the $k = 1$ case corresponds to unpaired, abelian states. For these, the K-matrices were known for a long time. However, $so(5)$ is non simply-laced, and this implies that the state at level $k = 1$ has non-abelian statistics. Thus one expects pseudoparticles in the K-matrix description. This result was indeed obtained in [8]. We will first discuss this situation.

### 5.4.1 $\mathfrak{so}(5)_{k=1,M}$

It is well-known that the spin-polarized MR state is closely related to an abelian states at filling $\nu = \frac{1}{M+1}$: the two-layer $(M+2, M+2, M)$ Halperin state. The transition between these states have been discussed in the literature (see e.g. [58, 74, 82, 106, 23]). The connection on the level of K-matrices was described in the previous section.

To obtain the K-matrix description for the spin-charge separated states (3.36), we would like to have an abelian cover state like in the case of the Moore-Read state. This state was identified in [8], and has filling fraction $\nu = \frac{2}{2M+1}$, equal to the level $k = 1$ spin-charge separated states. The wave function for this two-layer state reads [8]

$$\begin{aligned}\widetilde{\Psi}^M_{2-\text{layer}}(\{z_i^{\uparrow t}, z_i^{\downarrow t}, z_i^{\uparrow b}, z_i^{\downarrow b}\}) = &\prod_{i<j}(z_i^{\uparrow t} - z_j^{\uparrow t})^{M+2} \prod_{i<j}(z_i^{\downarrow t} - z_j^{\downarrow t})^{M+2} \\ &\prod_{i<j}(z_i^{\uparrow b} - z_j^{\uparrow b})^{M+2} \prod_{i<j}(z_i^{\downarrow b} - z_j^{\downarrow b})^{M+2} \prod_{i,j}(z_i^{\uparrow t} - z_j^{\downarrow t})^{M+1} \prod_{i,j}(z_i^{\uparrow b} - z_j^{\downarrow b})^{M+1} \\ &\prod_{i,j}(z_i^{\uparrow t} - z_j^{\uparrow b})^{M} \prod_{i,j}(z_i^{\downarrow t} - z_j^{\downarrow b})^{M} \prod_{i,j}(z_i^{\uparrow t} - z_j^{\downarrow b})^{M-1} \prod_{i,j}(z_i^{\downarrow t} - z_j^{\uparrow b})^{M-1}\,, \end{aligned} \tag{5.54}$$

where the indices $t, b$ refer to the top and bottom layers. This wave function arises as a correlator of two-layer spin full electron operators which, in the case $M = 0$, are the currents of the $\mathfrak{so}(6)_1$ affine Kac-Moody algebra

$$\psi(\uparrow_t) = :e^{i\varphi_l}::e^{\frac{i}{\sqrt{2}}\varphi_s}::e^{\frac{i}{\sqrt{2}}\sqrt{2M+1}\varphi_c}:\,, \tag{5.55}$$

$$\psi(\downarrow_t) = :e^{i\phi_l}::e^{-\frac{i}{\sqrt{2}}\varphi_s}::e^{\frac{i}{\sqrt{2}}\sqrt{2M+1}\varphi_c}:\,, \tag{5.56}$$

$$\psi(\uparrow_b) = :e^{-i\phi_l}::e^{\frac{i}{\sqrt{2}}\varphi_s}::e^{\frac{i}{\sqrt{2}}\sqrt{2M+1}\varphi_c}:\,, \tag{5.57}$$

$$\psi(\downarrow_b) = :e^{-i\phi_l}::e^{-\frac{i}{\sqrt{2}}\varphi_s}::e^{\frac{i}{\sqrt{2}}\sqrt{2M+1}\varphi_c}:\,, \tag{5.58}$$

where $\varphi_l$ is a chiral boson, corresponding to the 'layer' degree of freedom. We expect that the relation between the paired spin-charge separated state (3.36) and the state (5.54) is on the same footing as the relation between the (paired) Moore-Read state and the corresponding 2-layer state. The MR state (with $M = 0$), is obtained from the Halperin $(2, 2, 0)$ state eq. (2.21) by symmetrizing over all the electron coordinates. In this procedure, the layer structure is lost, as one would like.

To obtain the spin-charge separated state from the 2-layer state (5.54), one has to take into account that we are dealing with spin full electrons, and we want to make a spin-singlet state. To obtain a spin-singlet, one has to do perform the following symmetrization procedure (see, for instance, [56]). First, anti-symmetrize over $(z_1^\uparrow, z_1^\downarrow)$, then over $(z_2^\uparrow, z_2^\downarrow)$, etc. After this, the result has to be symmetrized over all the spin up particles, and finally over all the spin down particles. This indeed gives a spin-singlet state, which for 4 particles is indeed the spin-charge separated state. For more particles, this construction does not seem to work. However, we feel that a construction similar to the construction to obtain the clustered states of sections 3.4 and 3.5 should be possible.

Coming back to the K-matrix description of the 2-layer state related to $\mathfrak{so}(6)$, naively, one would write down the ansatz

$$\mathbb{K}_e = \begin{pmatrix} 2+M & 1+M & M & -1+M \\ 1+M & 2+M & -1+M & M \\ M & -1+M & 2+M & 1+M \\ -1+M & M & 1+M & 2+M \end{pmatrix} , \tag{5.59}$$

based on the wave function (5.54). However, this matrix is not invertible. To find an 'invertible' description, we apply the operator $\mathcal{C}_{1,4}$, which adds a composite of the 'first' and 'last' particle. We proceed by removing the spin down particles to find

$$\mathbb{K}'_e = \begin{pmatrix} 2+M & M & 1+2M \\ M & 2+M & 1+2M \\ 1+2M & 1+2M & 2+4M \end{pmatrix} , \qquad \mathbb{K}'_\phi = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1+M}{1+2M} \end{pmatrix} ,$$

$$\mathbf{t}'_e = -(1,1,2) , \qquad \mathbf{t}'_\phi = (0,0,\frac{1}{1+2M}) ,$$

$$\mathbf{s}'_e = (1,1,0) , \qquad \mathbf{s}'_\phi = (-1,-1,1) ,$$

$$\mathbf{l}'_e = (+,-,\cdot) , \qquad \mathbf{l}'_\phi = (\frac{-}{2},\frac{+}{2},\cdot) . \tag{5.60}$$

where we used the standard duality equations to find the quantum Hall data for the $\phi$-sector. The vector $\mathbf{l}$ denotes the 'layer' degree of freedom, and is only used as a bookkeeping device. The fact that the K-matrix in eq. (5.59) is not invertible, is caused by the fact that we were trying to describe the rank 3 algebra $\mathfrak{so}(6)$ by four degrees of freedom. The method to 'cure' this resembles the method used in section 5.2.2.

To obtain the quantum Hall data for the $\mathfrak{so}(5)_{k=1,M}$ states, we need to do a projection, as was the case for the wave functions, which could be obtained form the abelian states by projecting the two layers onto each other (compare with the 'Ho reduction' for the MR state [58]).

On the level of K-matrices, this reduction take the form of introducing a composite in the electron sector, and the introduction of a pseudoparticle in the quasihole sector. To do this projection, we apply the W-transformation

$$\mathbb{W} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.61}$$

on the quantum Hall data (5.60). This results in the following quantum Hall data

$$\begin{aligned} \mathbb{K}_e &= \begin{pmatrix} 2+M & 2+2M & 1+2M \\ 2+2M & 4+4M & 2+4M \\ 1+2M & 2+4M & 2+4M \end{pmatrix} , & \mathbb{K}_\phi &= \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1+M}{1+2M} \end{pmatrix} , \\ \mathbf{t}_e &= -(1,2,2) , & \mathbf{t}_\phi &= (0,0,\frac{1}{1+2M}) , \\ \mathbf{s}_e &= (1,2,0) , & \mathbf{s}_\phi &= (0,-1,1) . \end{aligned} \tag{5.62}$$

In this description, the first quasihole is neutral and spinless, and is interpreted as a pseudoparticle. This results in a reduction of the central charge by $\frac{1}{2}$, because of the fermionic statistics. So the central charge corresponding to the data (5.62) becomes $c = \frac{5}{2}$, which is indeed the central charge of the $\mathfrak{so}(5)_1$ theory. We will make one further change to the data (5.62), which will make the spin-charge separation of the fundamental quasiholes manifest. This can be done by applying another W-transformation (which leads to an equivalent description)

$$\mathbb{W} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} . \tag{5.63}$$

Finally, we obtain the following formulation, which can also be found in [8]

$$\begin{aligned} \mathbb{K}_e &= \begin{pmatrix} M+2 & 1 & 2M+1 \\ 1 & 2 & 0 \\ 2M+1 & 0 & 4M+2 \end{pmatrix} , & \mathbb{K}_\phi &= \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{2M+3}{8M+4} \end{pmatrix} , \\ \mathbf{t}_e &= -(1,0,2) , & \mathbf{t}_\phi &= (0,0,\frac{1}{2M+1}) , \\ \mathbf{s}_e &= (1,2,0) , & \mathbf{s}_\phi &= (0,-1,0) . \end{aligned} \tag{5.64}$$

Also in this formulation, the first particle in the $\phi$-sector is interpreted as a pseudoparticle. This pseudoparticle is similar to the pseudoparticle for the MR state (eq. (5.6) with $k = 2$), which was of course to be expected from the structure of the spin-charge separated state at $k = 1$. The central charge associated to the data (5.64) is also given by $\frac{5}{2}$. Of course, to prove that this description is correct, one should be able to reproduce the affine characters of $\mathfrak{so}(5)_1$ in the UCPF form with the K-matrices. At this point, this check has not been completed. However, we were able to reproduce the restricted Kostka polynomials for $\mathfrak{so}(5)_1$, which is also a highly non-trivial check [5].

### 5.4.2 $\mathfrak{so}(5)_{k,M}$

In this section, we will present the K-matrix description for the spin-charge separated states at general level $k$. Again, the results were not derived as rigorous as was the case for the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ related states. However, the fact that those matrices could be constructed quite naturally by taking the $k = 1$ formulations and taking composites, leads to the believe that a similar construction is possible in this situation. We now take this point of view, and present the results. On these results, we will perform some non-trivial checks, to show that they are indeed correct. More details will be provided in the paper [5].

To construct the K-matrices corresponding to the $\mathfrak{so}(5)_{k,M}$ state, we take $k$ copies of the $M = 0$ formulation for $\mathfrak{so}(5)_{k=1,M=0}$, eq. (5.64) and make the direct sum, resulting in a $(3k, 3k)$ matrix. Making the composites, needed in the level $k$ formulation, is done via the following W-transformation

$$\mathbb{W} = \begin{pmatrix}
1 & 0 & 0 & \vdots & 1 & 0 & 0 & & 1 & 0 & 0 \\
0 & 1 & 0 & \vdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \vdots & 0 & 0 & 0 & & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \vdots & 1 & 0 & 0 & & 1 & 0 & 0 \\
0 & 1 & 0 & \vdots & 0 & 1 & 0 & & 0 & 0 & 0 \\
0 & 0 & 1 & \vdots & 0 & 0 & 1 & & 0 & 0 & 0 \\
 & \vdots & & & & & & \ddots & & & \\
0 & 0 & 0 & \vdots & 0 & 0 & 0 & & 1 & 0 & 0 \\
0 & 1 & 0 & \vdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 1 & \vdots & 0 & 0 & 1 & & 0 & 0 & 1
\end{pmatrix} . \tag{5.65}$$

To display the resulting matrix, we first define the matrix $\mathbb{D}_3(a, b)$ in the following way

$$\mathbb{D}_3(a, b) = \begin{pmatrix} a & b & b \\ b & a & 0 \\ b & 0 & a \end{pmatrix} . \tag{5.66}$$

Note that $\mathbb{D}_3(2, -1)$ is the Cartan matrix of $so(6)$. After a suitable permutation of the particles, the K-matrix for the electron sector, obtained from the W-transform, takes the

following form

$$\mathbb{K}_e = \begin{pmatrix} \mathbb{D}_3(2,0) & \mathbb{D}_3(2,0) & \cdots & \mathbb{D}_3(2,0) & \mathbb{D}_3(2,1) \\ \mathbb{D}_3(2,0) & \mathbb{D}_3(4,0) & \cdots & \mathbb{D}_3(4,1) & \mathbb{D}_3(4,2) \\ \vdots & & \ddots & & \vdots \\ \mathbb{D}_3(2,0) & \mathbb{D}_3(4,1) & \cdots & \mathbb{D}_3(2(k-1),k-2) & \mathbb{D}_3(2(k-1),(k-1)) \\ \mathbb{D}_3(2,1) & \mathbb{D}_3(4,2) & \cdots & \mathbb{D}_3(2(k-1),k-1) & \mathbb{D}_3(2k,k) \end{pmatrix} . \tag{5.67}$$

Note that the structure of this K-matrix is very similar to the matrix $\mathbb{K}_e$ for the electron sector of the $\mathfrak{su}(3)_k$ states. The corresponding spin and charge vectors are given by

$$\begin{aligned} \mathbf{t}_e &= -(1,2,0;2,4,0;\cdots;k-1,2(k-1),0;k,2k,0)\,, \\ \mathbf{s}_e &= (1,0,2;2,0,4;\cdots;k-1,0,2(k-1);k,0,2k)\,. \end{aligned} \tag{5.68}$$

Using the charge vector, and the shift map (4.33), on easily constructs the K-matrix for general $M$. The matrix for the $\phi$-sector is just the inverse of $\mathbb{K}_e$. To bring it in a nice form, we again perform a suitable permutation on the order of the particles

$$(\mathbb{K}_\phi)^{\mathrm{perm}} = \left(\begin{array}{c|c|cccccc|cc} 1 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ \hline -1 & & & & & & & 0 & 0 \\ -\frac{1}{2} & & & & & & & \vdots & \vdots \\ -\frac{1}{2} & & & & \mathbb{D}_3^{-1}\otimes\mathbb{A}_{k-1} & & & 0 & 0 \\ 0 & & & & & & & -\frac{1}{2} & -\frac{1}{2} \\ \vdots & & & & & & & -\frac{3}{4} & -\frac{1}{4} \\ 0 & & & & & & & -\frac{1}{4} & -\frac{3}{4} \\ \hline 0 & 0 & \cdots & 0 & -\frac{1}{2} & -\frac{3}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \cdots & 0 & -\frac{1}{2} & -\frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{(6k-4)M+3}{8kM+4} \end{array}\right) , \tag{5.69}$$

where $\mathbb{D}_3^{-1}$ denotes the inverse Cartan matrix of $so(6)$. The charge and spin vectors are given by

$$\begin{aligned} \mathbf{t}_\phi &= (0,\cdots,0;0,\frac{1}{2kM+1})\,, \\ \mathbf{s}_\phi &= (0,\cdots,0;-1,0)\,. \end{aligned} \tag{5.70}$$

It is important to realize that in this formulation, the first $3k-2$ particles in the $\phi$-sector are to be interpreted as pseudoparticles. It is this interpretation that causes the reduction from

the cover system to the (non-abelian) $\mathfrak{so}(5)_{k,M}$ states. It is easily checked that the filling factors obtained from the quantum Hall data is

$$\nu = \frac{2k}{2kM+1} , \qquad \sigma = 2k . \tag{5.71}$$

Note that in calculating the filling factors, we should keep in mind that the matrices (5.67) and (5.69) are each inverses only after a permutation on the order of the particles.

The first check that the quantum Hall data above are correct concerns the central charge. This is most easily done by computing, numerically, the central charge reduction due to the presence of pseudoparticles. For some moderate sizes (up to $k = 5$), we found that the pseudoparticles reduce the central charge by $\frac{(3k-1)k}{k+3}$. Using the formula (4.22), we indeed find the correct central charge for the $\mathfrak{so}(5)_{k,M}$ affine Lie algebra $c_{\rm cft} = \frac{10k}{k+3}$. Moreover, for $k = 1, 2$, the pseudoparticle matrices are shown to occur in the restricted Kostka polynomials for $\mathfrak{so}(5)$. These results will be presented in [5].

The non-abelian statistics of the quasihole excitations over the spin-charge separated states at level $k = 1$ manifests itself at the level of the partition function $\lambda_{\rm tot} = \lambda_c\lambda_s$ ($\lambda_c, \lambda_s$ denote the partition functions of the holons and spinons, respectively). We derived an equation for $\lambda_{\rm tot}$ (at $k = 1$) from the IOW equations (4.4) using the statistics matrix $\mathbb{K}_\phi$ of equation (5.64) (with $M = 0$)

$$(\lambda_{\rm tot} - 1)^2 = (\lambda_{\rm tot} + 1)(\lambda_{\rm tot}^{\frac{1}{2}} + 1)x_s x_s + (\lambda_{\rm tot} + \lambda_{\rm tot}^{\frac{1}{2}})(x_c^2 + x_s^2) , \tag{5.72}$$

where $x_c$ and $x_s$ are the fugacities for the holons and spinons respectively. The small $x$ behaviour can easily be derived, and is given by

$$\lambda_{\rm tot} = 1 + \sqrt{2}(x_c + x_s) + o(x^2) . \tag{5.73}$$

The factor $\sqrt{2}$ signals the non-abelian statistics (see section 4.2) and it is the same as for the Moore-Read state (see eq. (5.8) with $k = 2$). Indeed, the Moore-Read state is defined in terms of the same parafermion and spin fields as the spin-charge separated states at level $k = 1$.

Also, the factor $\sqrt{2}$ follows from the quantum dimensions of the 4-dimensional representation of $so(5)$, which is calculated to be

$$\alpha_k = 2\Big(\cos\big(\frac{\pi}{k+3}\big) + \cos\big(\frac{2\pi}{k+3}\big)\Big) . \tag{5.74}$$

For $k = 2$, this equation gives $\alpha_2 = \sqrt{5}$, a result which also can be derived from the Bratteli diagram for the spin fields of the related parafermion theory at level $k = 2$.

In the next section, we will relate the K-matrices for the electron sector to the parafermionic CFTs, and write the corresponding characters in the form of a UCPF. As these characters are related to the so-called *string functions* of the affine Lie algebra theories, we can check the consistency of the results of this chapter with respect to these string functions.

## 5.5 The relation with parafermions

As an application of the K-matrices which were obtained for the various quantum Hall states, we will consider the relation with the K-matrices for the parafermion theories. In the next chapter, these will be extremely useful in determining the ground state degeneracy of the non-abelian quantum Hall systems in the presence of quasiholes. The K-matrices for the parafermions associated to simply-laced affine algebras are known already. However, for the non simply-laced cases, the results appear to be new.

To explain the relation between the K-matrices for the quantum Hall systems and the statistics of parafermions (which is encoded in the K-matrices), we will use the $\mathbb{Z}_k$ parafermions of the spin-polarized clustered quantum Hall states as an example. For convenience, we will repeat the matrix $\mathbb{K}_e$ and charge vector (eq. (5.10)) for this state

$$\mathbb{K}_e^M = \begin{pmatrix} M+2 & 2M+2 & \dots & kM+2 \\ 2M+2 & 2(2M+2) & \dots & 2(kM+2) \\ \vdots & \vdots & \ddots & \vdots \\ kM+2 & 2(kM+2) & \dots & k(kM+2) \end{pmatrix} , \qquad \mathbf{t}_e^M = - \begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix} . \tag{5.75}$$

The operators which can be associated to the particles described by this K-matrix are

$$V_l^{\rm rr} = \psi_l : e^{\frac{il}{\sqrt{k}}\sqrt{kM+2}\varphi_c} : , \tag{5.76}$$

for $l = 1, \dots, k$ and the operator with $l = 1$ is just the electron operator eq. (3.14). Note that the parafermion field $\psi_k = \psi_0 = \mathbf{1}$, which means that $V_k^{\rm rr}$ is just a chiral vertex operator.

The statistics properties which are due to the vertex operators are easily calculated, by taking products of the coefficients of the chiral boson fields

$$\mathbb{L} = \frac{kM+2}{k} \begin{pmatrix} 1 & 2 & \cdots & k \\ 2 & 4 & & 2k \\ \vdots & & & \vdots \\ k & 2k & \cdots & k^2 \end{pmatrix} . \tag{5.77}$$

Note that this matrix $\mathbb{L}$ is not a K-matrix or statistics matrix of a quantum Hall systems; it is the contribution of the chiral boson fields to the (mutual) statistics of the particles. To obtain the full statistics properties, the statistics due to the parafermion fields must be included. This leads to the conjecture that the K-matrix for the parafermions can be obtained by taking the difference of the matrices (5.75) and (5.77) and consider only the first $k-1$ particles. Doing this, we find the following matrix

$$(\mathbb{K}_{\rm pf})_{ij} = (\mathbb{K}_e - \mathbb{L})_{ij} = 2(\min(i,j) - \frac{ij}{k}) , \qquad i,j = 1, \dots, k-1 . \tag{5.78}$$

This is precisely twice the inverse of the Cartan matrix of $su(k)$, which is indeed the K-matrix corresponding to the $\mathbb{Z}_k$ parafermion theory, see eq. (6.35) and [68, 65, 41]. In particular, this matrix is used in the fermionic character formula for the $\mathbb{Z}_k$ parafermions. In this UCPF, all the particles are real particles.

The method described to obtain the K-matrices for the parafermion theories can be cast in a form which is more easy to handle. The 'decoupling' of the bosonic degrees of freedom from the parafermions can be achieved by applying a $SL(n,\mathbb{Q})$ transformation. For the case at hand, we have

$$\mathbb{X} = \begin{pmatrix} 1 & & & -\frac{1}{k} \\ & 1 & & -\frac{2}{k} \\ & & \ddots & \vdots \\ & & 1 & -\frac{k-1}{k} \\ & & & 1 \end{pmatrix},$$

$$\tilde{\mathbb{K}}_e = \mathbb{X}\mathbb{K}_e\mathbb{X}^T = \begin{pmatrix} & & & 0 \\ & 2\mathbb{A}_{k-1}^{-1} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & k(kM+2) \end{pmatrix}. \tag{5.79}$$

Thus the 'X-transformation' has the effect of decoupling the parafermion fields from the chiral vertex operators. In the $\phi$-sector, the pseudoparticles are decoupled from the quasihole excitations, as can be verified easily

$$\tilde{\mathbb{K}}_\phi = \begin{pmatrix} & & & 0 \\ & \frac{1}{2}\mathbb{A}_{k-1} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \frac{1}{k(kM+2)} \end{pmatrix}. \tag{5.80}$$

We immediately find that the K-matrix for the parafermion theory is just the inverse of the K-matrix for the pseudoparticles of the quasihole sector, where all particles are considered to be real particles, as was the case for the all the particles of the electron sector. This result is also true for the parafermions of $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ (see, for instance, [41]). We now make the conjecture that the K-matrix for the parafermions associated to the coset $\mathfrak{so}(5)_k/\mathfrak{u}(1)^2$ is also given by the inverse of the pseudoparticle matrix of the $\mathfrak{so}(5)_k$ affine Lie algebra theory. This matrix is easily obtained from eq. (5.69)

$$(\mathbb{K}^{\mathrm{pf}}_{\mathfrak{so}(5)_k})^{-1} = \begin{pmatrix} 1 & \vdots & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 \\ \cdots & & & & & & & \\ -1 & \vdots & & & & & & \\ -\frac{1}{2} & \vdots & & & & & & \\ -\frac{1}{2} & \vdots & & & \mathbb{D}_3^{-1} \otimes \mathbb{A}_{k-1} & & & \\ 0 & \vdots & & & & & & \\ \vdots & \vdots & & & & & & \\ 0 & \vdots & & & & & & \end{pmatrix}. \tag{5.81}$$

Using the result for the central charge of the K-matrix description of the $\mathfrak{so}(5)_k$ affine Lie algebra theory, it is easily checked that the central charge corresponding to this matrix is indeed given by $c_{\rm pf} = \frac{10k}{k+3} - 2$, which is the central charge of the parafermions related to $\mathfrak{so}(5)_k$. The central charge associated to the K-matrices of the parafermions and the pseudoparticles add up to $3k-2$, because these matrices are each others inverse and have dimension $3k-2$ (see eq. (4.15)). Also, the central charge associated to the pseudoparticles is such that $3k$ is reduced to the central charge of the affine Lie algebra, $c = \frac{10k}{k+3}$. Combining these results, we indeed find $c_{\rm pf} = (3k-2) - (3k - \frac{10k}{k+3}) = \frac{10k}{k+3} - 2$.

Another important check which we performed is exploiting the relation between the parafermion character formulas and the string functions associated to the affine Lie theories. This relation, which follows from the definition of the parafermionic CFTs [43], in general takes the following form

$$Z_{\rm pf}^{\Lambda,\lambda} = (\eta)^n c_\lambda^\Lambda \ , \tag{5.82}$$

where $\eta = q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1-q^k) = q^{\frac{1}{24}}(q)_\infty$ and $n$ is the rank of the affine Lie algebra. The quantities $c_\lambda^\Lambda$ are the string functions of the Lie algebra. These are the generating functions for the multiplicities of the (affine) weight $\lambda$ in the highest weight representation $\Lambda$ [34]. The parafermion fields correspond to $\Phi_\lambda^{\mathbf{1}}$, so the form of the parafermion matrices we conjectured are expected to be related to the character $Z_{\rm pf}^{\Lambda=\mathbf{1}} = \sum_\lambda (\eta)^n c_\lambda^{\mathbf{1}}$, where the sum over $\lambda$ is such that all the string functions of the form $c_\lambda^{\mathbf{1}}$ are included in the sum.

To show how this works, we will use the $\mathfrak{so}(5)_2$ parafermions as an example. Explicitly the corresponding K-matrix is (conjectured to be)

$$\mathbb{K}^{\rm pf} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix} \ . \tag{5.83}$$

The UCPF based on this K-matrix, namely

$$Z_{\rm pf}^{\Lambda=\mathbf{1}} = \sum \frac{q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}^{\rm pf}\cdot\mathbf{m}}}{\prod_i (q)_{m_i}} \ , \tag{5.84}$$

where the $m_i$ $(i=1,\ldots,4)$ run over the non-negative integers, can indeed be written as the sum over string-functions

$$Z_{\rm pf}^{\Lambda=\mathbf{1}} = \sum_\lambda (\eta)^2 c_\lambda^\Lambda \ . \tag{5.85}$$

The sum over $\lambda$ can also be characterized by saying that all the independent parafermion fields $\Phi_\lambda^{\mathbf{1}}$ must be 'present'. This correspondence has been checked on the computer, by comparing the result from eq. (5.84) to the string functions tabulated in [62], using the relation eq. (5.85)

The various string-functions $c_{(\lambda_1,\lambda_2)}^{(0,0)}$ are obtained by restricting the sum in eq. (5.84). Explicitly, we have

$$c_\lambda^{(0,0)} = \frac{q^{-\frac{1}{12}}}{(q)_\infty^2} \sum_{\rm res(\lambda)} \frac{q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}^{\rm pf}\cdot\mathbf{m}}}{\prod_i (q)_{m_i}} \ , \tag{5.86}$$

where $\mathrm{res}(\lambda)$ denotes the following restriction on the summation

$$\mathrm{res}(\lambda) = \begin{cases} \begin{matrix} 2m_1 + m_2 + 2m_3 & = 0 \bmod 4 \\ m_3 + m_4 & = 0 \bmod 2 \end{matrix} & \text{for } \lambda = (0,0) \\ \begin{matrix} 2m_1 + m_2 + 2m_3 & = 0 \bmod 4 \\ m_3 + m_4 & = 1 \bmod 2 \end{matrix} & \text{for } \lambda = (2,0) \\ \begin{matrix} 2m_1 + m_2 + 2m_3 & = 2 \bmod 4 \\ m_3 + m_4 & = 0 \bmod 2 \end{matrix} & \text{for } \lambda = (0,2) \\ \begin{matrix} 2m_1 + m_2 + 2m_3 & = 1 \bmod 4 \\ m_3 + m_4 & = 0 \bmod 2 \end{matrix} & \text{for } \lambda = (0,1)\ . \end{cases} \tag{5.87}$$

Again, we found that these forms correspond, up to the order checked, with the string functions obtained from [62]. Though we haven't got a proof, we believe that the explicit forms of the string functions (5.86) are indeed correct. To our best knowledge, these expressions are the first explicit expressions for some of the string functions of $\mathfrak{so}(5)_2$. In the forthcoming paper [5], we will give conjectures for the K-matrices corresponding to arbitrary affine Lie algebras, and the corresponding K-matrices for the related parafermion CFTs (which are again related to the string functions of the form $c^{\mathbf{1}}_{\lambda}$).

## 5.6 K-matrices: an outlook

In the previous sections, we identified K-matrices for the various clustered quantum Hall states. The main result is that the K-matrices can be written in the form $\mathbb{K}_e \oplus \mathbb{K}_\phi$, with $\mathbb{K}_\phi = \mathbb{K}_e^{-1}$. For the states with clustering (at level $k \geq 1$), composites are present at the 'electron sector'. These composites are accompanied by 'pseudoparticles' in the $\phi$-sector. For the non-simply laced cases, composites and pseudoparticles are also present at level $k = 1$. This 'duality' is present at every level of description of the clustered quantum Hall states. It was present at the level of electron and quasihole operators (see chapter 3). It will also be present in the next chapter, were the excitations over the states are studied by means of numerical diagonalization of model hamiltonians, and by means of the exclusion statistics properties of the parafermion fields.

The matrices obtained in this chapter also correspond to the matrices appearing in the universal chiral partition functions for the affine Lie algebra conformal field theories. The methods used to identify the K-matrices are not restricted to the cases motivated by the quantum Hall states. Thus we expect that K-matrices with a similar structure can be found for the other affine Lie algebra CFTs. This indeed turns out to be the case; more details on the K-matrices for the affine Lie algebra CFTs will be given in [5].

In identifying the K-matrices, we encountered some identities, based on character identities, to relate matrices for different theories. Of course, we would like to have a general scheme to find the matrices for general (rational) conformal field theories. An important class of these CFTs are the coset conformal field theories. A first step in finding a general scheme to obtain the K-matrices describing the coset conformal field theories will be given in [5]. Some of the matrices corresponding to these cosets were identified in the literature

before, and fit in the scheme presented in [5]. However, also new results will be given, for instance, results of K-matrices for parafermionic CFTs related to non simply-laced affine Lie algebras.

# Chapter 6

# State counting for clustered quantum Hall states

The statistics properties of the parafermion fields will be investigated in this chapter, with the intention of obtaining closed form expressions for the ground state degeneracy of clustered quantum Hall states in the presence of quasihole excitations, as described in [47, 9]. This state counting problem is interesting for the following reasons. The clustered quantum Hall states can be seen as ground states of a hamiltonian with an (ultra local) interaction between the electrons. Finding the ground state degeneracy of this hamiltonian can be done in a conformal field theory (CFT) approach, relying heavily on the statistics properties of the parafermionic fields. Another approach is by numerically diagonalizing the interaction hamiltonian for a small number of electrons. This method can serve as a check on the analytical results of the first approach. Thus, the quasihole degeneracies of a system of interacting electrons can be understood in terms of parafermionic statistics!

Also, the results for the state counting can be viewed as a justification of the K-matrices for the quantum Hall states presented in the previous chapter, because these matrices are closely related to the K-matrices for the parafermion theories, used in the state counting.

In the context of the spin-polarized states of Read and Rezayi, the $\mathbb{Z}_k$ (or $\mathfrak{su}(2)_k/\mathfrak{u}(1)$) parafermions are the relevant parafermions. For the non-abelian spin-singlet (NASS) states of [10], the relevant parafermions are the parafermions related to $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ (see [43] for a discussion on general parafermion CFTs).

This chapter is based on [9] and [4]. The plan of this chapter is as follows. In section 6.1 we will shortly indicate the setup of numerical diagonalization studies, because we need to adapt the calculations to the setup in which these studies are done. Also, we present some of the results of these numerical studies, which were done by E. Rezayi. These results can also be found in [9]. The general structure of the counting formulas will be indicated in section 6.2. It will become clear that the degeneracy consists of an intrinsic and an orbital part, which need to be combined in the right way. The intrinsic degeneracy factors need to be split to make this possible. The remainder of this chapter is devoted to this task, closely following the discussion of [4]. We will explain the procedure to obtain these expressions

using the $\mathfrak{su}(3)_2/\mathfrak{u}(1)^2$ parafermions of the NASS state at level $k = 2$ as an example (see also [9]). The first step is to find a basis for the (chiral) spectrum of the parafermion CFT. Here, we make contact with the K-matrix description of the clustered quantum Hall states from chapter 5. Using this basis, recursion relations for truncated characters will be derived (section 6.3). These recursion relations can be solved using the results of section 6.4, providing expressions for the truncated characters. From the explicit truncated characters, the 'split degeneracies' can be extracted. Finally the counting formula for the paired spin-singlet states is obtained in section 6.5, filling in some of the details of the discussion in [9]. In section 6.6, counting formulas for the RR states at general level $k$ are obtained, while section 6.7 deals with the counting formulas for the general $k$ NASS states. For all the cases checked, the results of the numerical diagonalization studies are exactly reproduced by the counting formulas.

## 6.1 The setup of the numerical studies

Though we will not describe numerical diagonalization studies in depth in this chapter, it is necessary to point out briefly in which setup they are done, because we need to adapt our calculations to be able to compare results. The numerical diagonalization is most easily done on the sphere. The interaction between the electrons is chosen such that the clustered state under investigation is the unique ground state (in the absence of quasihole excitations). Note that this interaction is an ultra local, many-body interaction, rather different from the long range Coulomb interaction. To 'tune' to the right filling fraction, a specific number of flux quanta needs to penetrate the sphere. States with quasiholes can be studied by increasing the number of flux quanta (but keeping all the other parameters the same); this results in the creation of quasiholes, as can be seen from the Laughlin gauge argument. The number of flux quanta needed for a state on the sphere with quasiholes is given by

$$N_\phi = \frac{1}{\nu} N - \mathcal{S} + \Delta N_\phi \,, \tag{6.1}$$

where $N$ is the total number of electrons, and $\Delta N_\phi$ the number of excess flux quanta, needed for the creation of the quasiholes. $\mathcal{S}$ is an integer constant depending on the state under investigation. Also, the number of quasiholes which are created by increasing the flux by one flux quantum depends on the state under investigation. For the spin-polarized RR states, this relation is given by $n = k\Delta N_\phi$, where $n$ is the number of quasiholes. For the NASS states of section 3.5, we have

$$n = n_\uparrow + n_\downarrow = 2k\Delta N_\phi \,. \tag{6.2}$$

For the clustered quantum Hall states with quasiholes present, the ground state is degenerate (for the ultra local interaction). The degeneracy consists of two parts. First of all, there is an orbital degeneracy, which is caused by the fact that in this setup, the quasiholes are non-local. This orbital degeneracy is not specific for clustered states; it is also present for the (unpaired) Laughlin states. For a system in which the quasiholes are localized, this degeneracy would not be present. Secondly, there is an intrinsic degeneracy, which stems from the non-trivial fusion rules of the spin fields, needed to create quasihole excitations.

This source of degeneracy is special for the clustered states. Here, we will focus on this intrinsic degeneracy and obtain analytical expressions, which allow the combination with the orbital degeneracy factors. This provides us with explicit expressions for the degeneracy of the ground states, in the presence of quasiholes.

As spin and angular momentum are good quantum numbers, all the states obtained from the numerical diagonalization fall into spin and angular momentum multiplets. The structure of the counting formulas is such that also the multiplet structure can be extracted.

The numerical diagonalization studies for the spin-singlet states eq. (3.5) were performed by E. Rezayi for $k = 2, M = 1$ in a spherical geometry. As discussed before the flux-charge relation for this state is $N_\phi = 7N/4 - 3$. The number of single-particle orbitals (the lowest Landau level degeneracy) is $N_\phi + 1$. In order to make contact with the results on more conventional geometries the radius $R$ of the sphere has to be chosen so that the number of flux is $N_\phi = 2R^2$ (where the magnetic field strength $B$ is fixed, such that the magnetic length is 1 in our units), so $R = \sqrt{N_\phi/2}$ [49]. The filling factor is $\nu = N/N_\phi = 2\pi\bar{n}$, where $\bar{n} = N/(4\pi R^2)$ is the particle number density.

For numerical purposes, it is best to re-express the interaction hamiltonian in terms of projection operators onto different values of the total angular momentum for different groups of particles [49]. For the $M = 1$, $k = 2$ case of the NASS states, the required hamiltonian can be written as

$$H = U \sum_{i<j<k} P_{ijk}(3N_\phi/2 - 3, 3/2) + V' \sum_{i<j} P_{ij}(N_\phi, 0), \tag{6.3}$$

with $U, V' > 0$. Here $P_{ijk}(L, S)$ ($P_{ij}(L, S)$) are projection operators for the three (resp., two) particles specified onto the given values of total angular momentum $L$ and spin $S$ for the three (resp., two) particles. Each projection is normalized to $P^2 = P$. To see that this is the required hamiltonian, that corresponds to the short range $\delta$-function interaction for $M = 0$, and gives the same numbers of zero-energy states found above, note the following. First, the maximal angular momentum for several particles corresponds to the closest approach of those particles [49]. In particular, the two-body term is a contact interaction, and $V' = V_0$, the zeroth Haldane pseudo-potential [49]. The two-body term implies that any zero-energy states must have no component with total angular momentum $N_\phi$ and total spin zero, which, since we are dealing with spin 1/2 fermions, means the wave function must vanish when any two particles coincide. The wave function must therefore contain a factor $\widetilde{\Psi}^1_{\rm L}$; multiplication by this factor defines a one-one mapping of the full space of states of spin 1/2 bosons in the lowest LL, with $N_\phi$ reduced by $N - 1$, onto the subspace of states of the fermions that is annihilated by the two-body term in $H$. Under this mapping, the three-body hamiltonian for the $M = 0$ case corresponds to the three-body term in $H$, and selects the corresponding states as zero-energy states. In particular, the total spin of the three bosons when they coincide (and hence of the fermions) must be $3/2$. Hence the zero-energy eigenstates of the present hamiltonian are given by the results derived earlier. Note also that $H$ can be rewritten in terms of $\delta$-functions and their derivatives. The zero-energy eigenstates of this hamiltonian were found for various $N$ and $N_\phi$ values, and analyzed in terms of $L$ and $S$.

The results for these numerical diagonalization studies for the spin-singlet states of [10] can be found in tables 6.1 and 6.2. The results are stated as a function of $N$ and $\Delta N_\phi$,

which is related to the number of quasiholes, eq. (6.2) with $k = 2$. Also, we indicated the spin and angular momentum multiplet structure.

As an example, we also include a typical energy spectrum for the systems without quasiholes in figure 6.1. Indeed, in this case, the ground state is non-degenerate, as one would expect.

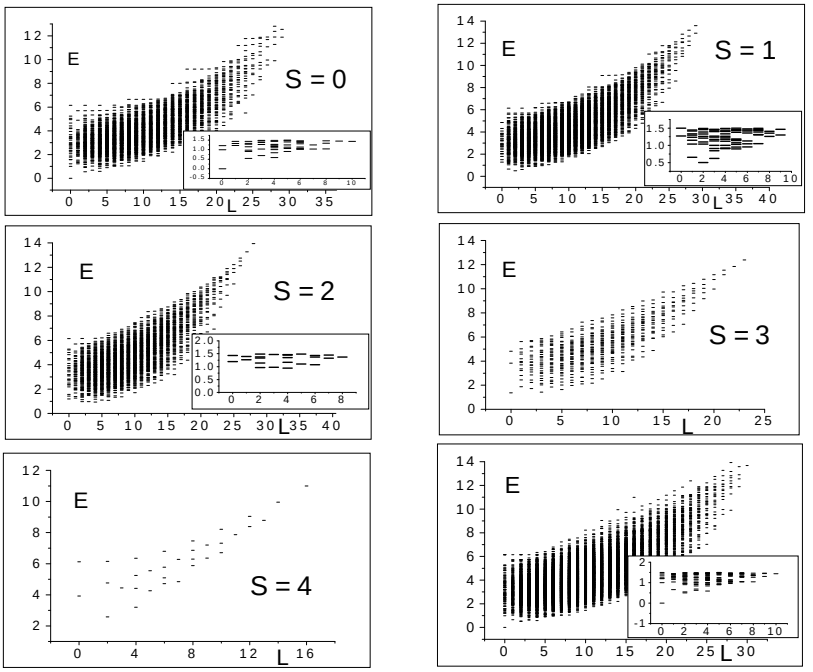


Figure 6.1: The spectrum of the NASS model ground state for $N = 8$ and 4/7 filling. The last panel shows all $S$ values combined. The insets are the low lying levels. Figure due to E. Rezayi.

## 6.2 Degeneracy factors and counting formulas

The intrinsic degeneracy is caused by the non-trivial fusion rules of the spin fields. As an example, we will use the spin fields of the $\mathfrak{su}(3)_2/\mathfrak{u}(1)^2$ parafermionic CFT. The fields and their fusion rules in this theory can be determined according to the methods of [43] and are summarized in table 6.3. We use the notation introduced in [9]. The parafermion fields are denoted by $\psi$, and all have conformal dimension $\Delta_\psi = \frac{1}{2}$. In particular, $\psi_1$,$\psi_2$,$\psi_{12}$ correspond to the roots $\alpha_1$,$-\alpha_2$ and $\alpha_1+\alpha_2$ of $su(3)$, respectively. The spin fields $\sigma_\uparrow$,$\sigma_\downarrow$,$\sigma_3$ and $\rho$ are related to the weights of $su(3)$ and their conformal dimensions are given by $\Delta_\sigma = \frac{1}{10}$ and $\Delta_\rho = \frac{3}{5}$. The fusion of an arbitrary number of $\sigma_{\uparrow,\downarrow}$ fields can be depicted

| | $\Delta N_\phi = 1$ | $\Delta N_\phi = 2$ |
|---|---|---|
| $N = 4$ | | |
| | $\#=20$ \| $S=0$ 1 2 | $\#=104$ \| $S=0$ 1 2 |
| | $L=0$ 1 0 1 | $L=0$ 1 0 1 |
| | $L=1$ 0 1 0 | $L=1$ 0 2 0 |
| | $L=2$ 1 0 0 | $L=2$ 2 1 1 |
| | | $L=3$ 0 1 0 |
| | | $L=4$ 1 0 0 |
| $N = 8$ | | |
| | $\#=105$ \| $S=0$ 1 2 | $\#=1719$ \| $S=0$ 1 2 3 4 |
| | $L=0$ 2 0 1 | $L=0$ 4 1 3 0 1 |
| | $L=1$ 0 2 0 | $L=1$ 1 7 2 1 0 |
| | $L=2$ 2 1 1 | $L=2$ 7 7 6 1 0 |
| | $L=3$ 0 1 0 | $L=3$ 3 9 3 1 0 |
| | $L=4$ 1 0 0 | $L=4$ 6 6 4 0 0 |
| | | $L=5$ 2 5 1 0 0 |
| | | $L=6$ 3 2 1 0 0 |
| | | $L=7$ 0 1 0 0 0 |
| | | $L=8$ 1 0 0 0 0 |
| $N = 12$ | | |
| | $\#=336$ \| $S=0$ 1 2 | |
| | $L=0$ 3 0 1 | |
| | $L=1$ 0 3 0 | |
| | $L=2$ 3 3 2 | |
| | $L=3$ 1 3 0 | |
| | $L=4$ 2 1 1 | |
| | $L=5$ 0 1 0 | |
| | $L=6$ 1 0 0 | |

| | $\Delta N_\phi = 3$ | $\Delta N_\phi = 4$ |
|---|---|---|
| $N = 4$ | | |
| | $\#=321$ \| $S=0$ 1 2 | $\#=755$ \| $S=0$ 1 2 |
| | $L=0$ 2 0 1 | $L=0$ 2 0 1 |
| | $L=1$ 0 2 0 | $L=1$ 0 3 0 |
| | $L=2$ 2 2 2 | $L=2$ 3 2 2 |
| | $L=3$ 1 3 0 | $L=3$ 1 4 1 |
| | $L=4$ 2 1 2 | $L=4$ 3 3 2 |
| | $L=5$ 0 1 0 | $L=5$ 1 3 0 |
| | $L=6$ 1 0 0 | $L=6$ 2 1 1 |
| | | $L=7$ 0 1 0 |
| | | $L=8$ 1 0 0 |

Table 6.1: Counting results for the NASS states at $k = 2$. $N$ is the number of electrons; $\Delta N_\phi$ is the number of excess flux quanta. The results are given as a function of the $L$ (angular momentum) and $S$ (total spin) quantum numbers. The total number of states is also indicated.

| | $\Delta N_\phi = \frac{1}{2}$ | $\Delta N_\phi = \frac{3}{2}$ |
|---|---|---|
| $N = 2$ | $\#=3$ \| $S=0$ $1$<br>$L=0$ \| $0$ $1$ | $\#=10$ \| $S=0$ $1$<br>$L=0$ \| $1$ $0$<br>$L=1$ \| $0$ $1$ |
| $N = 6$ | $\#=10$ \| $S=0$ $1$<br>$L=0$ \| $1$ $0$<br>$L=1$ \| $0$ $1$ | $\#=175$ \| $S=0$ $1$ $2$ $3$<br>$L=0$ \| $0$ $2$ $0$ $1$<br>$L=1$ \| $2$ $1$ $1$ $0$<br>$L=2$ \| $0$ $3$ $1$ $0$<br>$L=3$ \| $2$ $1$ $0$ $0$<br>$L=4$ \| $0$ $1$ $0$ $0$ |

| | $\Delta N_\phi = \frac{1}{4}$ | $\Delta N_\phi = \frac{5}{4}$ |
|---|---|---|
| $N = 5$ | $\#=0$ | $\#=48$ \| $S=\frac{1}{2}$ $\frac{3}{2}$<br>$L=\frac{1}{2}$ \| $1$ $1$<br>$L=\frac{3}{2}$ \| $1$ $1$<br>$L=\frac{5}{2}$ \| $1$ $0$ |

| | $\Delta N_\phi = \frac{3}{4}$ | $\Delta N_\phi = \frac{7}{4}$ |
|---|---|---|
| $N = 3$ | $\#=4$ \| $S=\frac{1}{2}$<br>$L=\frac{1}{2}$ \| $1$ | $\#=28$ \| $S=\frac{1}{2}$ $\frac{3}{2}$<br>$L=0$ \| $0$ $0$<br>$L=1$ \| $1$ $1$<br>$L=2$ \| $1$ $0$ |

Table 6.2: Counting results for the NASS states at $k = 2$ with fractional $\Delta N_\phi$ (symbols as in table 6.1).

| $\times$ | $\sigma_\uparrow$ | $\sigma_\downarrow$ | $\sigma_3$ | $\rho$ | $\psi_1$ | $\psi_2$ | $\psi_{12}$ |
|---|---|---|---|---|---|---|---|
| $\sigma_\uparrow$ | $\mathbf{1}+\rho$ | | | | | | |
| $\sigma_\downarrow$ | $\psi_{12}+\sigma_3$ | $\mathbf{1}+\rho$ | | | | | |
| $\sigma_3$ | $\psi_1+\sigma_\downarrow$ | $\psi_2+\sigma_\uparrow$ | $\mathbf{1}+\rho$ | | | | |
| $\rho$ | $\psi_2+\sigma_\uparrow$ | $\psi_1+\sigma_\downarrow$ | $\psi_{12}+\sigma_3$ | $\mathbf{1}+\rho$ | | | |
| $\psi_1$ | $\sigma_3$ | $\rho$ | $\sigma_\uparrow$ | $\sigma_\downarrow$ | $\mathbf{1}$ | | |
| $\psi_2$ | $\rho$ | $\sigma_3$ | $\sigma_\downarrow$ | $\sigma_\uparrow$ | $\psi_{12}$ | $\mathbf{1}$ | |
| $\psi_{12}$ | $\sigma_\downarrow$ | $\sigma_\uparrow$ | $\rho$ | $\sigma_3$ | $\psi_2$ | $\psi_1$ | $\mathbf{1}$ |

Table 6.3: Fusion rules of the parafermion and spin fields associated to the parafermion theory $\mathfrak{su}(3)_2/\mathfrak{u}(1)^2$ introduced by Gepner [43].

in a Bratteli diagram (see also [9]). Each arrow in the diagram in figure 6.2 stands for either a $\sigma_\uparrow$ or $\sigma_\downarrow$ field. The arrow starts at a certain field which can only be one of the fields on the left of the diagram at the same height. This last field is fused with the one

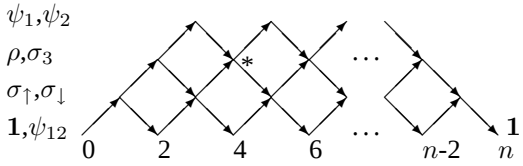


Figure 6.2: Bratteli diagram for the spin fields of $\mathfrak{su}(3)_2/\mathfrak{u}(1)^2$.

corresponding to the arrow, while the arrow points at a field present in this fusion. As an example, the arrows starting at the $*$ are encoding the fusion rules $\rho \times \sigma_{\uparrow(\downarrow)} = \psi_{2(1)} + \sigma_{\uparrow(\downarrow)}$ and $\sigma_3 \times \sigma_{\uparrow(\downarrow)} = \psi_{1(2)} + \sigma_{\downarrow(\uparrow)}$. One checks that the diagram is in accordance with the first two columns of table 6.3.

From figure 6.2, one immediately reads off that in general there is more than one fusion path of spin fields with leads to the identity (possibly the identity is reached only after the fusion with the parafermion fields $\psi_{1,2}$ of the electron operators). It is easily seen that the number of fusion channels starting from and terminating at **1** while $n_\uparrow$ $\sigma_\uparrow$ and $n_\downarrow$ $\sigma_\downarrow$ spin fields are fused is given by

$$d_{n_\uparrow,n_\downarrow} = \mathcal{F}(n_\uparrow + n_\downarrow - 2)\,, \tag{6.4}$$

where $\mathcal{F}(n)$ is the $n$-th Fibonacci number, defined by $\mathcal{F}(n) = \mathcal{F}(n-1) + \mathcal{F}(n-2)$ with the initial conditions $\mathcal{F}(0) = 1$ and $\mathcal{F}(1) = 1$. Next to this intrinsic degeneracy, there is an orbital degeneracy. These orbital degeneracy factors can be found in [47, 9] for the states discussed in sections 3.4 and 3.5. These factors have the general form

$$\prod_i \binom{\frac{n_i - F_i}{k} + n_i}{n_i}\,. \tag{6.5}$$

The product is over the types of quasiholes, while the numbers $F_i$ are interpreted as the number of 'unclustered' particles in the state. In the correlators, these correspond to the fundamental parafermions $\psi_i$. For each fusion path, these numbers can be different, implying that we have to split the intrinsic degeneracy according to these numbers. We denote these 'split degeneracy factors' by $\{\}_k$. Explicitly, we have $\{\,{}^{n}_{F}\,\}_k$ and $\{\,{}^{n_\uparrow\ n_\downarrow}_{F_1\ F_2}\,\}_k$ for the RR and NASS states, respectively.

Using the notation above, the counting formula for the clustered spin-singlet quantum Hall states take the following form

$$\#_{\mathrm{NASS}}(N, \Delta N_\phi, k) = {\sum_{N_{\uparrow,\downarrow}, n_{\uparrow,\downarrow}, F_{1,2}}}' \begin{Bmatrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{Bmatrix}_k \binom{\frac{N_\uparrow - F_1}{k} + n_\uparrow}{n_\uparrow} \binom{\frac{N_\downarrow - F_2}{k} + n_\downarrow}{n_\downarrow}\,, \tag{6.6}$$

where the prime on the summation indicates the presence of constraints (see below eq. (6.50)). The equivalent counting formula for the Read-Rezayi states is given in eq. (6.42).

The explicit split degeneracy factors for the $\mathbb{Z}_k$ and $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ parafermions at level $k > 2$ first appeared in [4] (for $k = 2$, these factors can be found in [84] and [9] respec-

tively). Previously, these factors for the $\mathbb{Z}_k$ parafermions ($k > 2$) could only be characterized via recursion relations, see [47, 22]. Note that the results in this chapter are easily extended to the more general $\mathfrak{su}(N)_k/\mathfrak{u}(1)^{N-1}$ parafermions.

We will now briefly outline in which way the split degeneracy factors are obtained. The starting point is the character of the parafermionic CFT. The symbols $\{\}_k$ can be extracted from finitized forms of these characters [47] (see also [90]). Recursion relations for these finitized characters can be derived from an explicit basis of the parafermionic CFT. These recursion relations will be written in a way that allows for an explicit solution, from which the symbols $\{\}_k$ can be extracted. In the sections 6.3-6.5, we will demonstrate this for the level $k = 2$ spin-singlet states of [10].

## 6.3 A basis for the $\mathfrak{su}(3)_2/\mathfrak{u}(1)^2$ parafermion theory

In this section, we briefly describe how an explicit basis for the chiral spectrum of the $\mathfrak{su}(3)_2/\mathfrak{u}(1)^2$ parafermion CFT is formed. The starting point is the chiral character for the parafermions in the $\mathfrak{su}(3)_2/\mathfrak{u}(1)^2$ conformal field theory. This character can be written in the form of a 'universal chiral partition function' (UCPF) see, for instance, [13, 5]. For the case at hand, this character reads [65]

$$\mathrm{ch}(x_1, x_2; q, k = 2) = \sum_{n_1,n_2} \frac{q^{(n_1^2+n_2^2-n_1n_2)/2}}{(q)_{n_1}(q)_{n_2}} x_1^{n_1} x_2^{n_2} \; . \tag{6.7}$$

In this character, $x_i = e^{\beta\mu_i}$ are fugacities of the particles, and $q = e^{\beta\varepsilon}$ ($\beta$ is the inverse temperature). $(q)_a$ is defined by $(q)_a = \prod_{k=1}^{a}(1-q^k)$ for $a > 0$ and $(q)_0 = 1$.

The bilinear form in the exponent of $q$ is described by the matrix

$$\mathbb{K} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \; . \tag{6.8}$$

The same matrix also describes the exclusion statistics of these parafermions. As we showed in section 5.5, it can also be obtained from the K-matrix of the electron sector for this quantum Hall state.

A basis for a CFT can be thought of as a set of states spanning the chiral Hilbert space. This set of states can be written as a (set of) vacuum state(s), on which creation operators act. The parafermions $\psi_{1,2}(z)$ in the $\mathfrak{su}(3)_2/\mathfrak{u}(1)^2$ theory can be expanded in modes as

$$\psi_{1,2}(z) = \sum_{m\in\mathbb{Z}} z^{-m} \psi^{1,2}_{m-\frac{1}{2}} \; . \tag{6.9}$$

As usual, the modes $\psi_m$ with negative index are the creation operators while the modes with positive index annihilate the vacuum

$$\psi_m|0\rangle = 0 \quad m > 0 \; . \tag{6.10}$$

The set of states

$$\psi^{a_n}_{-s_n}\psi^{a_{n-1}}_{-s_{n-1}} \cdots \psi^{a_1}_{-s_1}|0\rangle \tag{6.11}$$

is overcomplete, because of the (generalized) commutation rules of the parafermions. In the following, we will point out which restrictions on the indices $s_i$ will remove the 'over-completeness'. In doing so, we will follow the exclusion interpretation of the K-matrix as closely as possible and concentrate on the lowest possible 'energy' (given by $L_0 = \sum_i s_i$) for a certain number of applied fields first. The ordering of the modes $\psi^{1,2}$ is such that we apply the $\psi^1$ modes first. From (6.9) it follows that the simplest non-trivial state is

$$\psi^1_{-1/2}|0\rangle \ . \tag{6.12}$$

Interpreting the matrix (6.8) as the exclusion statistics matrix, the minimal spacing between two $\psi^1$ modes is 1, thus the state with two $\psi^1$'s acting on the vacuum with minimal energy is

$$\psi^1_{-3/2}\psi^1_{-1/2}|0\rangle \ . \tag{6.13}$$

The extension to $n_1$ $\psi^1$ modes is simple

$$\psi^1_{-(2n_1-1)/2}\cdots\psi^1_{-3/2}\psi^1_{-1/2}|0\rangle \ . \tag{6.14}$$

Note that if this were the whole story, we would describe the (free) Majorana fermion. The spacing between $\psi^2$ modes is the same as for the $\psi^1$ modes. However, if one acts with $\psi^2$ on a state in which $\psi^1$ modes are already present, one has to take into account the mutual statistics between $\psi^1$ and $\psi^2$ modes, which is $-1/2$, according to (6.8). Thus the energies of the $\psi^2$ modes have an extra shift of $-n_1/2$, resulting in the following states (with minimal energy)

$$\psi^2_{-(2n_2-1-n_1)/2}\cdots\psi^2_{-(3-n_1)/2}\psi^2_{-(1-n_1)/2}\psi^1_{-(2n_1-1)/2}\cdots\psi^1_{-3/2}\psi^1_{-1/2}|0\rangle \ . \tag{6.15}$$

The (dimensionless) energy associated to this state is $\frac{n_1^2+n_2^2-n_1n_2}{2}$, precisely the exponent of $q$ in the character (6.7). To obtain all the possible states, one has to allow states with higher energies as well. As usual [22], the energies of all the modes can have integer shifts, under the restriction that modes acting on a state have larger energies than the modes of the same type which have been applied earlier. This results in the following set of states

$$\begin{aligned}\psi^2_{-(2n_2-1-n_1)/2-t_{n_2}}\cdots\psi^2_{-(3-n_1)/2-t_2}\psi^2_{-(1-n_1)/2-t_1}\times \\ \times\,\psi^1_{-(2n_1-1)/2-s_{n_1}}\cdots\psi^1_{-3/2-s_2}\psi^1_{-1/2-s_1}|0\rangle \ ,\end{aligned} \tag{6.16}$$

with $s_{n_1} \geq \ldots \geq s_2 \geq s_1 \geq 0$ and $t_{n_2} \geq \ldots \geq t_2 \geq t_1 \geq 0$ ($s_i, t_j \in \mathbb{N}$).

Up to now, we used the special ordering of applying modes to the vacuum, namely, all the $\psi^1$ modes first. This is in fact enough to span the whole chiral spectrum, as can be seen if we perform the trace over all basis states. More or less by construction, we obtain the character (6.7). However, we also can allow a general ordering of the modes. As an example, we take the following state

$$\psi^2_{-0}\psi^1_{-1/2}|0\rangle \ . \tag{6.17}$$

The energy of the $\psi^2$ mode is zero because it gets an extra shift of $-1/2$ due to the presence of the $\psi^1$ mode. In spanning the whole chiral spectrum, we can also choose to use the state,

with the order of the modes changed

$$\psi^1_{-0}\psi^2_{-1/2}|0\rangle \ . \tag{6.18}$$

In this case, the $\psi^1$ mode gets an extra shift of $-1/2$, because of the presence of the $\psi^2$ mode. Thus, the $L_0$ value is the same for both states. In general, changing the order of two neighbouring $\psi^1$ and $\psi^2$ modes does not change the $L_0$ value if the extra shifts are changed in the appropriate way. The extra shift of a field is given by $-1/2$ times the number of preceding modes of the other type. In general, two states related by a reordering of modes are different, but we can use either of them (but not both) to span the chiral spectrum. Note that the rules of the spacing between the various fields is in accordance with the (exclusion) statistics interpretation of the matrix $\mathbb{K}$. The character (6.7) is obtained by taking the trace over all the states in the basis (6.16)

$$\mathrm{ch}(x_1, x_2; q) = \mathrm{Tr}\, x_1^{n_1} x_2^{n_2} q^{L_0} \ . \tag{6.19}$$

We can now define the finitized characters needed in the derivation of the symbols $\{\}_k$ by using the basis described above. These finitized characters are polynomials which will be denoted by $Y_{(l,m)}$. These polynomials are traces over the basis (6.16), but restricted to the states in which the energy of the modes of the $\psi_1$ ($\psi_2$) fields are smaller or equal to $l$ ($m$). Though the total energy of a state does not depend on the ordering of the modes, the energies of the individual modes do depend on the ordering, as can be seen by comparing the states (6.17) and (6.18). By restricting the trace over states in which the labels of the modes are bounded, we must include a state if there is at least one ordering in which all the modes satisfy the bounds imposed. Note that there may be other orderings, in which these bounds are not satisfied. We write the finitized characters as

$$Y_{(l,m)}(x_1, x_2; q) = \underset{\leq l, \leq m}{\mathrm{Tr}'}\, x_1^{n_1} x_2^{n_2} q^{L_0} \ . \tag{6.20}$$

The prime on the trace denotes an important restriction on the number of modes (denoted by $n_1$ and $n_2$) present in the states. These numbers must satisfy $n_1 = 2l \pmod 2$ and $n_2 = 2m \pmod 2$. This restriction takes into account that after fusing the spin fields, one ends up in the right sector, which can be $\mathbf{1}, \psi_1, \psi_2$ or $\psi_{12}$ depending on the number of spin up and down electrons. This is necessary, because after fusing the spin fields and the parafermion fields of the electron operators, on has to end with the identity $\mathbf{1}$, to obtain a non-zero correlator.

The finitized characters $Y_{(l,m)}$ can be written in terms of recursion relations of the following form

$$\begin{aligned} Y_{(l,m)} &= Y_{(l-1,m)} + x_1 q^{l-\frac{1}{2}} Y_{(l-1,m+\frac{1}{2})} \ , \\ Y_{(l,m)} &= Y_{(l,m-1)} + x_2 q^{m-\frac{1}{2}} Y_{(l+\frac{1}{2},m-1)} \ . \end{aligned} \tag{6.21}$$

Note that the recursion relations above are stated in terms of the energy labels of the modes. The aim we have is finding the number of possible states when a certain number of extra flux is added. We therefore need to make a change to labels which depend on the additional

flux. In fact, we will use the number of particles (given by $n_\uparrow$ and $n_\downarrow$ in this case) created by this flux as labels for the finitized partition functions. Explicitly, we have $l = \frac{n_\uparrow}{2}$ and $m = \frac{n_\downarrow}{2}$. In terms of the number of created quasiholes, the recursion relations become

$$\begin{aligned} Y_{(n_\uparrow,n_\downarrow)} &= Y_{(n_\uparrow-2,n_\downarrow)} + x_1 q^{\frac{n_\uparrow-1}{2}} Y_{(n_\uparrow-2,n_\downarrow+1)} \ , \\ Y_{(n_\uparrow,n_\downarrow)} &= Y_{(n_\uparrow,n_\downarrow-2)} + x_2 q^{\frac{n_\downarrow-1}{2}} Y_{(n_\uparrow+1,n_\downarrow-2)} \ . \end{aligned} \tag{6.22}$$

The initial conditions for these recursion relations are as follows

$$\begin{aligned} &Y_{(1,0)} = Y_{(0,1)} = 0 \ , \\ &Y_{(0,0)} = Y_{(2,0)} = Y_{(0,2)} = 1 \ , \\ &Y_{(1,1)} = q^{\frac{1}{2}} x_1 x_2 \ . \end{aligned} \tag{6.23}$$

The finitized characters are completely described by (6.22) and (6.23). In the next section, we will solve these recursion relations and thereby provide explicit expressions for the finitized characters.

## 6.4 Recursion relations and solutions

The recursion relations of the previous section can be solved explicitly; we will follow the approach of [16]. The key observation is that the recursion relations can be matched to general recursion relations, which are solved in terms of finitizations of universal chiral partition functions. For convenience, we repeat the finitized partition functions eq. (4.51)

$$P_{\mathbf{L}}(\mathbf{z};q) = \sum_{\mathbf{m}} \Big( \prod_i z_i^{m_i} \Big) q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m}+\mathbf{Q}\cdot\mathbf{m}} \prod_i \begin{bmatrix} \big(\mathbf{L} + (\mathbb{I}-\mathbb{K})\cdot\mathbf{m} + \mathbf{u}\big)_i \\ m_i \end{bmatrix} . \tag{6.24}$$

In this equation, $\mathbb{I}$ is the identity matrix, $\mathbb{K}$ the statistics matrix and $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ the $q$-deformed binomial ($q$-binomial)

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} \frac{(q)_a}{(q)_b (q)_{a-b}} & a,b \in \mathbb{N} \, ; \, b \leq a \\ 0 & \text{otherwise} \ . \end{cases} \tag{6.25}$$

Note that we defined the $q$-binomial to be non-zero only if both entries are integers greater or equal to zero, to avoid additional constraints on the sums in the counting formulas.

From the definition of the $q$-binomials, the following identity is easily derived

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad = \quad \begin{bmatrix} a-1 \\ b \end{bmatrix} + q^{a-b} \begin{bmatrix} a-1 \\ b-1 \end{bmatrix} . \tag{6.26}$$

Replacing the $i$'th $q$-binomial factor in (6.24) by the right hand side of (6.26), one finds the following recursion relations

$$P_{\mathbf{L}}(\mathbf{z};q) = P_{\mathbf{L}-\mathbf{e}_i}(\mathbf{z};q) + z_i q^{-\frac{1}{2}\mathbb{K}_{ii}+\mathbf{Q}_i+\mathbf{u}_i+\mathbf{L}_i} P_{\mathbf{L}-\mathbb{K}\cdot\mathbf{e}_i}(\mathbf{z};q) \ . \tag{6.27}$$

The vector $\mathbf{e}_i$ represents a unit vector in the $i$'th direction. We will use the equivalence between (6.24) and (6.27) frequently, because the recursion relations we encounter in deriving the counting formulas are all of type (6.27). Of course, upon deriving polynomials from recursion relations, one has to take the initial conditions into account. For the counting we need to know the finitizations of the character formulas, and these can be written in the form (6.24). Thus, when we solve recursion relations by polynomials of the form (6.24), the proper initial conditions are automatically taken into account.

We start by applying the above to the recursion relations (6.21), resulting in the following expressions for the truncated characters $Y_{(n_\uparrow,n_\downarrow)}$

$$Y_{(n_\uparrow,n_\downarrow)}(x_1,x_2;q)=\sum_{a,b} q^{(a^2+b^2-ab)/2}x_1^a x_2^b \begin{bmatrix} \frac{n_\uparrow+b}{2} \\ a \end{bmatrix} \begin{bmatrix} \frac{n_\downarrow+a}{2} \\ b \end{bmatrix} . \tag{6.28}$$

In this sum, $a$ and $b$ have to be restricted such that $a=n_\downarrow \pmod 2$ and $b=n_\uparrow \pmod 2$. Taking the limit $(n_\uparrow,n_\downarrow)\to(\infty,\infty)$, and summing over the four possibilities of the parity for $n_\uparrow,n_\downarrow$, gives back the untruncated character (6.7).

The result eq. (6.28) for the truncated characters will be needed for the final counting formula, which we give in the next section.

## 6.5 A counting formula for the NASS state at $k=2$

From the truncated characters of the previous section, we can obtain the symbols $\{\}_2$, needed in the counting formula eq. (6.6). In fact, the symbols $\{\}_2$ are obtained by taking the limit $q\to 1$ of the coefficient of $x_1^{F_1}x_2^{F_2}$ in eq. (6.28) (see, for instance, [47, 9])

$$Y_{(n_\uparrow,n_\downarrow)}(x_1,x_2;1)=\sum_{F_1,F_2} x_1^{F_1}x_2^{F_2} \begin{Bmatrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{Bmatrix} . \tag{6.29}$$

In this limit, the $q$-binomials in (6.28) become 'ordinary' binomials and we find

$$\begin{Bmatrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{Bmatrix}_2 = \begin{pmatrix} \frac{n_\uparrow+F_2}{2} \\ F_1 \end{pmatrix} \begin{pmatrix} \frac{n_\downarrow+F_1}{2} \\ F_2 \end{pmatrix} . \tag{6.30}$$

The fact that the finitized characters indeed provide the symbols $\{\}$ is rather non-trivial. This connection was first proposed in [47]. Some (restricted) 'solid on solid' (SOS) models (see, for instance [3]) can be mapped to the Bratteli diagrams of the spin fields of the quasiholes. Recursion relations for the partition functions for these models (at finite size) are in general equivalent to recursion relations for finitized characters in certain CFTs. In the case at hand, the corresponding CFT is the parafermion CFT. This provides a link between the Bratteli diagrams and the parafermion theories. As a check, on can calculate the number of fusion paths for the spin fields by summing over the symbols $\{\}$ and compare to the result obtained from the diagram itself. In this specific case, the equivalence follows from the structure of the recursion relations (see for instance [9]), giving rise to the identity

$$\sum_{F_1,F_2}{}' \begin{pmatrix} \frac{n_\uparrow+F_2}{2} \\ F_1 \end{pmatrix} \begin{pmatrix} \frac{n_\downarrow+F_1}{2} \\ F_2 \end{pmatrix} = \mathcal{F}(n_\uparrow+n_\downarrow-2) . \tag{6.31}$$

The prime on the summation denotes the constraints $F_1 \equiv n_\downarrow \pmod 2$ and $F_2 \equiv n_\uparrow \pmod 2$. At the level of the wave functions, the degeneracy is due to the presence of particles which do not belong to a cluster any more. At the level of correlators, these unclustered particles correspond to parafermions $\psi^1$ and $\psi^2$, which act as 'cluster breakers'. In the case of the Moore-Read state, this was made explicit in [84].

The counting formula for the NASS state at $k = 2$ is obtained by inserting the symbol (6.30) in the general counting formula (6.6)

$$\#(N, \Delta N_\phi, k = 2) = \sum_{N_{\uparrow,\downarrow}, n_{\uparrow,\downarrow}, F_{1,2}}{}' \binom{\frac{n_\uparrow + F_2}{2}}{F_1} \binom{\frac{n_\downarrow + F_1}{2}}{F_2} \binom{\frac{N_\uparrow - F_1}{2} + n_\uparrow}{n_\uparrow} \binom{\frac{N_\downarrow - F_2}{2} + n_\downarrow}{n_\downarrow}, \tag{6.32}$$

where the prime on the sum indicates the constraints $N_\uparrow + N_\downarrow = N$, $n_\uparrow + n_\downarrow = 4\Delta N_\phi$ and $N_\uparrow - N_\downarrow = n_\downarrow - n_\uparrow$.

We will now comment on the spin and angular momentum multiplet structure. As an example, we will write out the polynomials $Y_{(n_\uparrow, n_\downarrow)}$ in the case of two added flux quanta, giving eight quasiholes

$$\begin{aligned}
Y_{(8,0)} &= 1 + (q^2 + q^3 + 2q^4 + q^5 + q^6)x_1^2 + q^8 x_1^4 + (q^6 + q^7 + q^8 + q^9 + q^{10})x_1^4 x_2^2 \,, \\
Y_{(7,1)} &= (q^{\frac{1}{2}} + q^{\frac{3}{2}} + q^{\frac{5}{2}} + q^{\frac{7}{2}})x_1 x_2 + (q^{\frac{7}{2}} + 2q^{\frac{9}{2}} + 2q^{\frac{11}{2}} + 2q^{\frac{13}{2}} + q^{\frac{15}{2}})x_1^3 x_2 \\
&\quad + q^{\frac{19}{2}} x_1^5 x_2^3 \,, \\
Y_{(6,2)} &= 1 + (q^2 + q^3 + q^4)x_1^2 + (q^2 + q^3 + 2q^4 + q^5 + q^6)x_1^2 x_2^2 \\
&\quad + (q^6 + q^7 + q^8)x_1^4 x_2^2 \,, \\
Y_{(5,3)} &= (q^{\frac{1}{2}} + 2q^{\frac{3}{2}} + 2q^{\frac{5}{2}} + q^{\frac{7}{2}})x_1 x_2 + (q^{\frac{7}{2}} + q^{\frac{9}{2}} + q^{\frac{11}{2}})x_1^3 x_2 \\
&\quad + (q^{\frac{9}{2}} + q^{\frac{11}{2}} + q^{\frac{13}{2}} + q^{\frac{15}{2}})x_1^3 x_2^3 \,, \\
Y_{(4,4)} &= 1 + q^2 x_1^2 + q^2 x_2^2 + (q^2 + 2q^3 + 3q^4 + 2q^5 + q^6)x_1^2 x_2^2 + q^8 x_1^4 x_2^4 \,, \\
&\text{etc.}
\end{aligned} \tag{6.33}$$

After multiplying the coefficient of $x_1^{F_1} x_2^{F_2}$ with (in general) $q^{-(n_\uparrow F_1 + n_\downarrow F_2)/4}$, one obtains a sum of terms of the form $q^{l_z}$, which together form a collection of angular momentum multiplets with quantum numbers $l_z$. Taking the polynomial $Y_{(5,3)}$ as an example, we find the following non-zero symbols

$$\begin{aligned}
\begin{Bmatrix} 5 & 3 \\ 1 & 1 \end{Bmatrix}_2 &= 6 && (L = \frac{3}{2}\,,\, L = \frac{1}{2})\,, \\
\begin{Bmatrix} 5 & 3 \\ 3 & 1 \end{Bmatrix}_2 &= 3 && (L = 1)\,, \\
\begin{Bmatrix} 5 & 3 \\ 3 & 3 \end{Bmatrix}_2 &= 4 && (L = \frac{3}{2})\,.
\end{aligned} \tag{6.34}$$

An alternative way to obtain these results is by associating angular momentum multiplets to the binomials in eq. (6.32). The binomials $\binom{a}{f}$ forming the symbols $\{\}_2$ need to be

interpreted as the number of ways on can put $f$ fermions in $a$ boxes, which are labeled with $l_z = -\frac{(a-1)}{2}, -\frac{(a-1)}{2}+1, \ldots, \frac{(a-1)}{2}$ angular momentum quantum numbers. Each way of putting the $f$ fermions in $a$ boxes has an $l_z^{\rm tot}$ associated with it. Together, these $l_z^{\rm tot}$ quantum numbers fall into angular momentum multiplets. In this way, angular momentum multiplets can be associated with the binomials. The angular momentum multiplets of the various binomials in the counting formula need to be added in the usual way.

Though the parafermion theory does not have a proper $SU(2)$ spin symmetry, one can associate spin quantum numbers to every state by taking $S_z = \frac{N_\uparrow - N_\downarrow}{2}$. Combining the spin and angular momentum, one finds that all the states fall into spin and angular momentum multiplets.

The numerical diagonalization studies for the NASS state at level $k = 2$ are described in [9]. It is very gratifying to see that the counting formula eq. (6.32) does in fact exactly reproduce the quasihole degeneracies, as well as the multiplet structure.

In order to find the counting results for the spin-singlet states at general level-$k$, we first take a closer look at the counting of the Read-Rezayi states, which was in fact done in [47]. Those results however, were stated in terms of recursion relations which are difficult to solve. The advantage of the recursion relations presented in the next section is that they can easily be solved in terms of ($q$-deformed) binomials, and thus provide explicit expressions for the symbols $\{\}_k$.

## 6.6 Counting formulas for the Read-Rezayi states

The derivation of the counting formulas for the RR states goes along the same lines as the derivation for the NASS k=2 states as explained in the previous sections. Therefore, we do not go into full detail, but concentrate on the parts which need more explanation.

We start with the character of the $\mathfrak{su}(2)_k/\mathfrak{u}(1)$ parafermionic theory (see [115]), which can be obtained from [68, 65, 41]

$$\mathrm{ch}(x;q,k) = \sum_{a_i} \frac{q^{\frac{1}{2}(\mathbf{a}\cdot\mathbb{C}_{k-1}\cdot\mathbf{a})}}{\prod_i (q)_{a_i}} x^{ia_i} \,, \tag{6.35}$$

where $\mathbf{a} = (a_1, \ldots, a_{k-1})$ and $\mathbb{C}_{k-1} = 2\mathbb{A}_{k-1}^{-1}$, $\mathbb{A}_{k-1}$ being the Cartan matrix of $su(k)$. In components, these matrices are given by

$$(\mathbb{A}_{k-1})_{i,j} = 2\delta_{i,j} - \delta_{|i-j|,1} \,, \tag{6.36}$$

$$(\mathbb{A}_{k-1}^{-1})_{i,j} = \min(i,j) - \frac{ij}{k} \,, \qquad i,j = 1,\ldots,k-1 \,. \tag{6.37}$$

In fact, $\mathbb{C}_{k-1}$ is the K-matrix for the $\mathbb{Z}_k$ parafermions and can also be obtained from the matrix $\mathbb{K}_e$ in eq. (5.10) by the methods described in section 5.5.

The parafermions in this theory are $\psi_0, \psi_1, \ldots \psi_{k-1}$ ($\psi_0$ is the identity $\mathbf{1}$ and the labels are defined modulo $k$). By writing $x^{ia_i}$ in the character (6.35), we take care of the fact that the fugacity of species $i$ is $i$ times the fugacity of the first type of particle. In fact, the $i$'th species can be thought of as a 'composite' of $i$ particles of species 1. This point of view is

supported by the fusion rules for these parafermions $\psi_1 \times \psi_p = \psi_{p+1}$, with $p = 1, \ldots, k-1$. This structure is also present in the K-matrix structure describing the Read-Rezayi states, presented in section 5.1 (see also [48, 6, 7]).

A basis for the chiral spectrum can be constructed in the same way as described in section 6.3. The shifts in modes between the various fields are given by the elements of the matrix $\mathbb{K} = \mathbb{C}_{k-1}$. We will now proceed by directly giving the corresponding recursion relations

$$Y_{\mathbf{l}}(x;q,k) = Y_{\mathbf{l}-\mathbf{e}_i} + x^i q^{l_i - \frac{i(k-i)}{k}} Y_{\mathbf{l}-\mathbb{C}_{k-1}\cdot\mathbf{e}_i} \,. \tag{6.38}$$

The factor $\frac{i(k-i)}{k}$ is the conformal dimension of the $i$'th parafermion in the $\mathbb{Z}_k$-parafermion theory. These recursion relations are solved by the following polynomials

$$Y_{\mathbf{l}}(x;q,k) = \sum_{a_i} q^{\frac{1}{2}(\mathbf{a}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{a})} \prod_{i=1}^{k-1} x^{i a_i} \begin{bmatrix} \left(\mathbf{l} + (\mathbb{I}_{k-1} - \mathbb{C}_{k-1})\cdot\mathbf{a}\right)_i \\ a_i \end{bmatrix}, \tag{6.39}$$

where $\mathbb{I}_{k-1}$ denotes the $(k-1)$-dimensional unit matrix. To obtain the counting results, we have to specify the truncation parameters $l_i$. As in the NASS case with $k = 2$, we will do this in terms of the number of particles created by the extra flux, given by $n = k\Delta N_\phi$ for the states under consideration. Because the chemical potential of species $i$ is $i$ times the chemical potential of species 1, the truncation parameter $l_i$ has to be scaled with a factor $i$ with respect to $l_1$ (see, for instance [7]), which is found to be $l_1 = \frac{n}{k}$. This leads to the following truncation parameters $l_i = \frac{in}{k}$, and the truncated characters needed for the counting become

$$Y_n(x;q,k) = \sum_{a_i} q^{\frac{1}{2}(\mathbf{a}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{a})} \prod_{i=1}^{k-1} x^{i a_i} \begin{bmatrix} \frac{in}{k} + \left((\mathbb{I}_{k-1} - \mathbb{C}_{k-1})\cdot\mathbf{a}\right)_i \\ a_i \end{bmatrix}. \tag{6.40}$$

To obtain the symbols $\left\{ {n \atop F} \right\}_k$ which are needed for the counting, one has to take the limit $q \to 1$ of the prefactor of $x^F$ in eq. (6.40). This results in

$$\left\{ {n \atop F} \right\}_k = \sum_{\sum i a_i = F} \prod_{i=1}^{k-1} \begin{pmatrix} \frac{in}{k} + \left((\mathbb{I}_{k-1} - \mathbb{C}_{k-1})\cdot\mathbf{a}\right)_i \\ a_i \end{pmatrix}. \tag{6.41}$$

With this result, we arrive at the following counting formula for the Read-Rezayi states (for general $k$)

$$\#_{\mathrm{RR}}(N, \Delta N_\phi, k) = \sum_F \left\{ {n \atop F} \right\}_k \begin{pmatrix} \frac{N-F}{k} + n \\ n \end{pmatrix}, \tag{6.42}$$

with $n = k\Delta N_\phi$. To make the above (in particular the symbols $\{\}_k$ of eq. (6.41)) more explicit, we will discuss the $k = 2$ (i.e. the Moore-Read state) and $k = 3$ cases. For the MR state counting, we need to know the symbol $\{\}_2$. Eq. (6.41) with $k = 2$ gives $\left\{ {n \atop F} \right\}_2 = \binom{\frac{n}{2}}{F}$. Of course, this is just the result already found in [84]. Note that our notation is slightly different with respect to the one used in [84, 47]. In our notation, we denote the number of created quasiholes by $n$. In [84, 47], $n$ denoted the number of extra fluxes, which is denoted by $\Delta N_\phi$ in our notation.

Although the method described above seems to be unnecessarily complicated to reproduce this result, it is very useful for obtaining closed expressions for $k > 2$. As an illustration, we will discuss the case $k = 3$, and compare our results with [47]. For $k = 3$, the polynomials are given by the following expression

$$Y_n(x;q,3) = \sum_{a,b} q^{\frac{2}{3}(a^2+b^2+ab)} x^{a+2b} \begin{bmatrix} \frac{n}{3} - \frac{a+2b}{3} \\ a \end{bmatrix} \begin{bmatrix} \frac{2n}{3} - \frac{2a+b}{3} \\ b \end{bmatrix} . \tag{6.43}$$

Indeed, these polynomials reduce to the ones in [47], upon setting $q = 1$. The symbols $\{\}_3$ are now easily written down

$$\begin{Bmatrix} n \\ F \end{Bmatrix}_3 = \sum_{a+2b=F} \binom{\frac{n}{3} - \frac{a+2b}{3}}{a} \binom{\frac{2n}{3} - \frac{2a+b}{3}}{b} . \tag{6.44}$$

Note that only a finite number of terms contribute to the sum in eq. (6.44). In fact, this is true for all the symbols (6.41) with $n$ finite.

The fusion rules for the spin field $\sigma$ which is part of the quasihole operator at level $k = 3$ (see [85]), can be encoded in a Bratteli diagram with the same structure as the diagram 6.2 (note that the fields differ, of course). This is a consequence of the rank-level duality $\mathfrak{su}(2)_3 \leftrightarrow \mathfrak{su}(3)_2$ (see §16.6 in [34]). Thus the total intrinsic degeneracy for the $k = 3$ Read-Rezayi state with $n$ quasiholes is given by $d_n = \mathcal{F}(n-2)$. Indeed, by summing the symbols $\{ \begin{smallmatrix} n \\ F \end{smallmatrix} \}_3$ over $F$, this result is reproduced.

The angular momentum multiplets can be found in the same way as described in section 6.5. Let us note that for $k = 1$ the only degeneracy factor remaining in eq. (6.42) is $\binom{N+n}{n}$, which is precisely the orbital factor for the Laughlin states with quasiholes present. This was of course to be expected, as the $k = 1$ Read-Rezayi states are in fact the Laughlin states.

To conclude the discussion on the counting for the Read-Rezayi states, we would like to mention that the numerical studies as presented for $k = 3$ in [47] are in complete agreement with the counting formulas. At this point, no numerical results are available for $k \geq 4$. In the following section we will turn our attention to the counting of the NASS states for general level.

## 6.7 Counting formulas for the NASS states

In this section, we describe the counting for the NASS states at general level $k$. We will closely follow the procedure of the previous sections, that is, we start by writing down the chiral character corresponding to the $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ parafermions [65, 41]

$$\mathrm{ch}(x_1,x_2;q,k) = \sum_{a_i,b_j} \frac{q^{\frac{1}{2}(\mathbf{a}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{b})}}{\prod_{i,j}(q)_{a_i}(q)_{b_j}} x_1^{ia_i} x_2^{jb_j} , \tag{6.45}$$

where we used the same notation as in eq. (6.35). This character is of the UCPF form with the K-matrix equal to the K-matrix of the $\mathfrak{su}(3)_k/\mathfrak{u}(1)^2$ parafermions, which is given by

$\mathbb{K} = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) \otimes \mathbb{A}_{k-1}^{-1}$. The recursion relations corresponding to the basis of this theory can be written in the following way

$$\begin{aligned} Y_{(\mathbf{l},\mathbf{m})}(x_1,x_2;q,k) &= Y_{(\mathbf{l}-\mathbf{e}_i,\mathbf{m})} + x_1^i q^{l_i - \frac{i(k-i)}{k}} Y_{(\mathbf{l}-\mathbb{C}_{k-1}\cdot\mathbf{e}_i,\mathbf{m}+\frac{1}{2}\mathbb{C}_{k-1}\cdot\mathbf{e}_i)}\ , \\ Y_{(\mathbf{l},\mathbf{m})}(x_1,x_2;q,k) &= Y_{(\mathbf{l},\mathbf{m}-\mathbf{e}_j)} + x_2^j q^{m_j - \frac{j(k-j)}{k}} Y_{(\mathbf{l}+\frac{1}{2}\mathbb{C}_{k-1}\cdot\mathbf{e}_j,\mathbf{m}-\mathbb{C}_{k-1}\cdot\mathbf{e}_j)}\ . \end{aligned} \tag{6.46}$$

Once again, we solve the recursion relations by matching these recursions to eq. (6.27). The truncated characters take the form

$$\begin{aligned} Y_{(\mathbf{l},\mathbf{m})}(x_1,x_2;q,k) = \sum_{a_i,b_j} & q^{\frac{1}{2}(\mathbf{a}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{b})} \times \\ & \prod_{i=1}^{k-1} x_1^{ia_i} \begin{bmatrix} \left(\mathbf{l}+(\mathbb{I}_{k-1}-\mathbb{C}_{k-1})\cdot\mathbf{a}+\frac{1}{2}\mathbb{C}_{k-1}\cdot\mathbf{b}\right)_i \\ a_i \end{bmatrix} \times \\ & \prod_{j=1}^{k-1} x_2^{jb_j} \begin{bmatrix} \left(\mathbf{m}+(\mathbb{I}_{k-1}-\mathbb{C}_{k-1})\cdot\mathbf{b}+\frac{1}{2}\mathbb{C}_{k-1}\cdot\mathbf{a}\right)_j \\ b_j \end{bmatrix}\ . \end{aligned} \tag{6.47}$$

We continue by specifying the parameters $l_i$ and $m_j$. We have to use the same construction as in the RR case, with the difference that we now need the number of spin up and down particles (denoted by $n_\uparrow$ and $n_\downarrow$) created by the excess flux. Using $l_i = \frac{in_\uparrow}{k}$ and $m_j = \frac{jn_\downarrow}{k}$ results in

$$\begin{aligned} Y_{(n_\uparrow,n_\downarrow)}(x_1,x_2;q,k) = \sum_{a_i,b_j} & q^{\frac{1}{2}(\mathbf{a}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbb{C}_{\mathbf{k-1}}\cdot\mathbf{b})} \times \\ & \prod_{i=1}^{k-1} x_1^{ia_i} \begin{bmatrix} \frac{in_\uparrow}{k} + \left((\mathbb{I}_{k-1}-\mathbb{C}_{k-1})\cdot\mathbf{a}+\frac{1}{2}\mathbb{C}_{k-1}\cdot\mathbf{b}\right)_i \\ a_i \end{bmatrix} \times \\ & \prod_{j=1}^{k-1} x_2^{jb_j} \begin{bmatrix} \frac{jn_\downarrow}{k} + \left((\mathbb{I}_{k-1}-\mathbb{C}_{k-1})\cdot\mathbf{b}+\frac{1}{2}\mathbb{C}_{k-1}\cdot\mathbf{a}\right)_j \\ b_j \end{bmatrix}\ . \end{aligned} \tag{6.48}$$

From eq. (6.48) we obtain the symbols $\left\{ \begin{smallmatrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{smallmatrix} \right\}_k$ by taking the limit $q \to 1$ of the coefficient of $x_1^{F_1} x_2^{F_2}$

$$\begin{aligned} \begin{Bmatrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{Bmatrix}_k = \sum_{\substack{\sum ia_i = F_1 \\ \sum jb_j = F_2}} & \prod_{i=1}^{k-1} \begin{pmatrix} \frac{in_\uparrow}{k} + \left((\mathbb{I}_{k-1}-\mathbb{C}_{k-1})\cdot\mathbf{a}+\frac{1}{2}\mathbb{C}_{k-1}\cdot\mathbf{b}\right)_i \\ a_i \end{pmatrix} \times \\ & \prod_{j=1}^{k-1} \begin{pmatrix} \frac{jn_\uparrow}{k} + \left((\mathbb{I}_{k-1}-\mathbb{C}_{k-1})\cdot\mathbf{b}+\frac{1}{2}\mathbb{C}_{k-1}\cdot\mathbf{a}\right)_j \\ b_j \end{pmatrix}\ . \end{aligned} \tag{6.49}$$

We know have specified all the ingredients of the counting formula for the NASS states

$$\#_{\mathrm{NASS}}(N,\Delta N_\phi,k) = {\sum_{N_{\uparrow,\downarrow},n_{\uparrow,\downarrow},F_{1,2}}}' \begin{Bmatrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{Bmatrix}_k \begin{pmatrix} \frac{N_\uparrow - F_1}{k} + n_\uparrow \\ n_\uparrow \end{pmatrix} \begin{pmatrix} \frac{N_\downarrow - F_2}{k} + n_\downarrow \\ n_\downarrow \end{pmatrix}\ , \tag{6.50}$$

where the prime on the sum indicates the constraints $N_\uparrow + N_\downarrow = N$, $n_\uparrow + n_\downarrow = 2k\Delta N_\phi$ and $N_\uparrow - N_\downarrow = n_\downarrow - n_\uparrow$. The last constraint is a necessary condition for the state to be a spin-singlet (for more information on the constraints, see [9]).

The case $k = 2$ was explicitly discussed in section 6.5. For $k = 1$ only the orbital degeneracy factors remain, and we obtain the counting formula for a particular class of Halperin states [55]. Indeed, for $k = 1$, the NASS states reduce to the spin-singlet Halperin states. As already mentioned in section 6.5, the counting formula (6.50) with the symbols (6.49) exactly reproduces the results of the diagonalization studies for $k = 2$ [9]. For $k \geq 3$, no numerical results are available at the moment.

# Epilogue

After the results of the previous chapters, this is a good point to sit back, and reflect upon the issues which were discussed. Mainly, this thesis has been about using conformal field theory to give a description of (possible) quantum Hall states. Motivation came form the observation of an even-denominator fractional quantum Hall state, which is believed to be related to a trial wave function which consists of the usual Laughlin factor and a factor which is best characterized by a pairing structure. This pairing (or in general, clustering) opens the possibility for an extra fractionalization of the quantum numbers for the quasi-holes, compared to the Laughlin quasiholes. Moreover, the quasiholes of the clustered states satisfy what is called non-abelian statistics. These statistics properties are a consequence of the clustering properties of the electrons.

This duality between the properties of the electrons and quasiholes is present at the level of the conformal field theory description as well. On this level, it was found that a basis of the underlying conformal field theory (of affine Lie algebra type), can be split into two dual parts, one related to the quasiholes, the other to the electron like particles. Though such a description of conformal field theories came about in the study of fractional quantum Hall states, it can be used in a more general setting, to find similar bases for other conformal field theories.

In defining new quantum Hall states (see chapter 3), we used the following procedure. We assumed the existence of a quantum Hall state with certain characteristics (for instance a paired spin-singlet state) and then used conformal field theory to describe such a state. Of course, one can not hope to give an exhaustive list of all possible states, as this would require a full classification of conformal field theories. Also, not all conformal field theories are related to quantum Hall states. However, if one finds quantum Hall states in this way, one can study its properties, having quite powerful conformal field theory techniques at hand. As stated above, this has led to new ways of looking at bases for conformal field theories.

Starting from the spin-polarized Moore-Read state, we have included the electron spin in a natural way. This has led to some interesting spin-singlet quantum Hall states, which have a similar clustering structure as the spin-polarized variants. Also, the quasiholes obey non-abelian statistics. In addition, in one type of these spin full states, the fundamental quasiholes show a separation of the spin and charge degrees of freedom, for the first time in the context of the quantum Hall effect.

To test the ideas of using conformal field theories in the description of clustered quantum Hall states, one can try to find a model interaction for which the state under consideration is the ground state. The degeneracy obtained by diagonalizing this interaction (for instance on the sphere), for systems with quasiholes present, can be understood in terms of the underlying conformal field theories. The fact that the quasihole states are degenerate, lie at heart of the non-abelian statistics satisfied by the quasiholes. For the NASS states we carried out the program outlined above in full detail, see chapter 6. Important to note here is that the only things specified in the numerical diagonalization are the ground state properties, for the system in the absence of quasiholes. This again points to the duality between the clustering properties of the electrons and the non-abelian statistics of the quasiholes. Also here, it is to be expected that this relation holds in more general situations, not only the ones related to quantum Hall states.

An important issue concerning the clustered quantum Hall states is an effective field theory description. This issue is not addressed in this thesis. To further study the behaviour of the excitations, and calculate in more detail the tunneling behaviour, it would be nice to have a full quantum field theory description for these states, as is available in the case of the Laughlin states. Though some progress has been made in this respect, more research in this area is needed for a full understanding of these states.

On the experimental side, there are a lot of unanswered questions. In general, it is hard to distinguish experimentally between the various proposed states. One of the possibilities is to measure the $I-V$ characteristics for various tunneling processes, as these are, in many cases, different for the various proposals. Also the charge of the fundamental quasiparticles is an option, because of the additional fractionalization of quantum numbers in comparison to the competing hierarchical states, which exist in the cases of odd-denominator clustered states. However, measuring these properties on samples where the relevant plateaux are observed is not possible at this point.

In relation to the spin-singlet states studied in this thesis, it is to be expected that these states are most relevant in the regime where the Zeeman energy goes to zero. This can be achieved experimentally by applying pressure on the sample. This changes the band structure and in effect sends the Zeeman energy to zero. This behaviour has been demonstrated in experiments which showed the existence of skyrmions at the $\nu = 1$ plateau. These skyrmions are topological excitations with many reversed spins, and can have lower energy than ordinary spin-flips in the low Zeeman energy limit.

A very interesting experimental development is the observation of a fractional quantum Hall effect at $2$ Kelvin in *organic molecular semiconductors*. In the more conventional materials, the fractional quantum Hall states are only observed at temperatures below $0.5$ Kelvin. It would be interesting to cool these organic molecular semiconductors to the milli-Kelvin regime. One can hope that in these systems the Hall plateaux at even denominator filling fraction are observed at temperatures higher compared to the conventional systems. Possibly, quantum Hall states are observed at new filling fractions as well.

One has to realize that the study of the quantum Hall effect is much broader than the study of the clustered states presented in this thesis. An important aspect which I would like to mention here is the issue of the transitions between quantum Hall states, and the transition to the insulating state at high magnetic fields. In certain types of samples, there have been clear observations of scaling with universal scaling exponents, pointing at quantum critical

behaviour.

In the single layer quantum Hall systems, a lot of different states are formed. In double layer systems, there is a whole plethora of states, due to the extra degree of freedom and a lot of research is done in this area (both experimental and theoretical). There is much more interesting physics in the quantum Hall regime, but describing it all would go beyond the scope of this epilogue.

To conclude, the research described in this thesis covers only a small area of quantum Hall physics, but is nevertheless important, as it provides systems in which (quasi)particles satisfy new forms of statistics, which are special to the types of quantum Hall states discussed. Also, the properties of these states and the corresponding quasiparticles were studied in various ways, giving interesting results from the physical as well as the mathematical point of view.

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# Samenvatting

Dit proefschrift beschrijft mogelijke toestanden gevormd door de electronen in systemen waarin het *fractionele quantum Hall effect* optreedt. Door analogieën te maken met alledaagse verschijnselen zal ik in deze samenvatting proberen duidelijk te maken wat ik tijdens mijn promotie heb onderzocht.

Laat ik beginnen met een zeer alledaagse stof: water. Water kan voorkomen in drie toestanden (fases), namelijk in vaste vorm, in vloeibare vorm en als gas. Door de temperatuur (of de druk) te veranderen kunnen deze vormen in elkaar overgaan, en spreken we van een *fase overgang*. In dit proefschrift heb ik mij niet bezig gehouden met water, maar met *electronen*. Electronen kunnen, op dezelfde manier als water moleculen, verschillende toestanden vormen, afhankelijk van de externe omstandigheden, waaronder de temperatuur. In dit proefschrift heb ik een aantal van deze toestanden onderzocht; voordat ik die toestanden beschrijf, wil ik eerst de belangrijkste eigenschappen van electronen duidelijk maken.

Electronen kunnen, net als water moleculen, opgevat worden als deeltjes. Een van de belangrijke eigenschappen van electronen is dat ze geladen zijn. Omdat alle electronen dezelfde lading hebben (die we op $-1$ stellen; het minteken is historisch bepaald), stoten electronen elkaar af, op ongeveer dezelfde manier als gelijknamige polen van magneten. Bovendien worden electronen beïnvloed door magnetische velden. Hiervan wordt gebruik gemaakt in beeldbuizen: electronen worden afgeschoten en daarna afgebogen door magnetische velden, waardoor ze op de juiste plaats van het scherm komen. Daar produceren ze licht, en zo wordt het beeld gevormd dat we zien.

Er zijn nog twee eigenschappen van electronen die ik wil noemen. De eerste is de *spin* van het electron. Globaal gesproken kunnen electronen in twee 'soorten' voorkomen, die we spin-op en spin-neer noemen. Deze soorten kunnen in elkaar overgaan. De laatste eigenschap die zeer belangrijk is, is de *statistiek* van electronen. Kort gezegd komt het er op neer dat electronen van dezelfde soort niet op dezelfde plaats voor kunnen komen, zoals we ook gewend zijn van voorwerpen in het dagelijks leven. Deze eigenschap van electronen wordt het *uitsluitingsprincipe* genoemd. Golven, zoals bijvoorbeeld aan het oppervlak van water, maar ook radiogolven, hebben een heel andere statistiek. Dat dat zo moet zijn blijkt uit het feit dat deze golven ongehinderd door elkaar heen bewegen, zonder elkaar te beïnvloeden; golven kunnen zich dus wel op dezelfde plaats bevinden.

Zoals reeds opgemerkt is, kunnen electronen verschillende fases vormen, op dezelfde manier als water moleculen. Een voorbeeld is de toestand van electronen in een stuk metaal.

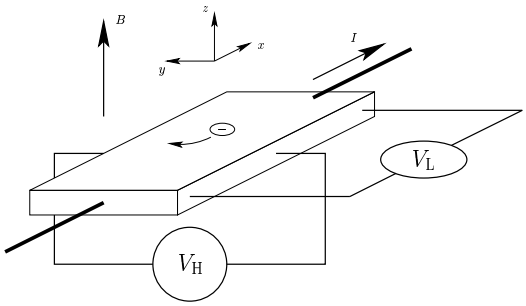


Figure S.1: Schematische weergave van een (quantum) Hall weerstands meting

In een (naïef) beeld kunnen de electronen in metaal als koper beschreven worden als zouden ze zich als een gas gedragen. In deze beschrijving wordt ervan uit gegaan dat de electronen elkaar niet beïnvloeden, een benadering die in sommige gevallen redelijk goed blijkt te werken.

Echter, de interactie tussen electronen is cruciaal voor de systemen waar ik de afgelopen jaren onderzoek naar heb gedaan. Dit zijn geen alledaagse systemen. Eén van de ongewone dingen is dat in de systemen waar ik naar gekeken heb, namelijk het grensvlak van een isolator en een halfgeleider, de electronen zich alleen maar in een plat vlak kunnen bewegen. In normale metalen kunnen electronen zich in alle drie de dimensies bewegen, maar in deze systemen is de vrijheid beperkt tot het zojuist genoemde grensvlak.

Tevens wordt er op deze systemen (of *samples*) een sterk magneetveld aangelegd en worden ze sterk afgekoeld. Het zijn dus niet de meest alledaagse omstandigheden, maar in een laboratorium kunnen ze gerealiseerd worden. Het is onder deze omstandigheden dat de electronen eigenaardige toestanden vormen, die zich nog het beste laten vergelijken met een vloeistof, echter wel een zeer eigenaardige vloeistof.

Om de vreemde eigenschappen van deze vloeistoffen duidelijk te maken, zal ik eerst een experiment beschrijven dat reeds aan het eind van de $19^{\mathrm{de}}$ eeuw werd uitgevoerd. Men neemt een dunne strip metaal, bijvoorbeeld van koper. Deze strip wordt in een magneetveld geplaatst, en er wordt een stroom door deze strip gestuurd (zie figuur S.1 voor een schematisch overzicht). Merk op dat in dit experiment de electronen in alle drie richtingen kunnen bewegen. De stroom door de strip gaat gepaard met een *spanningsverschil* $V_{\mathrm{L}}$ in de richting van de stroom, zoals gebruikelijk is voor normale metalen. Echter, door het magneetveld onstaat er ook een spanningsverschil $V_{\mathrm{H}}$ in de richting loodrecht op de stroom, de Hall spanning, genoemd naar de ontdekker Edward H. Hall [53]. Deze Hall spanning wordt veroorzaakt doordat de electronen door het magnetisch veld naar één kant van de strip worden afgebogen en er dus een ladingsverschil tussen beide kanten van de strip wordt opgebouwd. Als het magneetveld sterker wordt gemaakt worden de electronen sterker afgebogen, en daarmee wordt ook de Hall spanning groter (en wel recht evenredig); in formule

$$V_{\mathrm{H}} = cB \,, \tag{S.1}$$

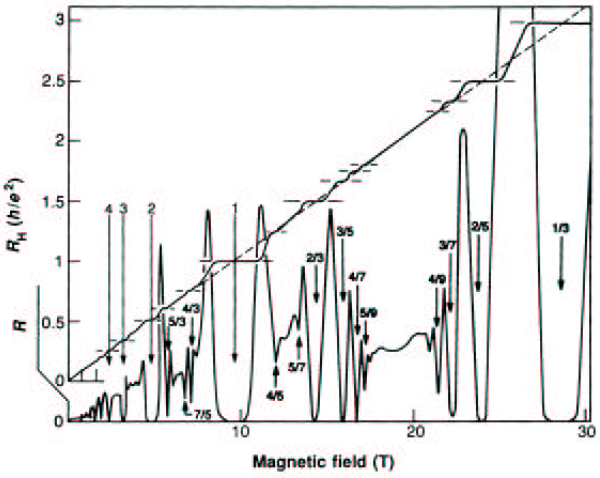


Figure S.2: Weerstands metingen aan een quantum Hall sample

waarbij $c$ een constante is die onder meer afhangt van de dichtheid van de electronen. Dit effect wordt het (klassieke) Hall effect genoemd.

Nu kunnen we de overstap maken naar de vloeistoffen die electronen kunnen vormen in samples waarin de electronen zich daadwerkelijk in een plat vlak bewegen. Deze vloeistoffen kunnen alleen ontstaan bij zeer lage temperatuur, minder dan één graad boven het absolute nulpunt. Bovendien moet het magneetveld zeer sterk zijn, zo'n miljoen maal zo sterk als het magnetisch veld van de aarde. Als we onder deze omstandigheden dezelfde soort metingen doen als E.H. Hall, dan vinden we iets opmerkelijks. De Hall spanning vertoont een heel ander beeld, er treden plateaus op, zoals te zien is in figuur S.2. De waarde van het Hall geleidingsvermogen (gegeven door $\sigma_{\rm H} = \frac{1}{R_{\rm H}} = \frac{V_{\rm H}}{I}$) op deze plateaus is zeer bijzonder, namelijk

$$\sigma_{\rm H} = \frac{p}{q}\frac{e^2}{h}\ , \tag{S.2}$$

waarbij $p$ een geheel getal is en $q$ een oneven getal. $e$ is de electrische lading, terwijl $h$ de constante van Planck is; beide zijn natuurconstanten. Dus de waarde van de Hall geleiding is een eenvoudige breuk vermenigvuldigd met natuurconstanten. Belangrijk om op te merken is dat dit gedrag voor verschillende samples, die qua details van elkaar kunnen verschillen, hetzelfde is. Ook de vorm van het sample is hierbij niet van belang, in tegenstelling tot *klassieke* systemen, waarin de geleiding sterk afhangt van de vorm van het systeem. Behalve de plateaus die waargenomen worden in de Hall weerstand, gaat de longitudinale weerstand bij dezelfde waardes van het magneetveld naar nul; het is alsof de stroom ongehinderd door het sample kan lopen, een effect dat lijkt op het gedrag van supergeleiders. Voor het *integer* quantum Hall effect, ontdekt in 1980, geldt dat $q = 1$. In het geval dat $q > 1$, maar wel geheel, spreken we van het *fractionele* quantum Hall effect. Om dit effect te kunnen verklaren is het van essentieel belang dat de electron interacties worden meegenomen in de

beschrijving. De toestanden die de electronen vormen onder deze omstandigheden worden ook wel *quantumvloeistoffen* genoemd.

De quantumvloeistoffen die de quantum Hall systemen beschrijven met $p = 1$ en $q$ een oneven geheel getal zijn bedacht en onderzocht door R.B. Laughlin en worden Laughlin toestanden genoemd. Zoals bij normale vloeistoffen ook het geval is, zijn deze quantumvloeistoffen moeilijk samen te drukken. Een andere analogie die met gewone vloeistoffen getrokken kan worden is het bestaan van golven aan het oppervlak van vloeistoffen. Voor de quantumvloeistoffen in twee dimensies kan een soortgelijk verschijnsel optreden, maar dan aan de rand van het sample. Dit soort golven noemen we *excitaties* van het systeem.

De eigenschappen van deze excitaties zijn niet te vergelijken met de eigenschappen van gewone golven. De quantumvloeistoffen zijn opgebouwd uit electronen, die geladen zijn. Echter, de lading die kan worden toegekend aan de excitaties is een slechts een fractie van de lading van de electronen. In het geval van de Laughlin toestanden is de lading $\frac{1}{q}$ maal de lading van het electron. Deze fractionele lading is waargenomen in experimenten, waarbij de excitaties van een kant van het sample naar de ander kant *tunnelen*. Hierbij veroorzaken de excitaties ruis, die vergeleken kan worden met de ruis die te horen is er hagelstenen vallen op een dak. De lading van de excitaties is van invloed op de ruis. Door deze ruis te onderzoeken is de lading van de excitaties gemeten, en deze blijkt inderdaad een fractie te zijn van de electronlading.

Ook binnen de quantumvloeistof kunnen geladen excitaties voorkomen; wederom kan deze lading een fractie zijn van de lading van het electron. Deze excitaties, of ook wel quasideeltjes, kunnen, althans in theorie, gecreëerd worden door locaal het magneetveld langzaam iets te verhogen. Het gevolg is dat de electronen zich 'een beetje' van deze plaats vandaan bewegen, en er een gat achter blijft. De lading van dit gat hangt nauw samen met het geleidingsvermogen op van het systeem, dat gegeven wordt door vergelijking (S.2). Zo is bijvoorbeeld de lading van deze deeltjes voor het systeem met $p/q = 1/3$ gelijk aan $1/3$ maal de lading van het electron. Dat er quasideeltjes kunnen bestaan met een lading die kleiner is dan de lading van de electronen die de toestand opbouwen kan alleen doordat heel veel electronen tezamen een toestand vormen, en als het ware samen werken om dit gedrag voor elkaar te krijgen.

Een andere, zeer belangrijke eigenschap van deze quasideeltjes is dat ze *topologisch* zijn. Dat houdt in dat hun aanwezigheid het gehele sample op een globale manier veranderd, en niet alleen in de buurt waar deze deeltjes zich bevinden. Het gevolg is dat kleine verstoringen in het systeem de eigenschappen van de quasideeltjes niet kunnen veranderen. Bovendien is de statistiek die we aan deze deeltjes toekennen bijzonder. Deze statistiek zit als het ware in tussen de statistiek van electronen, die elkaar volledig uitsluiten, en de golven, die ongehinderd op dezelfde plaats kunnen komen.

Lange tijd is het zo geweest dat het fractionele quantum Hall effect alleen werd waargenomen met plateaus waarbij het geleidingsvermogen gegeven wordt door formule (S.2), waarbij $q$ een oneven getal is. Dit is ook duidelijk terug te zien in figuur S.2. Het feit dat $q$ in al deze gevallen oneven is, is gerelateerd aan een fundamentele eigenschap van electronen, namelijk het uitsluitingsprincipe. Groot was dan ook de verbazing toen er een quantum Hall effect alleen werd waargenomen bij een waarde van $\frac{p}{q} = \frac{5}{2}$, zie figuur 1.2 op pagina 15.

Tegenwoordig wordt dit begrepen door aan te nemen dat in deze toestanden de electro-

nen als het ware paren vormen (ondanks het feit dat electronen elkaar afstoten). Voor deze paren geldt het uitsluitingsprincipe niet, en deze paren kunnen een toestand vormen die optreedt bij de waargenomen waarde van het geleidingsvermogen. Deze toestand is bedacht door G. Moore en N. Read, en wordt wel de Moore-Read toestand genoemd.

Het is gebleken dat de quasideeltjes die in deze toestand voor kunnen komen speciale eigenschappen hebben. Net zoals de quasideeltjes in de Laughlin toestanden hebben de quasideeltjes een lading die een fractie (namelijk $1/4$) van de lading van het electron is. Belangrijker is echter een eigenschap die we *ontaarding* noemen. Dit houdt in dat het plaatsen van identieke quasideeltjes op bepaalde posities op verschillende manieren kan gebeuren. Stel we plaatsen vier quasideeltjes (op vier vaste plaatsen), dan blijkt dit op twee verschillende manieren kan worden gedaan. Laten we deze twee manieren voor het gemak 'rood' en 'blauw' noemen. Door nu meerdere quasideeltjes achtereenvolgens om elkaar heen te draaien (om daarna weer uit te komen op de oorspronkelijke posities) kan het zo zijn dat het systeem overgegaan is van 'rood' naar 'blauw'. De volgorde waarin het de deeltjes om elkaar zijn gedraaid is van belang voor de uiteindelijke uitkomst. Dit zeer opmerkelijke gedrag (dat volgt uit de theoretische beschrijving van de toestand, en (nog) niet is waargenomen), komt niet voor bij de quasideeltjes die horen bij de Laughlin toestand. De mogelijkheid voor dit gedrag komt, uiteindelijk, voort uit het gegeven dat de electronen paren gevormd hebben. Ook het feit dat we systemen bekijken in twee dimensies onder invloed van een sterk magneetveld is van essentieel belang.

Met deze toestand zijn we aanbeland bij het onderzoek dat ik tijdens mij promotie heb gedaan. Dit gaat ook over speciale toestanden waarbij de electronen paren, of meer algemeen, clusters vormen. We hebben onderzocht welke toestanden er, in principe, mogelijk zijn. Het is gebleken dat de spin van het electron verschillende toestanden mogelijk maakt. Van de toestanden die we gevonden hebben, hebben we de eigenschappen in kaart gebracht. Hierbij hebben we speciaal gelet op de eigenschappen van de quasideeltjes, omdat deze afwijkend zijn van de 'normale' quasideeltjes in quantum Hall systemen. Eigenschappen waar we speciaal naar gekeken hebben zijn de topologische eigenschappen van de quasideeltjes, en ook de ontaarding, zoals die ook optreedt bij de toestand die hierboven is beschreven. Een van de mooie resultaten, die in het geval van de Laughlin toestanden al bekend was, is dat de topologische eigenschappen nauw verbonden zijn met de statistiek van de deeltjes. Deze statistiek op zijn beurt is weer nauw verbonden met de ontaarding. Deze verbanden hebben het mogelijk gemaakt om expliciete formules af te leiden die de ontaarding van de toestanden, waarin zich quasideeltjes bevinden, beschrijven. Het mooie is dat deze ontaardingen ook uit computersimulaties gehaald kunnen worden. Het is gebleken dat de resultaten van de computerberekeningen perfect overeen komen met de formules die gegeven worden in hoofdstuk 6.

Ondanks dat de meeste toestanden die in dit proefschrift beschreven zijn, (nog?) niet zijn waargenomen, is het onderzoek ernaar belangwekkend, omdat deze toestanden eigenschappen hebben, die nieuw zijn in quantum Hall systemen. Ook voorzien ze in mogelijke systemen, waarin deeltjes voorkomen met vormen van statistiek die nog niet zijn waargenomen. Dit onderzoek is dus niet zozeer van praktisch nut geweest, maar heeft wel bijgedragen aan het begrip van de mogelijke toestanden van materie die in de natuur voor kunnen komen.

# Dankwoord

Dit proefschrift zou niet tot stand zijn gekomen zonder de hulp van een aantal mensen, die ik hier graag wil bedanken. Veel dank gaat uit naar promotor, Kareljan Schoutens, die de ideale begeleider heel dicht heeft benaderd. Hij kwam altijd met interessante problemen en met ideeën om ze op te lossen. Het onderzoek ging vaak in sneltreinvaart. Zo gebeurde het tijdens het eerste jaar van mijn promotie dat Kareljan na een mooie ingeving zei dat er binnen tien dagen een artikel zou liggen. In eerste instantie wist ik niet of ik dat serieus moest nemen, maar binnen de gestelde termijn was het artikel een feit. Kortom, ik heb veel geleerd in de afgelopen vier jaar. Ook de ‘coaching’ voor de periode na dit boekje heb ik zeer gewaardeerd.

Het werken in de groep ‘Theorie van de gecondenseerde materie’ van Aad Pruisken en Kareljan Schoutens heb ik aangenaam gevonden. In het bijzonder wil ik Ronald noemen, voor de vele levendige discussies en de bijbehorende misverstanden over statistiek en alles wat daarmee te maken heeft. Ook de overige leden, Boris, Frank, Jasper, Mischa, Sathya en Stéphane hebben bijgedragen aan interessante discussies.

Peter Bouwknegt heeft mij de mogelijkheid geboden om interessant onderzoek te doen in Australië, waar ik veel heb geleerd en een mooie tijd heb gehad. Dichter bij huis, dat wil zeggen, gewoon in Amsterdam, was de sfeer op het instituut goed. Ondanks dat Jeroen en ik regelmatig ‘van huis’ waren, vond ik het leuk om hem als kamergenoot te hebben. Veel werd besproken, op het gebied van de natuurkunde, maar ook daarbuiten. Het lijkt er helaas op dat op het laatste moment asbest ons toch nog uit elkaar drijft.

Discussies met Joost waren zeer nuttig, omdat hij vaak met goede vragen kwam. Mark en Mischa stonden altijd klaar om mijn gebrekkige kennis op het gebied van computers aan te vullen.

Buiten het instituut om hebben vele vrienden bijgedragen aan dit proefschrift, zei het op indirecte wijze. Vaak was ik te vinden op het Parachutisten Centrum Midden Nederland, waar ik op prettige wijze mijn hobby heb kunnen uitoefenen.

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