Royal Institute of Technology

Non-abelian anyons in Toric-code like models

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Master of Science Thesis

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Stockholm, Sweden 2017

Typeset in LaTeX

Examensarbete inom ämnet Teoretisk Fysik för avläggande civilingenjörsexamen inom utbildningsprogrammet Teknisk Fysik.
Graduation thesis on the subject Theoretical Physics for the degree of Master of Science in Engineering from the School of Engineering Physics.

Cover illustration: Braided world lines traced out by anyonic quasi-particles in space-time.

TRITA-FYS 2017:39
ISSN 0280-316X
ISRN KTH/FYS/–17:39—SE

Printed in Sweden by Universitetsservice US AB

Abstract

The toric-code is an example of a stabilizer error correcting code that hypothetically can be used in topological quantum computing. More specifically, it is defined on a two dimensional $\mathbb{Z}_2$ lattice wrapped around a 2-torus. Due to the special boundary conditions possessed by objects of such topology, one feature of the system is that it's topologically ordered. It also has a gapped degenerate ground state that can be used to represent qubits, which simply is short for quantum bits, in which topologically protected quantum information can be encoded. The excitations of the model can be regarded as errors which is detected through measurements by means of local operators. The system can also be considered as a string-net condensate where the excitations are located in the ends of the strings. These quasi-particle excitations are of anyonic nature with non-trivial braiding statistics. Hence, when tracing out the world lines by excitations that are interchanged, the paths will form knots with non-trivial topology which allows for an arbitrary complex phase to be acquired by the quantum state. Because of this one can use different braids as implementations of logical operators so that the information can be processed. In this thesis work, however, the main focus lies on generalizations and manipulations of the original toric-code. In order to perform powerful quantum computations the braiding statistics has to be non-abelian and since this requirement is not met by the original toric-code, the system has to be manipulated somehow. It has been shown that topological defects, or twists, can be created by performing dislocations in the lattice which mimics the behaviour of non-abelian anyons. In this system the concept of fusion can be used to represent protected qubit states since the fusion outcome that corresponds to two or more twists always is indefinite. Further we will generalize the lattice from having only two degrees of freedom per lattice site, to a system which comprises an arbitrary number degrees of freedom per lattice site. The excitation spectrum will be derived, and we will also see how to move these excitations around which allows us to compute the braiding rules and the fusion category of the model.

Acknowledgement

This thesis work was conducted towards the fulfillment of a masters degree in the subject of Theoretical Physics and I want to thank Prof. Eddy Ardonne for agreeing to supervise me through the project, and for always taking time out to answer questions and for helping me unravel various issues I encountered along the way. I also want acknowledge some of the lecturers and teachers whom I found especially inspiring during my time as a physics student for many interesting lectures and tutorial sessions. These persons are: Nicholas Apazidis, Jack Lidmar, Erik Lindborg, Patrik Henelius, Mats Wallin, Tomas Karlsson, Mats Boij, Sushant Raut, Egor Babaev and Edwin Langmann among others. Moreover I want to take this opportunity to show my appreciation to my buddy Jesper Lindfors for sharing his expertise and helping me out with some graphics related problems and to Mikael Scott for many interesting discussions and for showing me around in his cool laser physics lab. Finally I want to thank my family for all support over the years and for believing in me.

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Chapter 1

Introduction

The last century has truly been an era characterized by success in the field of physics. Not only did Einstein formulate the mathematical foundation of the theories of relativity, which forced us to abandon our view of space and time, but also the theory of quantum mechanics, which perhaps was even more groundbreaking, was developed as well. These theories were two crucially important milestones in the development of physics, which today constitute the basis of the modern theory. Just like relativity, quantum mechanics is a beautiful subject, which has provided us with a peephole into the microscopic world, and revealed that the very nature of our physical reality reaches far beyond the human intuition. Nevertheless, apart from all philosophical aspects, it does also make up one of the cornerstones of modern technology and engineering. Almost all electronic devices are built up from components, such as semi-conducting transistors, in which quantum mechanics plays a key role. Another interesting, and ingenious, application is in the field of computation which has given rise to the emergence of a new discipline in physics, which we nowadays refer to as *quantum computing*. The field of quantum computing is a vibrant research area, in which physicists are trying to make use of the superposition feature that is possessed by systems in the quantum regime, and use such system to represent quantum bits, or *qubits*, of information. This would make it possible to process a huge amount of data if compared to a classical computer, which will result in an explosive amplification in terms of computational power. However, in order to wrap ones head around the main idea behind quantum computing, one will probably have to abandon the way one thinks about a conventional computer to fully grasp the concept. Not only is the construction and the components different from a classical computer, but the basic principles at a theoretical level as well. The main approach is way more physical than in classical computer science, which means that one will have to regard the subject from the eyes of a physicist rather than from the ones of a computer scientist.

In this thesis work I have studied a special type of hypothetical quantum computer, which is at its heart based on the mathematical principle of topology, hence the name *topological quantum computer*. The field of topological quantum computing is an extremely wide and diverse subject which combines several disciplines of physics and mathematics at a fundamental level together with cutting edge engineering. For this particular reason I find the subject not only to be challenging, but also very rewarding and fruitful. It may be natural, however, to question the whole idea about quantum computing, and why it might benefit us. The variety of amazing things that can be achieved with a modern classical computer is numerous

and it is without doubt that the computer has shaped out society ever since the British mathematician Alan Turing invented it during the second world war. It was indeed a great achievement which had a major impact on the war and how it finally could be brought to an end. Aside from this historical remark, the computer has continued to serve the human society ever since, and one could maybe even say that it is responsible for the paradigm shift that has taken place in science and technology during the second half of the nineteenth century. So what do we need quantum computers for? Are not classical computers enough? Even if modern computers are powerful enough in most cases, the quantum computer would certainly open doors to a completely new world with new opportunities. Basically, a quantum computer can tackle any problem on a scale far beyond the best classical computers that we have today. The potential is enormous. But of course, there is a catch. Just like a classical computer is prone to errors, so is a quantum computer. This means that even if we succeed to build the computer itself, we will still need error correcting codes that correct the errors in order for the computer to operate at a satisfactory level. The so called *Toric-code* is a well studied example of such a code which is defined on a torus-shaped two dimensional spin-$\frac{1}{2}$ lattice. One powerful features of this system is that it possesses topological order, which makes the phases gapped and the ground state degenerate [20]. These properties make the realization of topologically protected qubits possible so that information can be stored safely. It also possesses *anyonic* excitations with fractional exchange statistics which, as we shall see, can be used to implement logical gates that allows the computer to process the information.

The study of topological phases is a remarkably interesting research area which has proven that matter is even more complex than what we first thought. According to Landau's theory, different states of matter can be characterized by an order parameter which value will vary with the temperature so that the free energy is minimized, and the system remains in equilibrium. Moreover, as the temperature is lowered, the order parameter will go from zero to a non-vanishing value when the system leaves the disordered phase, and enters the the ordered one. When such transitions take place, certain symmetries possessed by the disordered phase will brake which thus act as a signature of the phase transition. By studying broken symmetries, group theory has allowed us to classify 230 different three dimensional crystal structures in nature [8]. One good, and well known example of this, is when water freezes to ice. In the liquid state the system possesses continuous transnational symmetry, but as fast as the temperature is lowered below $0°C$, the liquid will start to crystallize into an ordered structure which only have discrete transnational symmetry. Landau's theory is also applicable to more exotic states of matter such as Bose-Einstein condensates, superfluids and superconducting phases. For a long time we thought that all types of phases could be distinguished by means of Landau's theory, but that idea had to be abandoned when topological phases of matter, and general quantum phases, were discovered which gave rise to some new types of quantum materials that couldn't be classified in this way. Such phases are very different in their nature which sometimes requires a more delicate mathematical framework. Quantum phases that cannot be described adequately with Landau's theory are distinguished by projective symmetry groups, or *PSG's*, which means that different quantum phases still can possess the same symmetry but different PSG's [37]. In the same way as broken symmetries have allowed us to classify different

crystal structures, the PSG description has allowed us to identify over 100 different spin liquids which all have the same symmetry [37]. Anyway, the excitement of quantum and topological order does not stop here. Some indications are pointing towards the existence of new quasi-particles that might be even more fundamental than those we know today in the Standard model [37]. It has also been proven that wave equations, such as the Navier and Euler equations for quantum solids and liquids, can be recovered by considering different kinds of organizations, or orders, of particles [35]. Even Maxwell's equations of light in electrodynamics can be found by organizing bosons into string-net condensates, so that the motion of the waves in the string-net liquid is governed by just this set equations [35, 38]. It seems like different types of order leads to different kinds of waves and particles, so maybe we have looked at it wrong and all elementary particles originates from more exotic orders at the most fundamental level. If this is the case, we will face a new revolution in physics, and once again, we will be forced to fundamentally change the way we look at the universe.

Chapter 2

Quantum mechanics as a means for computation

The main purpose of this Chapter is to review the most important aspects of quantum computing and error-correction theory. We will start off by comparing classical bits, and the their quantum mechanical counterpart, the qubits, and motivate why qubits are superior to bits in terms of information storage capacity. We will then continue the Chapter by providing a review of classical error-correction theory and then generalize the idea to a quantum system, in which the information is encoded in qubits, and see how errors in such systems can be detected and treated.

2.1 Bits vs. Qubits

2.1.1 Bits

In a conventional computer the fundamental concept of information are known as *bits* [25], which constitute the basic units. A bit can take any of the two values 0 or 1 and can thus be used to answer binary questions where we, at least, will half the uncertainty of the answer for each step [13]. A very basic example where one single bit is enough to answer the question completely, is what the outcome will be from one single coin toss. We already know in beforehand that the answer will be either heads or tails, so if we assign, for example, 1 to the answer heads and 0 to the answer tails, we won't need to ask any further questions since even if we answer wrong, we know that the correct answer must be the other. In terms of binary trees, this process will only have two branches. However, in more complicated cases we will only be able to reduce the uncertainty to half of the initial. To illustrate this with a simple example, imagine that you are asked to guess a number between one and six. Then you can use the numbers 1 and 0 to ask whether the value is equal to 4 or greater, or if the value is less than four. If you, for example, ask if it's less than four and you are right, you will know that the correct answer will be either one, two or three. This procedure can be repeated until you are only left with two choices, just like in the coin tossing example, and you will find out the correct answer in the next step. This example would require three bits in order to reach the correct answer with hundred percent certainty and in the corresponding binary tree there would exist $2^3 = 8$ distinct paths or possibilities. Thus, if we know how many possibilities there are we can go backwards to figure out how many questions that

are needed in order to find the right answer. Since there always are 2 possibilities in each step, and we know that the total number of possibilities is k, then we just have to solve the equation $2^n = k$ for n, which can easily be done by taking the logarithm in base 2 of each side which yields $n = log_2(k)$. So in conclusion, if the total number of possibilities k is known, the number of questions that have to be asked in order to arrive at the correct answer is given by $n = log_2(k)$ for any value of k. However, in order for the computer to execute the requested computation, the information has to be processed properly which is usually done by means of logic circuits. Such circuits are built up from logic gates connected by wires through which the information-encoded current is running. These logic gates are, in theory, based on Boolean logic and are implemented by transistors which determine the output signal. Basically, there are three fundamental logic gates known as the OR-gate, the AND-gate and the NOT-gate. The OR-gate takes two signals as input and if either of the signals has value 1 (true in Boolean logic), the output signal will be 1 as well. If, on the other hand, both inputs are 0 (both are false in Boolean logic), the output will be 0. The AND-gate also takes two inputs and the output will only be 1 if both inputs are 1, and 0 otherwise. The NOT-gate, however, works a little bit different. This gate only takes one signal as input and its operation flips the input value. Hence, if the input is a 1 the output will be a 0, and vice versa.

2.1.2 Qubits

Quantum information is based upon an analogous concept called *qubits*, which is simply short for quantum bit [25]. Just like a bit, a qubit has states as well, but as we shall see it's much richer in the sense of information. We are also forced to employ some more mathematics due to the more abstract nature of qubits. If we start by just regarding a qubit as a mathematical object without, yet, any physical significance, we can define it as

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \tag{2.1}$$

where $|0\rangle$ and $|1\rangle$ correspond to different quantum states (in the Dirac notation) and α and β are two complex numbers [25] satisfying $|\alpha|^2 + |\beta|^2 = 1$, due to the probabilistic normalization. So how is Equation (2.1) to be interpreted? We see that in the special cases when either α or β is zero, the qubit must either be $|1\rangle$ or $|0\rangle$, respectively, and hence it behaves like a bit. But α and β can take any value in the complex plane as long as $|\alpha|^2 + |\beta|^2 = 1$ is satisfied which means that a qubit, mathematically speaking, in general is a linear combination of the two orthogonal states which, in this context, also are known as the *computational basis states* [25]. We say that a qubit can be in a simultaneous quantum superposition of two states, which entails that if we generalize to a multiple qubit system with n quibits, it can simultaneously be in 2^n states, whereas n bits still only can be in one state, so n qubits corresponds to 2^n classical bits in total. Hence we may say that qubits obey "quantum mechanical" logic rather than Boolean logic. Moreover, since a qubit state is constructed from two basis states, it means that it lies in a plane, or in a two dimensional Hilbert space $\mathcal{H}^2$. Thus, in the general case with n qubits the full 2^n-dimensional Hilbert space is given by the tensor product of each one of the

individual qubit Hilbert spaces, i.e.

$$\mathcal{H}^{2^n} = \bigotimes_{i=1}^{n} \mathcal{H}_i^2 = \underbrace{\mathcal{H}_1^2 \otimes \mathcal{H}_2^2 \otimes ... \otimes \mathcal{H}_n^2}_{n\,times}, \tag{2.2}$$

which is spanned by all possible direct products of the computational basis states on the form $|i_1, i_2, .., i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes ... \otimes |i_n\rangle$, where each $i_k \in \{0, 1\}$ [14]. Just like in a classical computer, a quantum computer also needs logic gates that have to be implemented somehow. The OR- and the AND-gates basically work, at least at a mathematical level, analogous to the classical ones [25]. The NOT-gate, however, requires some more cautiousness. From a mathematical point of view we want to define an operator U that turns the $|0\rangle$ into the $|1\rangle$ in the qubit state, and vice versa [25]. The question we have to ask ourselves is how such an operator acts on the qubit, and the answer is that it acts linearly [25], i.e.

$$U: \quad |\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \rightarrow |\psi'\rangle = \alpha'\,|1\rangle + \beta'\,|0\rangle\,. \tag{2.3}$$

This linearity may seem natural but the truth is that it's a non-trivial feature which lies in the nature of quantum mechanics, and is by no means obvious[1] [25]. If we further think about the mathematical properties such a transformation must have, one may conclude that it has to be unitary, due to the normalization condition: $|\alpha'|^2 + |\beta'|^2 = 1$. In other words it must satisfy $UU^\dagger = I$ where $U^\dagger$ being the Hermitian conjugate of U and I being the dientity operator. This is because all unitary transformations have eigenvalues lying on the unit circle in the complex plane so that the transformation won't affect the normalization. However, there is a catch. In the end of the computational process the quantum state is measured which means that it will be forced to collapse into one of its basis states with a certain probability, due to one of the fundamental postulates of quantum mechanics. This is unwanted since all the other information will get lost so the trick is to protect the information in a way so that the qubit is in a basis state when the measurement is carried out.

Worth to point out though is that it's not the speed of each individual computation that is faster in a quantum computer than in a classical one, it's the number of computations required to arrive to the final result that is exponentially higher in a classical computer which makes the quantum computer way more efficient, in most cases. It is now about time to go into the physics and ask ourselves how a quantum bit can be realized physically. As we already have discussed a qubit is a superposition of two distinct quantum states so to realize a qubit we have to initialize a quantum system in such a way that this is satisfied. This can be achieved in many ways and to get a feeling of how such a realization can be obtained a few examples will be mentioned. For example one could prepare a system of atomic nuclei in a uniform magnetic field such that their spins are pointing along the direction of the field which thus corresponds to the ground state. Then an orthogonal oscillating field with the resonance frequency of the nuclei can be applied for a short period of time, so that the spins will be excited "half way" to the full excited states. Thus, each nucleus in the system will end up in a quantum superposition of the ground state and the excited state which results in a system that realizes a set of qubits. One could also use the two polarizations of a photon as states [25, 7], or excite an

[1] It is also very well motivated empirically [25].

electron in an atom half way between two quantized energy states by radiating a beam of light of suitable frequency on it [25, 7]. Nevertheless, in this thesis work we are going use a concept called *fusion* to represent qubit states. Briefly one can say that the main idea is built upon bringing two anyons together, or fuse them together, which can result in several distinct outcomes that defines the two qubit states. This idea will be discussed more rigourusly in Chapter 4. Also note that since it's possible to initialize a quantum system in not only two states, but in an arbitrary number of states, the concept of a qubit can be further generalized to a state living in a Hilbert space of arbitrary dimension so that

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle + ... + \alpha_m |m\rangle = \sum_{i=0}^{m} \alpha_i |i\rangle . \tag{2.4}$$

This generalization of a qubit is known as a *qudit*, and since the corresponding qudit Hilbert space is m^n dimensional the amount of information that can be encoded in a general qudit increases dramatically with m which thus allows for even more powerful computations.

2.2 Algebraic coding theory

Before proceeding to the Quantum double models in Chapter 5, which might be considered to be the essential part of this thesis work, a brief introduction to error correction theory is provided here. We shall start our discussion by introducing some basic concepts in classical error correction theory and then we will advance towards quantum error correction theory. One of the major problems that need to be solved in order to succeed with the quest of building a well working quantum computer, is to overcome the challenge of constructing an error correction code that allows the computer to store clean information in the memory. In general, it's impossible to isolate a physical system completely from its surroundings which, of course, will lead to irreducible noise, due to various interactions with the environment. Hence, we must approach this problem from another angle and instead aim at constructing a scheme that automatically deals with errors and tries to correct them. Such schemes are known as *error correcting codes* and these codes play a crucial role in the process which makes it possible perform satisfactory computations. Throughout the Sections in this Chapters we will mainly follow Chang (Ref. [7]) and Preskill (Ref. [28]).

2.2.1 Classical error correction theory

We shall start this Section by providing an extremely simple example to keep in mind along the way. Imagine that you receive a letter from an acquaintance and when you open the envelope and starts to read it, you realize that for some reason a few letters are missing. In that case, would you be able to read it anyway? The answer is probably yes, even if it of course depends on how many missing letters there are. When you try to read the text your brain will automatically try to fill in the gaps in such a way that each word makes sense in the sentence it corresponds to. So even if pieces of information are missing, or somehow distorted, it can still be possible to decode a message. This is the very essence of what error correction

coding really is about. First the code has to identify the error and then it will, logically, figure out the most probable pieces of information that is missing. In that way the computer can store information safely whilst it waits for the opportunity to come when it might be needing it. When a computer is working it often has to compare data obtained from computations that are separated in time, which means that it has to store some information temporary. Therefore, if some information get lost or distorted whilst the computer is processing other computations that are necessary, it is highly likely that the final output will be complete rubbish. In classical information systems, bits of information are transmitted through channels [7] in which they are subject to noise, which can cause the bits to flip so that a 0-bit becomes a 1-bit or the other way around [7]. Therefore, the received piece information might not be the same as the one that was sent. This process can be illustrated in a so called block-diagram which is depicted in Figure 2.1.

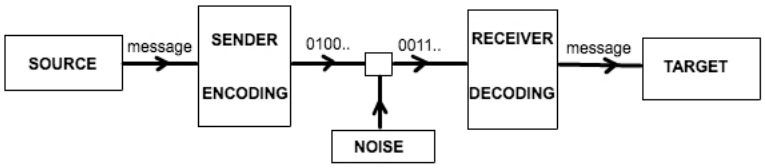


Figure 2.1: *Block-diagram illustrating the effect of noise that the channel is subject to.*

Furthermore, by assigning the probability p to the event corresponding to a flip of the bit, we can formulate a model of the noisy channel (see Figure 2.2), called the *binary symmetric channel* [7], to illutrate the main idea behind error correction.

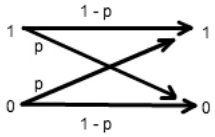


Figure 2.2: *Diagram over the binary symmetric channel. Each bit will flip with probability p and thus remain the same with probability $1 - p$.*

At this point we have only discussed error correction in a superficial way and it's time to phrase it in mathematical terms, which can naturally be done by invoking some linear algebra. If we let $\mathbf{A}$ be a set containing information which constitutes of strings of bits, and $\mathbf{E}$ be an encoding operator which maps $\mathbf{A}$ onto a code space $\mathbf{C}$, then the elements contained in $\mathbf{C}$ are the *code words*. We may also introduce a set of noise oparators $\mathbf{N} = \{I, N_1, N_2, .., N_m\}$ [7], where I is an identity operator which simply maps each element back onto itself. This set of noise operators transforms the set of code words $\mathbf{C}$ into a new set $\mathbf{C}_{noise}$ which is a collection of noise corrupted code words [7]. We also need a decoding operator $\mathbf{D}$ which decodes the noise corrupted code words in $\mathbf{C}_{noise}$ into strings of bits [7]. Hence, the received decoded code words will be subject to errors which have to be corrected. Therefore an error correcting code need to be designed and a basic concept that such codes often are based on is the concept of *adding redundancy* [7]. This basically means that additional information is added so if some strings get distorted by noise, they can be recovered again. An example could be that a 1 is added to the end of the message when the sum of the

bits in the string is odd, and a 0 otherwise. Thus by summing up the bits in the received string we know if the string has been distorted or not. Now, let the set $v_1, v_2, .., v_k$ constitute a basis of $\mathbf{C}$ such that any code word in $\mathbf{C}$ can be written as an n-dimensional linear combination in this basis, i.e. [28]

$$c(\alpha_1, \alpha_2, .., \alpha_k) = \sum_{i=1}^{k} \alpha_i v_i, \tag{2.5}$$

where $\alpha_i \in \mathbb{Z}_2 \; \forall \; i$. Moreover, we can introduce a so called *generator matrix* $\vec{G}$ [28, 7] in which we put the basis vectors v_i so that

$$\vec{G} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix}. \tag{2.6}$$

Now, with this matrix notation introduced, each code word in Equation (2.5) can instead be expressed as [28]

$$c(\alpha) = \alpha \vec{G}, \tag{2.7}$$

where $\alpha = (\alpha_1, \alpha_2, .., \alpha_k)$. The transformation in Equation (2.7) can be regarded as a map f that takes a k-bit message α and maps it onto a code word of length n [27]

$$f: \quad \mathbb{Z}_2^k \rightarrow \mathbb{Z}_2^n. \tag{2.8}$$

In other words, the generator matrix encodes the k-bit message α in the set of code words $\mathbf{C}$. Furthermore, we may introduce an $(n-k)\times n$ matrix $\vec{H}$ which satisfies [28, 7]

$$\vec{H} c(\alpha) = 0, \tag{2.9}$$

for all vectors $c \in \mathbf{C}$, and only those vectors [28]. If we can find such a matrix, it holds for any element in $\mathbf{C}$ that is subject to noise, and hence mapped onto $\mathbf{C}_{noise}$, that Equation (2.9) will not be satisfied and we thus know that the information is damaged. This matrix is a so called *parity check matrix* of $\mathbf{C}$ [28]. Formally this means that if we add an error vector e to a string c and act with $\vec{H}$, we have that

$$\vec{H}(c+e) = \vec{H}c + \vec{H}e = 0 + \vec{H}e = \vec{H}e \neq 0, \tag{2.10}$$

which is called the *syndrome* [28]. Moreover, if we form a set of errors $\epsilon = \{e_1, e_2, .., e_m\}$ it is always possible to recover the original information if and only if all syndromes $\vec{H}e_i$ are unique, or equivalently, for any two distinct errors e_j and e_k, it must hold that $\vec{H}e_j \neq \vec{H}e_k$ [28]. The message can thus be recovered by flipping back the bits which simply is done by adding the errors to the noise affected bits and then take that value modulo 2, i.e.

$$c(\alpha) + e_i \rightarrow (c(\alpha) + e_i) + e_i \equiv c(\alpha) \; (mod \, 2). \tag{2.11}$$

So if two syndromes corresponding to two distinct errors e_i and e_k are equal, there is a possibility that we are led to the "wrong" error so when we try to fix it, we may perform the following operation

$$c(\alpha) + e_j \rightarrow (c(\alpha) + e_j) + e_k \not\equiv c(\alpha)\ (mod\ 2), \tag{2.12}$$

and hence we fail to recover the original message.

2.2.2 Quantum error correcting theory

Just like a classical computational processes suffers from errors originating from various sources, so does a quantum computational process. The fact is that a process executed by a quantum computer is extremely fragile since it's not only prone to errors coming from imperfections in the operations performed on the information, but errors, or information loss, due to quantum mechanical decoherence as well. Thus, in order to build a feasible quantum computer one needs to construct error correcting codes that take care of both these types of errors from different origins. Similarly, as in the case of classical bits, the concept of redundancy can be used in quantum error correction where we have to construct a map between the k-dimensional qubit space $\mathcal{H}^k$ to a "redundant" n-dimensional code space $\mathcal{H}_c^n$, where $n > k$ [28]. This redundancy is implemented by adding qubits called *ancilla qubits* [7] to the qubit state by taking the tensor product. Thus if let $\mathcal{H}_a$ denote the ancilla qubit space, the full code space is given by $\mathcal{H}_c^n = \mathcal{H}^k \otimes \mathcal{H}_a$. A general state in this space is thus denoted by

$$|\psi\rangle \otimes |a\rangle = (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |a\rangle\,, \tag{2.13}$$

where $|a\rangle$ usually is set to $|0\rangle$. Before an error is to be corrected one has in some way detect it first, just like in classical error correction theory. So how do we do that? The task is to construct decoding operators D, such that if we let N be a noise operator and E be an encoding operator, we aim to obtain the following map [7]

$$DNE: \quad |\psi\rangle \otimes |a\rangle \rightarrow |\psi\rangle \otimes |s\rangle \tag{2.14}$$

where $|\psi\rangle$ is the piece of information we would like to retrieve and $|s\rangle$ is the syndrome we wish to associate to the noise N, so that the error can be corrected [7]. Now, the next question to answer is how we can design such a decoding operator. To answer this question we first need to introduce a set of matrices known as the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.15}$$

However, in the context of quantum computing it is customary to use the following notation: $\sigma_x = X$, $\sigma_y = Y$ and $\sigma_z = Z$. Therefore this will be employed form now on. Together this collection of matrices spans the $\mathfrak{su}(2)$ Lie algebra defined by the following commutation relations[2]

$$[X, Y] = 2i\epsilon_{xyz}Z \quad \{X, Y\} = 2\mathbb{1}\delta_{xy}, \tag{2.16}$$

[2]Here ϵ_{xyz} denotes the Levi-Civita permutation symbol.

which after performing an infinite power series expansion, exponentiates to the $SU(2)$ Lie group. The significance of this set of matrices, in this context, is that they represent observables in the qubit Hilbert space, which further implies that the pure qubit states are eigenstates of these matrices, with corresponding eigenvalues ± 1. Lets now focus on X and Z. If we consider these two matrices we note that when acting on a general qubit state $|\psi\rangle$, they will transform the state according to

$$X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle, \quad Z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle \tag{2.17}$$

so we see that the X operator flips the qubit whereas the Z operator contribute by a flipped phase. Furthermore we have that $Y = iXZ$, so the Y operator can be interpreted as phase flip followed by a qubit flip. In fact any second order unitary transformation can be regarded as a qubit error, but since the Pauli matrices spans the algebra, any error can be mathematically formulated as a linear combination of the Pauli basis. In conclusion, we thus only have two distinct types of errors: qubit flip and phase flip. Let us now consider the code space $\mathcal{H}_c$. In view of Equation (2.9) in the previous Subsection, where the parity-check matrix was defined, a similar object can be defined for qubits of quantum information in this space. Assume that we have a set of objects that, under action, leaves the states in the code space invariant if and only if the state in question is unperturbed. This would mean that errors could be detected upon action of such objects on the states in the code space, similarly as with parity-check matrices in the classical case. In fact, one can easily show that such a set of objects possesses group structure and if we let $|\psi\rangle$ be a state in the code space, we can define this group as

$$\mathcal{S} = \{S_i \mid S_i|\psi\rangle = |\psi\rangle \ \forall \ S_i\}. \tag{2.18}$$

In group theory one says that S_i stabilizes $|\psi\rangle$, since it leaves the state invariant under group action. Moreover, the stabilizer of $|\psi\rangle$ is the subset of all elements in the Pauli group [28] that maps $|\psi\rangle$ to itself when computing the orbit under the action of the group. Thus, since this set of Pauli operators possesses group structure we will for that reason call this set the *stabilizer group* of $|\psi\rangle$. In addition, error correcting codes that are based upon this idea are known as *stabilizer codes*. Now, for instance, let us assume that qubit 1 or 2 has been subject to an error X_1 or X_2. Thus by acting upon such a perturbed state with the operator Z_1Z_2, the error can be detected since X_i and Z_j anti-commutes if $i = j$ and commutes otherwise. This is because tensor products are implied so that

$$X_i = \underbrace{I \otimes I \otimes .. \otimes I}_{i-1 \ factors} \otimes X \otimes \underbrace{I \otimes .. \otimes I \otimes I}_{n-i \ factors}$$

and

$$Z_j = \underbrace{I \otimes I \otimes .. \otimes I}_{j-1 \ factors} \otimes X \otimes \underbrace{I \otimes .. \otimes I \otimes I}_{n-j \ factors}.$$

Hence, in general, if we have a perturbed state $|\chi\rangle = E_i|\psi\rangle$, where E_i being the error, we have that $S_i|\chi\rangle = S_i(E_i|\psi\rangle) = (S_iE_i)|\psi\rangle = -(E_iS_i)|\psi\rangle = -E_i(S_i|\psi\rangle) = -E_i|\psi\rangle$, so $E_i|\psi\rangle$ is an eigenstate of S_i with eigenvalue -1 [15]. As a result, if we

measure S_i of the perturbed state we will get [15]

$$\langle\chi| S_i |\chi\rangle = \langle\psi| E_i^\dagger S_i E_i |\psi\rangle = \langle\psi| (-S_i E_i^\dagger E_i) |\psi\rangle = -\langle\psi| S_i |\psi\rangle = -\langle\psi| |\psi\rangle = -1, \tag{2.19}$$

where the unitarity property of E_i, i.e. $E_i^\dagger E_i = E_i^{-1} E_i = 1$, and the orthogonality condition of the states $\langle\psi_a| |\psi_b\rangle = \delta_{ab}$, were applied. Thus if we measure the eigenvalue of the stabilizer operator we know that the state is perturbed if the eigenvalue is -1, and unperturbed if the eigenvalue 1. In conclusion the two types of errors can be detected by measuring the eigenvalues of the operators in the stabilizer group, which depends on whether the ancilla qubit was flipped or not [28], and can thus be corrected upon action of an appropriate Pauli matrix.

Chapter 3

Particle statistics and topology

In the Standard model of particle physics, we only allow for a certain collection of elementary particles to occur naturally in the universe. However, it has been proven that under very special conditions, such as in two dimensional materials, the emergence of other types of particles, or more exactly *quasi-particles*, becomes possible. These particles are called *anyons* which possess some rather special features that will be discussed in this Chapter.

3.1 Fermionic and bosonic statistics

The question "what is an anyon?" might be easiest to answer by explaining what an anyon is not. Already in quantum mechanics and particle physics at a very introductory level, students learn that elementary particles generally are categorized into two fundamental groups known as *fermions* and *bosons*, and what characterizes the particles in these two distinct groups are their spin properties and how their joint wave functions behaves under particle exchange. We have learned that fermions are half-integer spin particles and that their wave function exhibits anti-symmetry under particle interchange, while on the other hand, bosons are integer spin particles with a symmetric wave function under interchange of two identical particles [4]. This relation between the spin of the particles and the interchange symmetry is known as the *spin statistics theorem* [26] and is one of great achievements of relativistic quantum field theory. One says that fermions obey Fermi-Dirac statistics and bosons obey Bose-Einstein statistics. However, this theorem is only valid in three spatial dimensions or more, and it has been shown that in two dimensional physical systems[1], more exotic excitations are possible. These excitations, which are called *anyons*, can be regarded as types of quasi-particles and one feature that makes these particles so special is their nature under particle interchange. As already mentioned, the wave functions corresponding to fermions and bosons are very restricted in terms of interchange symmetry, since they have to be anti-symmetric and symmetric respectively under interchange. Formally this means that if we let a unitary permutation operator $P_{i,j}$, which swaps the particle positioned at x_i and the particle positioned at x_j, act on a fermionic n-particle state $\Psi_f(x_1, x_2, .., x_i, .., x_j, ..x_n)$, we get a new state

$$P_{i,j}\Psi_f(x_1, x_2, .., x_i, .., x_j, .., x_n) = -\Psi_f(x_1, x_2, .., x_j, .., x_i, .., x_n), \tag{3.1}$$

[1] Or in one dimension.

while for a bosonic n-particle state $\Psi_b(x_1, x_2, .., x_i, .., x_j, ..x_n)$ the new state becomes

$$P_{i,j}\Psi_b(x_1, x_2, .., x_i, .., x_j, .., x_n) = \Psi_b(x_1, x_2, .., x_j, .., x_i, .., x_n), \tag{3.2}$$

for any positive integers $i, j \leq n$. Furthermore, since the particles are *indistinguishable*, the Hamiltonian of the system will be left invariant under such permutation operations [4].

3.2 Exotic statistics

In our natural three dimensional world, according to the spin statistics theorem, the only possible excitations are either of fermionic or bosonic nature. Thus, it may be natural to ask: what happens if we create a two dimensional surface embedded in the three dimensional space? If this can be achieved, particles created on this surface will get trapped, and thus constrained to move around in two dimensions where the spin statistic theorem no longer is valid. Due to this, more exotic excitations are possible to appear which we will try to justify in this Section[2].

3.2.1 Topological considerations about the configuration space

In this thesis work we will only present a superficial motivation for the emergence of exotic excitations and will not go into the details so for a more detailed treatment, see Leinaas and Myrheims paper in Ref. [22]. Basically, the main idea that their argument was based upon, was that the particle configuration space will be affected by the special boundary conditions in two dimensions. A general configuration space V_n for n distinguishable particles simply is the cartesian product of n single particle configuration spaces $\mathbb{R}^2$

$$V_n = \underbrace{\mathbb{R}^2 \times \mathbb{R}^2 \times ... \times \mathbb{R}^2}_{\text{n times}}, \tag{3.3}$$

where $\mathbb{R}^2$ denotes the ordinary two dimensional Euclidean space. However, if the particles are identical this space will be redundant, since basic combinatorics yields $n!$ identical combinations which are the same up to permutation. Hence, the configuration space W for n identical particles should be given by

$$W_n = V/S_n, \tag{3.4}$$

where we have "divided out" the action of the symmetric group S_n on n letters [22]. Now if we, for simplicity, restrict our discussion to two particles, then it turns out that the wave function is singular in the center of mass point [22]. Therefore this singularity has to be removed from the space and as we shall see, this manipulation has some rather significant consequences depending on the dimensionality of the space. Thus, if we remove all configurations from V where the positions of these

[2] In 1977 the Norwegian physicists Jon Magne Leinaas and Jan Myrheim published a paper on this subject where they provided a rigorous mathematical explanation for the phenomenon, so for a more detailed treatment I refer to their paper in Ref. [22].

two particles coincide we'll get a new space $V - \{0\}$. Further with the identical permutations divided out we can write

$$W_2' = (V - \{0\})/S_2, \tag{3.5}$$

or in general for an n-particle system

$$W_n' = (V - \{0\})/S_n, \tag{3.6}$$

so the n-particle Hilbert space in which the n-particle state lives is thus defined by the tensor product of the n individual single particle Hilbert spaces

$$\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i = \underbrace{\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n}_{\text{n times}}, \tag{3.7}$$

and is spanned by a basis which constitutes of square integrable complex-valued functions over the punctured configuration space W', i.e. $L^2(W')$.

3.2.2 Why anyons?

Let us consider a three dimensional space in which we have removed a point. Then, if we regard an arbitrary pair of curves connecting any two point in this space, this pair of curves will form a closed loop and we will always be able to contract this loop continuously to a single point without crossing the removed point (see Figure 3.1). We then say that a set consisting of all such contractible loops form a *homotopy class*. This is, however, not the case in two dimensions. If we consider the same set-up, but now in a two dimensional plane, in which the removed point is enclosed by the loop, it is not very difficult to realize that the loop cannot be shrunk to a single point. The main reason for this is simply that the loop is trapped in the plane around the removed point and there is no way to pass it. In Figure 3.1 we can see that loop A around B, in three dimensions, can be lifted off the plane and thus be shrunk to a point [31], whereas the loop is confined to the plane in the two dimensional case which makes the loop non-contractible.

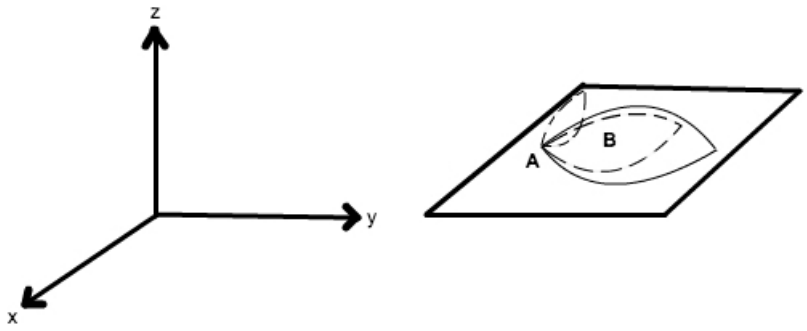


Figure 3.1: *In three dimensions the loop A around B can be lifted off the plane and contracted to a single point.*

Mathematically this means that if we remove a point from a two dimensional plane, the plane won't be *simply connected* anymore [22, 40]. In fact, the plane will be *infinitely connected* and has no longer a trivial topology [22, 40]. This is a

crucial insight since it entails that the topological properties of such a space strongly depends on the dimension of it. Therefore, it may seem natural to assume a connection between the dimensionality and the interchange symmetry of a multi particle state. As a consequence of the non-trivial topology in two dimensions, the order in which the quantization and symmetrization is performed does matter. Usually, when dealing with three dimensional systems, one does not have to bother about this but in two dimensions it is important that the symmetrization is performed before the quantization, due to the special topological features of the corresponding configuration space [22]. This will, however, result in fewer restrictions on the interchange properties, and a continuum of intermediate states between the symmetric and anti-symmetric ones will be allowed. Mathematically, what happens, is that the state picks up a complex phase under particle interchange so that

$$P_{i,j}\Psi_a(x_1, x_2, .., x_i, .., x_j, ..x_n) = e^{i\pi\alpha}\Psi_a(x_1, x_2, .., x_j, .., x_i, ..x_n), \tag{3.8}$$

where $\alpha \in [0, 2)$, since the modulus of the state must be invariant under interchange since the configuration is identical. Worth to point out is that we recover Fermi-Dirac statistics for $\alpha = 1$ and Bose-Einstein statistics for $\alpha = 0$. These types of states are called anyonic states and their statistics is thus ranging continuously between the Fermi-Dirac and the Bose-Einstein ones. Note that the emergence of anyonic statistics is purely a result of the topology of the manifold. This is truly a beautiful example of how something real and observable appears as a physical manifestation of an underlying abstract mathematical structure. Anyons are generally grouped into two types which we call abelian anyons and non-abelian anyons. The feature that distinguishes these two species is the structure of their respective permutation symmetry group, or more precisely if the group is commutative or not, hence the term abelian. As a result, the phase acquired in Equation (3.8) becomes non-trivial in the non-abelian case and may depend on the order in which the anyons are permuted. In this thesis work we will mainly focus on the non-abelian species and how it could be applied in the field of quantum computation.

3.3 Permutation of anyons

From a technical point of view, it turns out that non-abelian anyons are more suitable to use when encoding information, and in order to understand why, we need to discuss the differences between the abelian and non-abelian anyons with some more rigour. We shall start building our analysis on rather superficial considerations, which will lead us to a motivation for the emergence of the abelian phase, and then we will try to generalize our analysis to a non-abelian case. But before we go into further details about the mathematical structures that distinguish the abelian from the non-abelan case, a little bit of formalism needs to be introduced.

3.3.1 Feynman's path integral formalism and topological equivalence

A wave function that corresponds to a quantum mechanical system, roughly speaking, describes the probability of finding the system in a certain state in space and time. This function is usually found by solving the corresponding wave equation that

governs the system, but for our purpose, there is a more suitable way of interpreting this probability. In the mid-1900s, Richard P. Feynman developed an alternative way to express the wave function of a quantum system where he suggested that the probability amplitude for a particle to propagate from one space-time point to another, could be expressed as a sum over all possible paths which connect the two points [4]. By expressing the evolution from a starting point to an end point in terms of discrete propagators through time-slicing [4], and then taking the limit where each increment goes to zero, one finds that the full propagator from point (r_0, t_0) to point (r, t) can be written [4]

$$G(r, t; r_0, t_0) = \int_{r(t_0)}^{r(t)} e^{\frac{i}{\hbar} S(\vec{r}(\tau))} D[r(\tau)], \tag{3.9}$$

where the integral is carried out over the functional of all possible paths connecting $(\vec{r}_0, t_0)$ and $(\vec{r}, t)$, and $S[r(\tau)]$ is the action defined by $S[r(\tau)] = \int_{r(\tau)} \mathcal{L}(\tau) d\tau$ and $\mathcal{L}(\tau)$ is the Lagrangian of the system. Moreover, the wave function at $(\vec{r}, t)$, given the state at $(\vec{r}_0, t_0)$, can now be expressed as [4]

$$\psi(r, t) = \int_{(r_0, t_0)}^{(r,t)} G(r, t; r', t_0) \psi(r', t_0) dr'. \tag{3.10}$$

Now, as earlier discussed, the emergence of anyons can be deduced from the fact that the particle configuration space in two dimensions, with the singular point excluded, is multiply connected. Graphically this means that not all closed curves in the plane can be continuously deformed into one another without crossing the removed point. This phenomenon gives rise to a classification of paths where all paths that can be deformed into one another are grouped together into sets. A deformation, or transformation, of this type can be regarded as a map and thus constitute an *equivalence relation* amongst all such paths. In algebraic topology such classes of paths are referred to as *homotopy classes*, since two paths are said to be homotopic if and only if they obey the equivalence relation defined above [16]. Another way to prove whether two configurations of paths belong to the same homotopy class or not, is to compute the Kauffman invariants for the knots involved in the configurations, and thus see if the results are the same or if they differ. A graphical representation of topological equivalence is viewed in Figure 3.2.

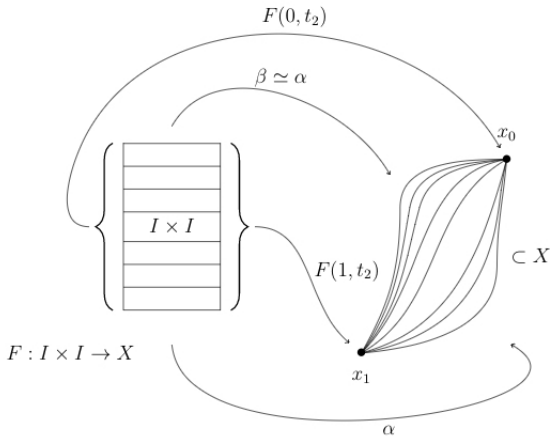


Figure 3.2: *Illustration of the concept of homotopy. A path-homotopic map* $F : I \times I \to X$ *maps elements in* $I \times I$ *to a set of homotopic paths in* X*.*

Thus, one may ask oneself what this has to do with the path integral formulation. As a matter of fact, the path integral is deeply related to the partitioning of the equivalence classes. We already know from the definition of Feynman's path integral that each path will contribute by a complex phase, and as it turns out, the phases derived from two homotopic paths will be identically the same [21]. This subtle insight basically entails that each homotopy class can be assigned a unique phase, so in a way, one can say that the path integral defines a partition on the set of paths by assigning each homotopy class a corresponding complex phase.

3.3.2 The fundamental group and braiding

When a multi-particle system of anyons undergoes a permutation, one can interpret these permutations as topological interactions and group theory provides the right language capture the mathematical structure and behavior. We are now familiar with the concept of homotopy and that paths can be categorized into classes defined by a certain equivalence relation. Furthermore, if we take one element from each homotopy class and form a set $\mathcal{O}$ of non-homotopic paths as well as defining an operation '$\circ$' that composes two paths in a suitable way, this set together with the operation '$\circ$' will form a group [16]. The natural way to define this operation is simply to let it compose two paths such that if a and b are two elements (loops) in the group, then these elements will form a new element $a \circ b = c$ which also lies in the group, in such a way that one first follows loop a and then loop b. Lets define this group by[3] [31]

$$\pi_1(W') = (\mathcal{O}, \circ), \tag{3.11}$$

where W' is the modified configuration space defined in Equation (3.6). In the field of algebraic topology, this group is known as the *fundamental group* [16]. Now, since each loop around the removed point can be identified by a *winding number*, which simply is an integer, this group should be isomorphic to the group of integers under

[3]For a formal mathematical definition, see Appendix A.

addition $(\mathbb{Z}, +)$ [31], i.e.

$$\pi_1(W') \simeq (\mathbb{Z}, +), \tag{3.12}$$

and from this relation it becomes clear that $\pi_1(W')$ has to be abelian since $(\mathbb{Z}, +)$ obviously is. This isomorphic relation to the group of integers also becomes transparent when spitting up the propagator defined in Equation (3.9) into an infinite number of discrete sums over paths ending at the same points (up to exchange) so that [31]

$$G(r, t; r_0, t_0) = \sum_{direct\ paths} e^{\frac{i}{\hbar}S} + e^{i\theta} \sum_{one\ exchange} e^{\frac{i}{\hbar}S} + e^{i2\theta} \sum_{two\ exchanges} e^{\frac{i}{\hbar}S} + .. \tag{3.13}$$

where the first sum is carried out over direct paths starting and ending in the same point without any exchange, the second sum corresponds to all paths involving one exchange, and so on. Thus, for the propagator defined in this way, we can identify a corresponding anyon, an abelian anyon. However, there is a more fruitful representation of this algebraic structure which provides a pictorial way of interpreting the interactions. This representation is known as the *braid group* and is also isomorphic to $\pi_1(W')$. In three dimensions when the configuration space with the singular point excluded is doubly connected [40], we know that there only exists two homotopy classes of paths, and therefore, according to the path integral formulation, we only have two distinct phases. This conclusion can also be deduced from graphical considerations since two exchanges correspond to a closed loop, and since the loop is contractible in three dimensions, the exchange operator squared P^2 should be the identity operator, which means that we must have $P = \pm I$. Mathematically, what this says, is that bosons and fermions transform as unitary representations of the permutation group S_N, which is the group consisting of all permutations of N fermions or bosons [31]. On the other hand, in the two dimensional case, when the configuration space is infinitely connected, anyonic statistics is allowed to emerge which introduces an arbitrary phase and thus yields a much richer classification [40]. As a result, anyons transforms as unitary representations of the braid group $\mathbb{B}_N$ [18, 31] instead of the ordinary symmetric permutation group. The elements of $\mathbb{B}_N$ form a set consisting of all unique and disconnected trajectories evolving through space-time, and the group multiplication law is defined in such a way that one simply connects two trajectories by linking them together. Or more formally, if we first act with an element s_a on a multi-particle state followed by acting with another element s_b, where $s_a, s_b \in \mathbb{B}_N$, we will first swap particle a and $a+1$ and then b and $b+1$. This is shown graphically in Figure 3.3 where we let an element act on its own inverse, which thus yields the identity element.

If we let $\left\{\sigma_k\right\}_{k=1}^{N-1}$ be the set of generators of $\mathbb{B}_N$ [40], where N is the number of particles, then we have to require that they obey the following algebra [40]

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \leq i \leq N-2 \end{cases}$$

where $|i|, |j| \leq N$. The first of these two relations simply says that two generators commute if they aren't neighboring elements in the set. This can easily be understood graphically by considering a chain of particles. If one exchanges particle

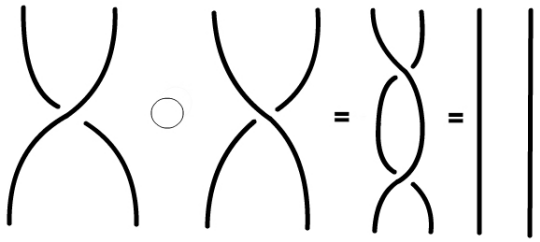


Figure 3.3: *An element s_a is acted upon by its own inverse $s_b = s_a^{-1}$ through composition which results in the identity element with time flowing in the upward direction. s_b is composed with s_a just by putting it on top on s_a. If one follows each strand in the resulting element one will come back to the original position without getting tangled up with the other strand which allows us to separate the strands completely, and thus get the identity.*

i with particle $i+1$, and then exchange particle j with particle $j+1$, one may quickly realize that this operation could just as well have been performed in the reversed order, and we would still end up with the same result if the particles aren't neighbors. Moreover, the second one is a consistency relation which is known as the Yang Baxter equation [40, 31] and is illustrated pictorially in Figure 3.4.

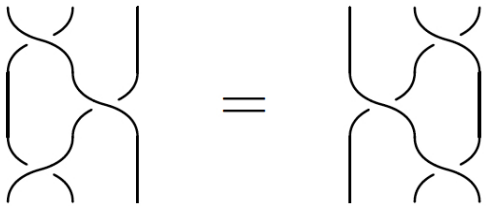


Figure 3.4: *A pictorial representation of the Yang Baxter equation with the time flowing in the upward direction.*

The fact is that it's actually these braids that implement the logic gates we discussed in Chapter 2. In order to process the information encoded in the qubits, one can perform braids on the anyons in the system in a suitable manner and then bring them together pairwise, or fuse them, when reading out the computer. The concept of *fusion*, however, will be discussed in more detail in Chapter 4 and will not be treated here.

3.3.3 The Chern-Simons Lagrangian

As already mentioned in Section 3.2.1, the phase can be non-trivial and may depend upon the path as well as the start and end point, which might give rise to even more exotic phenomena. Lets return for a wile to the path integral discussion. From our analysis we saw that the propagator could be split up into an infinite number of discrete sums over direct paths and multiple exchange paths, and that each path in a given homotopy class could be assigned a common complex phase. What assumption did we make that led us to that conclusion? Or more precisely, why did we acquire a trivial phase? If we study the propagator in Equation (3.9) it's easy to trace back the origin of the phase and see that it has to come from the action. Therefore, to

answer this we need to study the action and write down the Lagrangian explicitly. Generally, for bosonic or fermionic fields, the corresponding Lagrangian $\mathcal{L}_0$ takes the following form

$$\mathcal{L}_0 = \sum_{i=1}^{N} (\frac{m}{2} \dot{\vec{r}}_i^2 - V(\vec{r}_i)), \tag{3.14}$$

but in a punctured (2+1)-dimensional manifold, where '1' being the time dimension, an extra interaction term need to be added due to the topological effects in the system. This can be modelled by means of the Chern-Simons interaction which is a topological quantum field theory (TQFT), and is in (2+1)-dimensions defined as [40, 34]

$$\mathcal{L}_{CS} = \frac{\mu}{2} \epsilon^{\kappa\nu\rho} A_\kappa \partial_\nu A_\rho, \tag{3.15}$$

where $\epsilon^{\kappa\nu\rho}$ being the Levi-Civita permutation symbol and A_α a vector potential. Thus, it may be natural to draw the conclusion that the trivial phase acquired derives from this term, which is also the case. This can be shown simply by computing the factor $e^{\frac{i}{\hbar}S}$, where the action S is defined as $S = \int dx^3(\mathcal{L}_0 + \mathcal{L}_{CS})$. By doing so, one finds that $S = \int dx^3(\mathcal{L}_0 + \mathcal{L}_{CS}) = \int dx^3 \mathcal{L}_0 + \Delta\Phi$, so the exponent simply becomes the ordinary one plus a constant term $\Delta\Phi$, which will yield a constant complex phase $e^{i\Delta\Phi}$. So why is that? Well, the above introduced Chern-Simons interaction is an abelian gauge theory, and will thus contribute by a trivial phase factor. So in order to obtain a non-commutative phase we need to extend the theory to a non-abelian gauge theory. This can be done in a natural way by adding an extra term to the Lagrangian so that [40, 34]

$$\mathcal{L}_{CS} = \frac{\mu}{2} Tr[\epsilon^{\alpha\beta\gamma} A_\alpha (\partial_\beta A_\gamma + \frac{1}{3} g[A_\beta, A_\gamma])], \tag{3.16}$$

where g denotes the invariant volume factor [40]. Before we continue our discussion some rather interesting, and quite remarkable, features about the Chern-Simons term should be pointed out in order to justify its significance. The fact is that this term is not only Lorentz invariant, but generally covariant [40]. This means that it's not only invariant under a coordinate transformation to a system defined by a flat metric, it's actually invariant under a coordinate transformation to a system defined by an arbitrary metric, even systems in which the space-time is curved [40]. Another interesting property of the non-abelian Chern-Simons term is that if one performs a gauge transformation $A_\mu \to g^{-1} A_\mu g + i g^{-1} \partial_\mu g$ on the vector potential, something very interesting happens [34]. By doing so, an extra term will emerge in the Lagrangian and this term takes the form [34]

$$w(g) = \frac{\pi^2}{24} \int dx^3 \epsilon^{\mu\nu\rho} Tr[g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\rho g]. \tag{3.17}$$

So what is the interpretation of this function? The case is that by performing such a gauge transformation, what happens is that it winds around space-time and this function works as a counter [34]. In other words, it counts the number of windings. Now, if we return to the essentials, if one computes the corresponding action to the non-abelian Chern-Simons term one will find that the phase becomes non-trivial, as

we required. More precisely, it will take the form[4] $e^{i\Phi_\lambda T^\lambda_{\sigma\,\kappa}}$, and as a result the non-abelian statistics constitute a higher dimensional non-abelian representation of the braid group, whereas abelian statistics is a one dimensional abelian representation [40]. Therefore, the order in which the permutation is executed really does matter. Formally this means that if we let $U_{\alpha\beta}$ and $U_{\gamma\lambda}$ be two elements of a unitary representation of the braid group, which swaps particle α with particle β and particle γ with λ, respectively, then it does not holds that $[U_{\alpha\beta}, U_{\gamma\lambda}] = 0$ for all $\alpha, \beta, \gamma, \lambda$. Hence, the non-abelian statistics provides a far more exotic and rich representation which allows us to perform more complex operations.

[4]The object $T^\lambda_{\sigma\,\kappa}$ is a tensor of rank 3.

Chapter 4

The algebraic framework

In this Chapter we will discuss anyons from a field theoretic point of view where symmetries play a crucial role. Symmetry analysis in quantum field theory and particle theory has been proven to be very fruitful and one may even say that it constitutes one of the cornerstones of the subject. The idea of symmetries, through the concept of spontaneous symmetry breaking, will lead us to a classification of the different excitations in our model. In a gauge theory, the Lagrangian of the field in question is invariant under some symmetry group G, but the degenerate ground state may only be invariant under a subgroup H of G, $H \subset G$ [9]. Then in the low temperature regime, the invariance under G will, due to the *Higg's mechanism*, undergo breakdown to its residual subgroup H and according to the *Goldstone's theorem*, each generator of the broken group will be responsible for the emergence of a new excitation. Symmetry breaking may also manifest itself through "particle-like" excitations of a topological nature which can give rise to more exotic phenomena [9]. In addition, we will also account for the interactions among the excitations and formulate some rules, called *fusion rules*, which determine the outcome, or fusion product, when particles are brought together. We will also discuss how the quantum states that correspond to such fusion outcomes can be used as representations for qubits and qudits. In Section 4.1-4.2 we will mainly follow Wild Propitius (Ref. [3] and [9]) and Preskill in Section 4.3 (Ref. [29]).

4.1 Planar field theory and symmetry breaking

Let us consider a gauge theory in which we formulate a Lagrangian containing a potential $V(\Phi)$ of the Higgs field Φ , which is invariant under some continuous gauge group G of transformations [3]. This symmetry group can spontaneously brake down to a discrete subgroup $H \subset G$ by the non vanishing vacuum expectation value of the Higgs field Φ, which may only be invariant under H [3, 9, 1]. Then, new excitations emerge as fingerprints which in this case come in the form of topological defects, or *vortices*, which carry magnetic flux [3]. Note that these excitations are purely topological in nature and do not depend on whether the Lagrangian includes matter-coupled gauge fields or not. These topological defects are in (2+1)-dimensions (where '1' corresponds to the time) anyonic quasi-particles and are labeled by the elements of the homotopy group[1] $\pi_1(G/H)$ of G/H [9, 1]. This

[1]For definition see section 3.3.2.

statement is based on the assumption that each ground state can be reached from another ground state in the degenerate manifold by a transformation in G, which makes the ground state manifold isomorphic to the quotient group of cosets G/H [9, 1], and since the states can be transformed into one another by non-contractible maps, we can use $\pi_1(G/H)$ to label the vortices [2, 1]. But this group, however, is actually isomorphic to H [9], i.e. $\pi_1(G/H) \simeq H$, which entails that we just as well can characterize the vortex states by the elements h, where $h \in H$. However, as we later shall see, this label is not good in the case when H is non-abelian, since it will not be gauge invariant in general. Moreover, if we generalize the Lagrangian and include matter-coupled gauge fields [9], electric point charge excitations are also possible to appear which may be labeled by some irreducible representation Γ of H. However, note that the terms 'electric charge' and 'magnetic flux' are used merely as generalizations of the excitations involved Aharanov Bohm experiment, and may not be adequate descriptions in general. Now, let us consider the Aharanov Bohm setup in which we let an electric charge e circle around a magnetic flux tube h along an enclosed loop of arbitrary shape. This can, for example, be a achieved by interchanging the particles twice. Thus, due to the magnetic flux, the state will undergo a gauge transformation mediated by an operator $a \in H$. Imagine now that you measure the flux h before, respectively after e has circled around h. Since the measurement should be the same, it must hold that $ah = ha$, or equivalently

$$h = a^{-1}ha, \tag{4.1}$$

so more precisely one might say that the flux transforms by conjugation which suggests that it should be labeled by the *conjugacy classes* $C^c = \{a^{-1}ha \mid a \in H\}$ [3] of H, and not H itself, in order to obtain gauge invariance. Note that the transformation is trivial if the group in question is abelian since thus we get $h \to a^{-1}ha = ha^{-1}a = h$. Hence, the elements $h \in H$ still work as "good" labels for abelian anyons whereas in the non-abelian case, we have to use the conjugacy classes C^c of H as labels, since the flux measurement only will commute with the corresponding centralizer subgroup, so let us stick to the general case and let the conjugacy classes label the flux. Moreover the change in structure of the representation when the flux is non-abelian is known as flux *metamorphosis* [9, 3, 2] and can be observed experimentally.

4.1.1 Braiding of fluxes and charges

One interesting and for our purpose relevant feature of a pair of particles is the transformation under braiding, and especially braiding involving magnetic fluxes. Lets start by considering a pair of vortices denoted by the states $|v_1\rangle$ and $|v_2\rangle$, and let us act with an element $\mathcal{R}_{12}$ in the braid group defined in Section 3.3.2, then the total flux should be invariant so if we braid v_1 and v_2, the joint state should transform as

$$|v_1\rangle\,|v_2\rangle \to |v_1 v_2 {v_1}^{-1}\rangle\,|v_1\rangle \tag{4.2}$$

so that the ordered product $v_1 v_2 {v_1}^{-1} v_1 = v_1 v_2$ is preserved [3]. We have already concluded that fluxes are labeled by the conjugacy classes C^c and due to the degeneracy, the fluxes will carry a vector space V^c, which is spanned by the fluxes

labeled by the elements in C^c [3, 9]. Hence, each flux can be formulated as linear combination of an orthogonal basis of flux eigenstates $\{h_1, h_2, ..., h_n\}$ such that

$$\langle h_i | \, | h_j \rangle = \delta_{ij}, \quad \forall \; h_i, h_j \in C^c. \tag{4.3}$$

Moreover, if we want to figure out the exact statistics under braiding of two fluxes we need to study the action in this vector space. Let us now instead consider the interchange between an electric charge in the state $|e\rangle$ and a magnetic flux in the state $|v\rangle$. The electric charge is labeled by the irreducible representation Γ of H, and just as the label of the magnetic flux, this representation is not unique which suggests that the electric charges carry an intrinsic vector space as well, which we may denote by V^e. So by swapping an electric charge and a magnetic flux by the action of $\mathcal{R}_{12}$, the state of the electric charge might be rotated so the joint state should transform as [9]

$$|e\rangle \, |v\rangle \to |v\rangle \, |\Gamma_h e\rangle \, , \tag{4.4}$$

where $|\Gamma_h e\rangle$ is the rotated state.

4.1.2 Dyonic excitations and their representations

So far in this Chapter we have argued that two types of particles, electric charges and magnetic fluxes, emerge due to the symmetry breakdown and we have also discussed how their respective states transform under interchange. A natural question that arises now is if these particles are the only ones, or could there possibly exist other excitations? This question has a non-trivial answer, which in a sense, is both yes and no. It is indeed true that the full particle spectrum exclusively consists of particles of such a nature described earlier in this Section, but the fact is that they don't have to be pure [3]. What is meant by this is simply that particles with both magnetic flux and electric charge also are meaningful objects. Furthermore, in order to figure out how to label these particles, also known as *dyons*, we have to study the structure of H in more detail. Since a dyon is a composition of an electric charge and a magnetic flux, it's natural to think that the dyon should, in some way, be labeled by a composition of their respective representations. This is also the case, but not in the trivial way. If we consider Equation (4.1) we note that h is invariant if b and h commute. Of course these always exists at least one such an element b, namely the identity, but if there exist other non-trivial elements that commute with h, it means that h transforms invariantly under conjugation with all those elements. Mathematically speaking, if we form a set with all such elements, that set will actually possess group structure, and more specifically, it will be a *centralizer* subgroup $Z^h \subset H$ of h. Because of this, the adequate way to label the dyons, should be by composing the conjugacy classes C^c together with the irreducible representations ζ of the different irreducible components of Z^h, contained in Γ [9]. Hence, the dyons may adequately be labeled by (C^c, ζ) where C^c runs over the different conjugacy classes of the flux and ζ over the irreducible representations of each corresponding centralizer subgroup [9]. So in conclusion we can characterize the full excitation spectrum by

$$(C^c, \zeta), \tag{4.5}$$

in which we have pure magnetic flux sectors when the corresponding centralizer is trivial, i.e only the identity element, and pure electric charge sectors if the corresponding conjugacy class is trivial, and in all other cases we will have a mixture which we call dyons and denote by ϵ. No excitations, or a *vacuum*, is also possible when both representations are trivial. Hence, the pure sectors can be interpreted as special cases of the more general dyonic sector denoted by the states $\{|e^{\zeta_i}, v^{c_j}\rangle\}_{i,j=1}^{dim(\zeta),n}$, where the set $\{e^{\zeta_i}\}_{i=1}^{dim(\zeta)}$ constitutes the basis of the representation ζ. Similarly as the pure excitations carry an intrinsic vector space, so do the dyons. This vector space V^d can thus be formulated as the tensor product of V^ζ and V^v

$$V^d = V^\zeta \otimes V^v, \tag{4.6}$$

where V^ζ is the space spanned by the irreducible representations ζ of the corresponding centralizer subgroups. Hence, the pure sectors are confined to rotate in their corresponding subspaces of the full vector space V^d. However, we won't discuss the transformation properties of the dyons under braiding in this section since it requires some more theory that will be covered in the next Section.

4.2 A unified framework through the Hopf algebra

In the previous Sections in this Chapter we discussed how the symmetry breakdown of a group G to a discrete subgroup $H \subset G$ of our field theoretic model, lead to a certain spectrum of excitations. The first one we discussed was the topological flux vortex, and then we argued that if we were to add a mass-coupled gauge field to the Lagrangian, a second type of excitation would emerge, namely the electric point charge. We found that the former could properly be labeled by the conjugacy classes of the elements in the residual discrete subgroup H, whereas the latter were labeled by the unitary irreducible representations of H. Furthermore, we also saw how these excitations could be composed to form a third quasi-particle species, the dyon, and that the pure excitations could be considered as special cases of a more general and complex structure (C^c, ζ). This invites the idea of trying to find a more general framework with additional structure, such that the different excitations can be unified in a single representation Π_ζ^c. Luckily, this can be achieved by performing an extension of the discrete group H to the *quantum group* $D(H)$ [3, 9] by applying the Drinfeld quantum double construction to the Hopf algebra[2] and its dual [3, 11, 12]. As we shall see, each part of this structure has a physical interpretation that fits perfectly into our model. This algebra is spanned by a basis which constitutes of the gauge transformations corresponding to the electric charge $a \in H$ and the projection operators P_h which project out the flux of the corresponding state [9]. These projection operators are defined as

$$P_{h_i} P_{h_j} = \delta_{ij} P_{h_i}, \tag{4.7}$$

for any $h_i, h_j \in H$. Moreover, since we have showed that the gauge invariance of flux measurements requires the flux to transform under conjugation, we must also

[2]See Appendix B for definition.

have that

$$aP_h = P_{aha^{-1}}a, \tag{4.8}$$

so the quantum double is spanned by the set $\{P_h a\}_{a,h\in H}$. Thus, by means of Equation (4.7) and (4.8), we can form a product of two such element given by

$$P_{h_i}a_iP_{h_j}a_j = \delta_{h_i,a_ih_ja_i}P_{h_i}a_ia_j. \tag{4.9}$$

We are also interested in to see how such a state transforms under the joint action of a symmetry transformation a followed by projection P_h, but to do so, some more notation need to be introduced. Let us denote by $\{h_i^c\}_{i=1}^n$ the elements contained in the conjugacy class C^c so that $h_i^c = a_i^c h_j^c (a_i^c)^{-1}$, and by $\{|e_i^\zeta, v_j^c\rangle\}_{i,j=1}^{dim(\zeta),n}$ the basis of the vector space V^d defined in Equation (4.6). Thus, if we consider the action of the joint operator on some state $|e^{\zeta_i}, v^{c_j}\rangle$, it yields a new state according to [3, 9]

$$\Pi_\zeta^c(P_h a)\,|e^{\zeta_i}, v^{c_j}\rangle = \delta_{v,av_ia^{-1}}\,|\zeta((a_m^c)^{-1}ha_j)e^{\zeta_l}, av_ia^{-1}\rangle\,, \tag{4.10}$$

where a_m^c is defined by $h_m^c = ah_ja^{-1}$. So in conclusion we have that Π_ζ^c constitute a joint representation corresponding to the centralizer ζ of h in Γ and the conjugacy class C^c.

4.2.1 Coproduct of particle states

In order to generalize the action defined in Equation (4.10) to a many-particle state, we need to compose the action into a suitable tensor product. This can be done ideally by means of the coproduct that the Hopf algebra is equipped with. If we, for simplicity, start by considering a two particle state we want to generalize the operator in Equation (4.10) such that it acts within the vector space that is built up by the tensor product of the corresponding vector spaces of each respective particle. According to the definition of the single dyonic vector space defined in Equation (4.6), we must have in the case of two dyons, that the operator acts within

$$(V^\zeta \otimes V^v)_1 \otimes (V^\zeta \otimes V^v)_2 = V_1^d \otimes V_2^d. \tag{4.11}$$

Thus, by introducing the morphic coproduct defined by

$$\Delta: \quad D(H) \to D(H) \otimes D(H), \tag{4.12}$$

we can map the action corresponding to a gauge transformation followed by a flux projection, i.e. $P_h a$, through Δ so that [9, 3]

$$\Delta(P_h a) = \sum_{all\ \ h_i,h_j=h} P_{h_i}a \otimes P_{h_j}a, \tag{4.13}$$

where the sum is carried out over all projections where h_i and h_j are equal to h. Thus, the two particle generalization of Equation (4.10) is given by

$$\sum_{all\ \ h_i,h_j=h} \Pi_{\zeta_1}^{c_1}(P_{h_i}a) \otimes \Pi_{\zeta_2}^{c_2}(P_{h_j}a). \tag{4.14}$$

It may now be natural to ask how to interpret the action of this operator on a two particle state. When acting with this, it acts separately on the states so that

each electric part is gauge transformed and then the projection operator projects out the flux from each state [3]. This implies that even if the particles are fused together locally, the global properties will still be conserved. Now, before we go on and generalize Equation (4.14) to a system consisting of an arbitrary number of particles, we should start by adding just one in order to see how the representation of a third one can be composed with the initial two-particle representation. The map defined in Equation (4.12) also possesses a coassociative property so that

$$(\mathbb{1} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathbb{1}) \circ \Delta. \tag{4.15}$$

By applying this feature we can treat the initial quantum double basis elements, corresponding to the initial two particles, as a single one and then implement the coproduct once again with the initial joint two-particle elements together with the third which thus yields

$$(\mathbb{1} \otimes \Delta) \circ \Delta(P_h a) = (\Delta \otimes \mathbb{1}) \circ \Delta(P_h a) = \sum_{all\ \ h_i,h_j,h_k=h} (P_{h_1} a) \otimes (P_{h_2} a) \otimes (P_{h_3} a), \tag{4.16}$$

so the three particle generalization of the operator in Equation (4.10) is given by the operator

$$\sum_{all\ \ h_i,h_j,h_k=h} \Pi^{c_1}_{\zeta_1}(P_{h_i} a) \otimes \Pi^{c_2}_{\zeta_2}(P_{h_j} a) \otimes \Pi^{c_3}_{\zeta_3}(P_{h_k} a), \tag{4.17}$$

which now acts within the vector space defined by

$$(V^\zeta \otimes V^v)_1 \otimes (V^\zeta \otimes V^v)_2 \otimes (V^\zeta \otimes V^v)_3 = V_1^d \otimes V_2^d \otimes V_3^d. \tag{4.18}$$

Of course, the same technique can be used multiple times to build up an n-particle representation of the same operator. Hence, the action of an n-particle representation will simply be an n-particle generalization of the action on a two particle state, namely a gauge transformed electric part and the flux being projected out for each one of the particles, so that the global properties are conserved after each fusion process [3]. However, these multi-particle representations are in general reducible [3, 9], which entails that they can be decomposed into direct sums over irreducible ones. Thus, in the two particle case we have that

$$\Pi^{c_k}_{\zeta_k} \otimes \Pi^{c_l}_{\zeta_l} = \bigoplus_{c_m,\zeta_n} N^{c_k c_l \zeta_n}_{\zeta_k \zeta_l c_m} \Pi^{c_m}_{\zeta_n}, \tag{4.19}$$

where $N^{c_k c_l \zeta_n}_{\zeta_k \zeta_l c_m}$ is the multiplicity of each $\Pi^{c_m}_{\zeta_n}$, i.e. a positive integer, which we later shall see defines the dimension of the protected topological Hilbert space. This relation defines the so called *fusion rules* which describes the outcome $\Pi^{c_m}_{\zeta_n}$ as a result of $\Pi^{c_k}_{\zeta_k}$ and $\Pi^{c_l}_{\zeta_l}$ being fused together. It could also be used the other way around, namely to work out the decay products $\Pi^{c_k}_{\zeta_k}$ and $\Pi^{c_l}_{\zeta_l}$ of particle $\Pi^{c_m}_{\zeta_n}$. Furthermore, the multiplicity objects $N^{c_k c_l \zeta_n}_{\zeta_k \zeta_l c_m}$ can be computed as [9, 10]

$$N^{c_k c_l \zeta_n}_{\zeta_k \zeta_l c_m} = \frac{1}{|H|} \sum_{a,h} Tr[\Pi^{c_k}_{\zeta_k} \otimes \Pi^{c_l}_{\zeta_l}(\Delta(P_h a))] Tr[\Pi^{c_m}_{\zeta_n}(P_h a)]^* \tag{4.20}$$

where $|H|$ denotes the order of H and the star refers to complex conjugation. We will return and discuss more about this in Section 5.2.3 and 5.4.3 where we will motivate why fusion states are suitable as a basis of fault-tolerant quantum computation.

4.2.2 Braiding of dyons

At this stage, we have unified the descriptions of the pure sectors into a more general sector, the dyonic sector, in a single framework. Hence, we are now ready to discuss the transformation properties of dyonic particles under braiding. Again, if we restrict our discussion to the two particle case, we can write the two-particle braiding operator as a composition of a permutation operator σ and a two-particle $D(H)$-element such that [3]

$$\mathcal{R}^{\zeta_i\zeta_k}_{c_j c_l} = \sigma \circ (\Pi^{c_k}_{\zeta_k} \otimes \Pi^{c_l}_{\zeta_l})(R), \tag{4.21}$$

where $R \in D(H) \otimes D(H)$ is the universal R-matrix defiend by [9]

$$R = \sum_{a,h} P_a \otimes P_h a. \tag{4.22}$$

If we let this operator act on a state $|e^{\zeta_i}, v^{c_j}\rangle\,|e^{\zeta_k}, v^{c_l}\rangle$, the state will transform as [3, 9]

$$\mathcal{R}^{\zeta_i\zeta_k}_{c_j c_l}\,|e^{\zeta_i}, v^{c_j}\rangle\,|e^{\zeta_k}, v^{c_l}\rangle = |v^{c_j} v^{c_l} (v^{c_j})^{-1}, \zeta^k((a^{c_l}_m)^{-1} v^{c_j} a^{c_l}) e^{\zeta_k}_i\rangle\,|e^{\zeta_i}, v^{c_j}\rangle\,, \tag{4.23}$$

where $a^{c_l}_m$ is defined by $v^{c_l}_m = v^{c^j} v^{c_l} (v^{c^j})^{-1}$. If we consider this expression a few remarks can be pointed out. First, the magnetic part of the vector transforms under conjugation which means that the magnetic part of the dyon undergoes flux metamorphosis, just as in the case of two pure magnetic states in Equation (4.2). Secondly, we may also note that the electric part of the state is gauge transformed in the same manner as in Equation (4.4) when a pure electric charge encircled a pure magnetic vortex. Moreover, if we consider Equation (4.13) together with Equation (4.22) one can also check that the braid operator satisfies the following relations [9]

$$\mathcal{R}\Delta(P_h a) = \Delta(P_h a)\mathcal{R} \tag{4.24}$$

$$(\mathbb{1} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_1 \mathcal{R}_2 \tag{4.25}$$

$$(\Delta \otimes \mathbb{1})(\mathcal{R}) = \mathcal{R}_2 \mathcal{R}_1 \tag{4.26}$$

where $\mathcal{R}_1 = \mathbb{1} \otimes \mathcal{R}$ and $\mathcal{R}_2 = \mathcal{R} \otimes \mathbb{1}$. These relations are referred to as the *quasi-triangular conditions* which thus make the quantum group a quasi-triangular Hopf algebra. The first one of these relations basically tells that the braid operator commutes with the elements in $D(H)$, which means that the symmetry transformation will not be affected and that the total flux will be conserved. The last two relations, however, state that it does not matter if we braid one particle across two other ones before they are fused together or if you fuse them first and then braid the fused particle with the first one, the end result will still be the same. This property can be nicely demonstrated pictorially and is depicted in Figure 4.1. The quasi-triangular

conditions also ensures that $\mathcal{R}_1 = \mathbb{1} \otimes \mathcal{R}$ and $\mathcal{R}_2 = \mathcal{R} \otimes \mathbb{1}$ satisfy the Yang Baxter consistency equation [3, 9] which was discussed in Section 3.3.2 and depicted in Figure 3.4, i.e.

$$\mathcal{R}_1\mathcal{R}_2\mathcal{R}_1 = \mathcal{R}_2\mathcal{R}_1\mathcal{R}_2. \tag{4.27}$$

Hence, the braid operator introduced in Equation (4.23) forms a unitary representation of the braid group. However, since the objects Π^c_ζ generally are reducible, so are $\mathcal{R}^{\zeta_i\zeta_k}_{c_jc_l}$. As a result of this, when acting upon a multi-particle state, $\mathcal{R}^{\zeta_i\zeta_k}_{c_jc_l}$ will split the full multi-particle Hilbert space up into a direct sum of irreducible subspaces [9]. Moreover, the exact braiding properties further depend on the dimensionality of each respective subspace. If we are dealing with more than two particles the vector space spanned by their representations will have more than one dimension which results in a non-abelian braiding statistics, whereas if the number of particles is lesser than 3 the corresponding vector space will be one dimensional which makes the braiding statistics abelian [9].

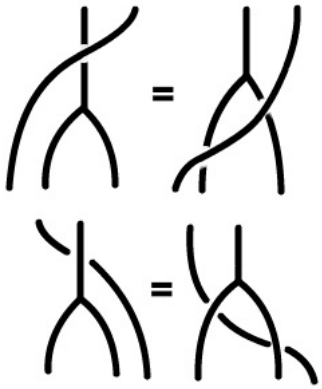


Figure 4.1: *The upper figure illustrates how the braiding between 1 and the fusion product of 2 and 3 yields the same result as if the braiding would take place before the fusion of 2 and 3, which is formally described in Equation (4.25). The lower picture illustrates the same principle described in Equation (4.26) in which 1 and 2 are fused together and braided before, and after, with 3.*

4.2.3 The counit and the antipode

So far we have discussed three types of excitations: the electric charge, the magnetic vortex and the dyon. We also saw that the electric charge and the magnetic vortex could be interpreted as special cases of the dyon, which is the fusion product of the two. Mathematically these pure excitations emerge when one of the representations in the composition is trivial. Hence, it might be natural to ask what kind of particle we get when both representations are trivial. The answer to this question is of course the vacuum, or no particles, which we denote by $\mathbb{1}$. Now, since the fusion between the vacuum and any particle should be trivial itself, we may demand that particles that are fused with the vacuum should be left unchanged. Furthermore, the Hopf algebra is equipped with a function called the *counit* defined by [30]

$$\varepsilon : \quad D(H) \to \mathbb{C}, \tag{4.28}$$

which we will see represents the vacuum. Let I be the identity map, then we have the following isomorphisms [30]

$$D(H) \xrightarrow{\Delta} D(H) \otimes D(H) \xrightarrow{\varepsilon \otimes I} k \otimes D(H) \simeq D(H) \tag{4.29}$$

and

$$D(H) \xrightarrow{\Delta} D(H) \otimes D(H) \xrightarrow{I \otimes \varepsilon} D(H) \otimes k \simeq D(H) \tag{4.30}$$

where $k \in \mathbb{C}$. Thus, if we interpret the above isomorphisms as equations we are demanding that [30]

$$(\varepsilon \otimes I) \circ \Delta = I = (I \otimes \varepsilon) \circ \Delta, \tag{4.31}$$

so the counit does indeed represent the vacuum in mathematical terms. Moreover, we know that each particle has an anti-particle, which if brought together with, will annihilate to a vacuum state. Such anti-particles also have a nice mathematical analogue in the Hopf algebra called the *antipode*. This function, which we will denote by $\mathcal{S}$, is an endomorphic map which thus maps the quantum group back onto itself, i.e.

$$\mathcal{S}: \quad D(H) \to D(H), \tag{4.32}$$

such that for any element $a \in D(H)$, $\mathcal{S}$ maps a to its corresponding inverse element so that $\mathcal{S}(a) = a^{-1}$. In other words, $\mathcal{S}$ takes $D(H)$ to its inverse which yields the following equivalence [30]

$$(\mathcal{S} \otimes I) \circ \Delta = \varepsilon = (I \otimes \mathcal{S}) \circ \Delta. \tag{4.33}$$

Hence, the antipode enables a mathematical treatment for anti-particles. We have now seen how the different parts of the Hopf algebra structure naturally provides a complete algebraic description of our model, where each part of the algebra have a natural physical interpretation.

4.3 Topologically protected fusion spaces

Before we jump right into the theory that will be discussed in this Section, we may remind ourselves what it is we want to achieve. A lot of theory have been discussed so far with one main objective, namely to find a protected space in which fault-tolerant quantum computations can be executed. Nevertheless, this has been necessary in order to complete the quest of finding such a space, that we are now starting to touch upon. If we return to Section 4.2.1 where the fusion rules in Equation (4.19) were introduced, a new object emerged which describes the multiplicity of the fusion product. This object was further defined in Equation (4.20) and will play a key role throughout this Section. Let us, for simplicity, consider a system consisting of two particles labeled by the letters a respectively b, which can be fused together and form a third particle labeled by the letter c. Analogous to Equation (4.19), we can now write the fusion of a and b as an ordinary sum over all possible outcomes c with multiplicity N_{ab}^{c}

$$a \times b = \sum_{c} N_{ab}^{c} c. \tag{4.34}$$

Since it does not matter in which order a and b are fused together, N_{ab}^c should be symmetric under interchange of a and b so that $N_{ab}^c = N_{ba}^c$. Moreover, this object describes in how many distinguishable ways particle c can be produced under a fusion process involving a and b. In other words, if $N_{ab}^c = 0$, c cannot be produced under such a process and if $N_{ab}^c = 1$, there is one unique way to form c from a and b and if $N_{ab}^c > 1$, there are N_{ab}^c distinguishable ways to produce c [29]. Hence we can form a vector space built up from the orthonormal states corresponding to the N_{ab}^c different distinguishable ways to produce c, which we will call the *fusion space*. Lets denote a fusion state in such a space by $|ab; c, \mu\rangle$, where μ is ranging from 1 to N_{ab}^c, and the fusion vector space constructed from the N_{ab}^c different fusion states by V_{ab}^c. These states can be represented graphically which provides a more concrete notation to interpret. A general two particle ket-vector together with its conjugate counterpart, the corresponding bra-vector, is depicted in Figure 4.2 and 4.3.

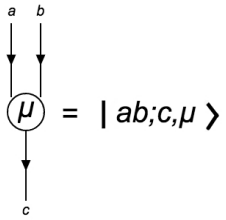


Figure 4.2: *Fusing a and b into c.*

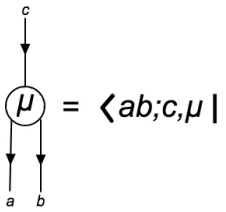


Figure 4.3: *Splitting c into a and b.*

In terms of fusion spaces we can now formulate a condition for a model to be non-abelian. If there only is one possible particle c that can be formed from a and b, and if this particle only can be formed in one unique way, the model will be abelian and non-abelian otherwise [29]. Hence, the model is non-abelian if and only if the following holds

$$dim(\bigoplus_c V_{ab}^c) = \sum_c N_{ab}^c > 1, \tag{4.35}$$

for some a and b. So in a non-abelian model we know that if two particles are fused together, there must be several outcomes c. Therefore the fusion spaces in non-abelian models will always be multi-dimensional which makes them more suitable for quantum computations than their abelian counterparts, which always have a definite fusion outcome, since much more information can be stored in a higher dimensional space. Furthermore, we can also define a unitary *charge conjugation* operator C which maps a label onto its conjugate so that [29]

$$C_a : \quad a \to \bar{a} \to a. \tag{4.36}$$

Graphically speaking, this operation reverses the direction of the arrow representing the world line of the corresponding particle [24]. In other words, if the fusion ket-vector depicted in Figure 4.2 is acted upon by C_a, the resulting state would correspond to the process in which particle b decays into particle $\bar{a}$ and c. Hence, the conjugation operator acts as a raising and lowering operator on the indices of the corresponding fusion space. This property induces a chain of isomorphisms among the fusion spaces [29]

$$V_{ab}^c \simeq V_b^{\bar{a}c} \simeq V_{\bar{c}}^{\bar{a}\bar{b}} \simeq V_{ab\bar{c}}^1 \simeq V_1^{\bar{a}\bar{b}c} \simeq ... \tag{4.37}$$

since the upper indices correspond to the labels of the outgoing particles and the lower ones to the labels of the ingoing particles. Furthermore, the index '1' corresponds to no particles, or the vacuum. Hence, the space $V^1_{ab\bar{c}}$ is spanned by all states that correspond to the annihilation of a,b,$\bar{c}$ when fused together, and the space $V_1^{\bar{a}\bar{b}c}$ is spanned by all states corresponding to the creation of $\bar{a}$,$\bar{b}$,c from the vacuum. Of course, if a particle with some label, say a, are trivially "fused" together with the vacuum 1, the outgoing particle will still be identical to a, i.e. $a \times 1 = a$, and the corresponding fusion space will thus be V^a_{a1}. Note that the fusion spaces only depend on the labels of the incoming particles and nothing else which means that any fusion state is completely independent of the paths of the incoming particles since the fusion product only will depend on the particle species and state. As we have earlier discussed in this Chapter, the joint state of the particles change under braiding processes which means that as long as we braid the particles properly, we will obtain the desired fusion product. So even if the particles are subject to small external perturbations, the end product will be the same as long as the braids traced out by the world lines belong to the correct homotopy class, and the computer will execute the requested computation. This non-local storage of information is the very principle that topological quantum computing is based upon. One initializes it by preparing pairs of anyons and then perform a sequence of exchanges on the system after which the anyons are brought together pairwise, and since we are guarantied to end up with the desired fusion product if we succeed to braid the particles accurately, the quantum information will not be affected by noise coming from the environment which otherwise would cause decoherence. In other words, the information will be protected by topological equivalence which results in a robust fault-tolerant system that is immune to errors.

4.3.1 Many-particle systems

Consider now instead an initial set consisting of three particles a, b and c, which can be fused together and thus form a fourth particle d. Since we know that the total charge is an intrinsic property of the three particles [29], one should be able to obtain d in two distinct ways. We could either fuse a and b first and then fuse the resulting particle with c, or we could fuse b and c first and then fuse their product with a. Mathematically speaking, this property is known as associativity and can formulated as

$$(a \times b) \times c = a \times (b \times c). \tag{4.38}$$

Hence, if we label the fusion product of a and b by e and the fusion product of b and c by e', the fusion space V^d_{abc} can be decomposed into a direct sums in two, up to isomorphism, identical ways [29]

$$V^d_{abc} \simeq \bigoplus_e V^e_{ab} \otimes V^d_{ec} \simeq \bigoplus_{e'} V^{e'}_{bc} \otimes V^d_{ae'}, \tag{4.39}$$

and to each one of these subspaces we may introduce a basis

$$|(ab)c \to d; e\mu\nu\rangle \equiv |ab; e, \mu\rangle \otimes |ec; d, \nu\rangle \tag{4.40}$$

$$|a(bc) \to d; e'\mu'\nu'\rangle \equiv |ae'; d, \nu'\rangle \otimes |bc; e', \mu'\rangle \tag{4.41}$$

where μ, ν, μ' and ν' denote in which way among the different N_{abc}^{d} distinct ways the respective fusion product is formed. These basis sets can be related through a matrix transformation F_{abc}^{d}, also known as the *F-matrix*, such that [29]

$$|(ab)c \to d; e\mu\nu\rangle = \sum_{e',\mu',\nu'} (F_{abc}^{d})_{e\mu\nu}^{e'\mu'\nu'} \, |a(bc) \to d; e'\mu'\nu'\rangle \, . \tag{4.42}$$

This action performed by the F-matrix can also be expressed graphically as in the figure below (Figure 4.4).

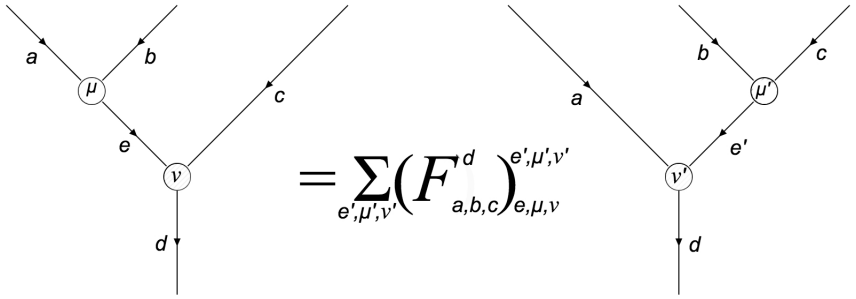


Figure 4.4: *The action of the F-matrix viewed graphically.*

In a real quantum computer, however, we will probably need more than two anyons in order to maximize the computational power. It is therefore of interest to consider larger systems of particles and how we can formulate a braiding operation on such a system. To do this we may, for convenience, define a standard basis for an n-particle Hilbert space $V^{c}_{a_1,a_2,..,a_n}$ with total charge c [29] so that the basis vectors are constructed in such a way that if we put the anyons on a line, we first fuse a_1 and a_2 which yields a new particle b_1, which is fused together with a_3 to form b_2, and so on until we finally fuse b_{n-2} together with a_n. This basis is visualized in a pictorial notation in Figure 4.5.

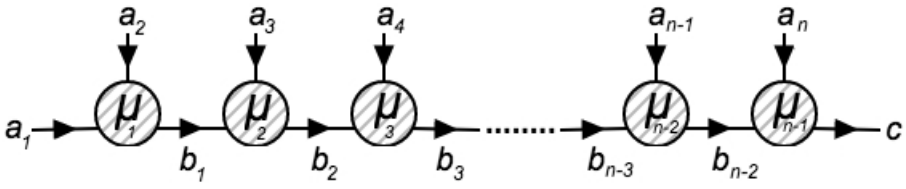


Figure 4.5: *Chain consisting of n anyons which are fused together from left to right.*

Note that our choice of basis was nothing but a matter of convenience. In fact, the order in which we fused the particles could have been arbitrarily chosen. However, the vector space $V^{c}_{a_1,a_2,..,a_n}$ spanned by this basis is isomorphic to the decomposition into a multiple direct sum of the tensor product among all fusion subspaces, over all possible fusion outcomes b_i [29], so that

$$V^{c}_{a_1,a_2,..,a_n} \simeq \bigoplus_{b_1} \bigoplus_{b_2} \cdots \bigoplus_{b_{n-2}} V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes \cdots \otimes V^{c}_{b_{n-2} a_n}. \tag{4.43}$$

The dimension of this vector space grows exponentially with the number of particles in the system which makes the many particle topological Hilbert space a suitable arena for powerful quantum information processing [29]. Therefore, it is of highest

importance to understand how braiding can be performed in such a complex space in order to realize topological quantum computing. The key insight is to observe that when permuting the particles (or labels), we go from one basis to another which means that a permutation operator works as a map from one vector space spanned by the initial states to another spanned by the states that correspond to the permuted labels. Up to this point, however, we have only discussed braiding in the two particle case where the R-matrix was introduced (see Equation (4.22)). Hence we must find a way to generalize the action of R so that we can permute the labels in a many particle state. So how can this be achieved? Well, we can use the F-matrix to go to a basis in which R is block diagonal, perform the braiding, and then go back to the initial basis by the action of the inverse of F [29]. Thus, the braiding operator B_n, acting on an n-particle state, can be defined as $B_n = F_n^{-1} R_n F_n$. Now, if we for example consider a three particle space in which the particles are labeled by a, b and c and we want to braid particle b and c by the action of B, we have the following map

$$B: \quad V_{acb}^{d} \to V_{abc}^{d}, \tag{4.44}$$

or more explicitly[3] [29]

$$B\,|(ac)b \to d; e\rangle = \sum_{g}\sum_{f} |(ab)c \to d; g\rangle\, ([F_{abc}^{d}]_{f}^{g})^{-1} R_{bc}^{f} (F_{acb}^{d})_{e}^{f} = \sum_{g} |(ab)c \to d; g\rangle\, (B_{abc}^{d})_{e}^{g} \tag{4.45}$$

where

$$(B_{abc}^{d})_{e}^{g} = \sum_{f} ([F_{abc}^{d}]_{f}^{g})^{-1} R_{bc}^{f} (F_{acb}^{d})_{e}^{f}, \tag{4.46}$$

so in conclusion the representation of the braid group in an n-particle space is completely characterized by the R- and F-matrices [29]. Moreover, one single F-matrix can be decomposed in different ways which give rise the consistency relation [24]

$$\sum_{\delta} [F_{e}^{fcd}]_{(g\beta\gamma)(l\delta\nu)} [F_{e}^{abl}]_{(f\alpha\delta)(k\lambda\mu)} = \sum_{h\sigma\psi\rho} [F_{g}^{abc}]_{(f\alpha\beta)(h\sigma\psi)} [F_{e}^{ahd}]_{(g\sigma\gamma)(k\lambda\rho)} [F_{k}^{bcd}]_{(h\psi\rho)(l\mu\nu)} \tag{4.47}$$

which, for obvious reasons, is called the *pentagon equation* (see Figure 4.6).

[3] Note that we have suppressed the indices corresponding the different fusion states.

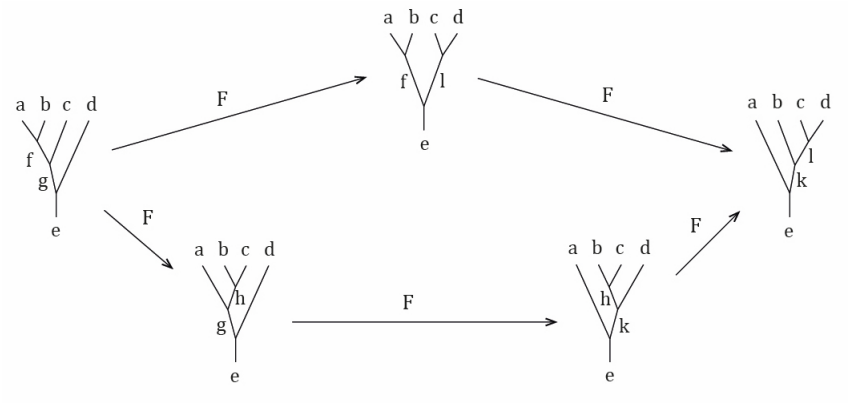


Figure 4.6: *The pentagon equation. If one starts from the left the end result will be the same independent of which one of the upper or lower paths one takes.*

Hence, by searching for solutions to the pentagon equation, given the fusion rules of the model, one can find all equivalent pair of sequences of transformations from one given space to another which together form the fusion categories [24]. A similar argument, involving the R-matrix, gives rise to two other important consistency relations known as the *hexagon equations* [24]

$$\sum_{\lambda\gamma}[R_e^{ac}]_{\alpha\lambda}[F_d^{acb}]_{(e\lambda\beta)(g\gamma\nu)}[R_g^{bc}]_{\gamma\mu} = \sum_{f\sigma\delta\psi}[F_d^{cab}]_{(e\alpha\beta)(f\delta\sigma)}[R_d^{fc}]_{\sigma\psi}[F_d^{abc}]_{(f\delta\psi)(g\mu\nu)} \tag{4.48}$$

and

$$\sum_{\lambda\gamma}[(R_e^{ca})^{-1}]_{\alpha\lambda}[F_d^{acb}]_{(e\lambda\beta)(g\gamma\nu)}[(R_g^{cb})^{-1}]_{\gamma\mu} = \sum_{f\sigma\delta\psi}[F_d^{cab}]_{(e\alpha\beta)(f\delta\sigma)}[(R_d^{cf})^{-1}]_{\sigma\psi}[F_d^{abc}]_{(f\delta\psi)(g\mu\nu)}. \tag{4.49}$$

These relations also have nice graphical representations (which their name obviously derives from) which are depicted in Figure 4.7.

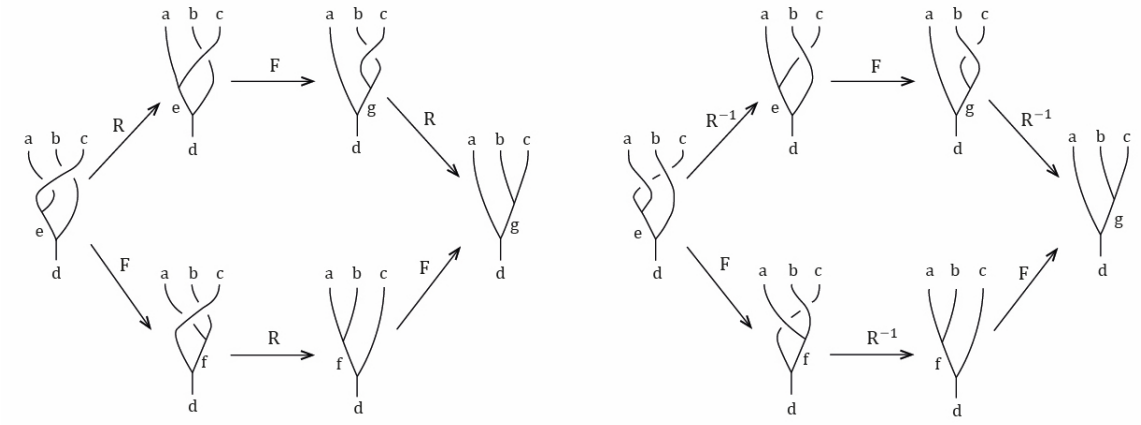


Figure 4.7: *The hexagon equations. The left one corresponds to Equation (4.48) and the right one to (4.49).*

Technically these relations illustrate that lines are allowed to slip over/under braids [24] which leads to compatibility between fusion and braiding. The consistencies enforced by the pentagon and hexagon equations dictate that any two sequences of R and F-transformations, that solve the pentagon and hexagon equations, and maps the initial space onto the final one within some category, must be equivalent [29]. This statement is known as *MacLane's coherence theorem* [24, 29] and is indeed very powerful. Roughly speaking, it states that no further requirements need to be enforced on the F and R-matrices other than that they solve the pentagon and hexagon equations, for the resulting morphic maps to be equivalent. Thus, a good starting point when an anyon model is to be constructed is to introduce a fusion rule by assumption and then search for solutions to the pentagon and hexagon equations. If one finds a unique set of solutions the assumption must have been correct and if the equations are unsolvable, the fusion rule in question must conflict with the fundamental principles of quantum physics and may thus be discarded [29]. It might also be possible to find several distinct sets of solutions which, in that case, correspond to different unique models but with the assumed fusion rule in common.

Chapter 5

Quantum double models

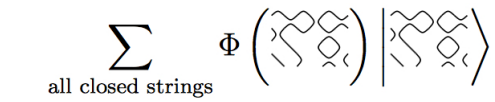
We have now come to the stage where we will introduce the concept of topological order and, from a mathematical point of view, discuss how to distinguish different topological phases. We know that classical phases can be studied by means of Landau's theory, but as we shall see, the nature of topological order is rather different from classical order which signals a need for a new mathematical framework. More specifically, we will study a well known model known as the *Toric-code* model, which exhibits topological order. We will also introduce topological defects to the model and see how these realize non-abelian quasi-particles, such as Majorana fermions, which can be used to encode protected quantum information. This will first be done in a $\mathbb{Z}_2$ lattice model which comprises 2 degrees of freedom per lattice site and then we will generalize to a $\mathbb{Z}_N$ model with N degrees of freedom per lattice site. However, we will start this Chapter by providing a brief introduction to topological phases in general before we go into the specific models.

5.1 Topologically ordered systems and string-net condensates

As we already know from Chapter 2 the fundamental units of information in a quantum computer are known as qubits (or qudits for higher dimensions). It is thus of highest interest to store the information encoded in these qubits as robustly as possible. In other words we want to protect it somehow which can be achieved if the qubit states are separated from the other non-computational states [20]. In many physical systems, e.g. a spin-$\frac{1}{2}$ particle, the degeneracy of the system is protected by symmetry [20]. Since, in absence of any perturbations, the particle can point both up and down, and none of the directions are energetically favourable, it is protected by $SU(2)$ symmetry. However, symmetries usually break due to noise, which in the case of the spin-$\frac{1}{2}$ particle can be an external magnetic field so that the ground state becomes the state pointing along the direction of the the field. So in order to obtain more reliable qubits the degeneracy must be protected more robustly. Thus, instead of symmetry, another way to provide stronger protection is to use the principle of topology. Therefore we will study systems that possess topological order, since a common feature of such systems in the thermodynamical limit is that their ground state degeneracy only depends on the topology of the manifold in which the system is embedded [19]. These degenerate ground states will span a protected subspace of the

full Hilbert space. In that way the degeneracy cannot be lifted by any perturbation of the Hamiltonian (assuming that the perturbation is smaller than the energy gap to the excited non-computational states) and as a result, the states will be well protected. This is, however, a rather phenomenological description of the principle of topological order, which at a deeper level has its origin in patterns of long range entanglements [36, 39]. This concept will not be discussed in a rigorous manner so we will merely aim to provide an intuitive explanation through an example that is presented in Ref. [39]. Let us consider a fermionic superfluid in which fermions move around in pairs, the two fermions in each pair move around according to the same rules and thus follows the same pattern. Imagine now that we have a system in which all particles are somehow connected so that every particle is moving around relatively to all other particles in a very systematic and organized way [39]. This behaviour will result in a global pattern that all particles participate in. This is, at least at a superficial level, the essence of topological order. Moreover, if two global patterns are not connected by some local unitary transformation, they correspond to two distinct topological phases [39]. On the other hand, if the patterns are connected by a unitary transformation, they belong to the same phase which makes it suitable to associate each topological phase with a tensor category [23]. Thus, for instance, if we consider a quantum spin liquid[1] consisting of interacting spin-$\frac{1}{2}$ particles, the ground state of the system is a superposition of all possible closed string configurations [19, 39]:

$$\sum_{\text{all closed strings}} \Phi \left(\right) \Big| \Big\rangle$$



where each loop is formed by particles with spins pointing in the same direction. We call such phase a *string-net condensate* [39, 23] which due to its global pattern of motion correspond to a non-trivial topological phase [39]. So how come such a system exhibits a global movement pattern? The answer lies in the rules that the strings are subject to as they move around. The first rule might be that the configurations always form closed loops in the ground state and the second rule could be that if we deform or reconnect loops, the amplitudes will not change in the wave function [39]. Hence, the ground state wave function that governs the system is a super position of all closed, and topologically inequivalent, loops which all have the same amplitude and thus gives rise to a global pattern. Note that these rules are merely an example of how a pattern, and thus a topological phase, can emerge. The rules could just as well be different which would result in a system possessing another pattern and topological phase. If the system contains non-closed strings it will be in an excited state in which the ends of the strings are topological defects which can be interpreted as quasi-particle excitations [39]. Thus since the closed loops and open strings never carry fractionalized quantum numbers, it might be the case that the topological excitations in the ends do [39]. This means that such strange excitation may have properties different from ordinary elementary particles, e.g. fractionalized charge and anyonic exchange statistics.

[1] A spin liquid can be regarded as disordered system of interacting spins. Similarly as liquid water is disordered state if compared to ice, a spin liquid is disordered state if compared to a ferromagnet.

Now, let us consider a $\mathbb{Z}_2$ spin liquid embedded on two 2-dimensional surfaces of different topology: a 2-sphere and a 2-torus with genus $g = 1$. On the 2-sphere we cannot form independent loops since all loops can be deformed into one another which will result in a unique ground state [39]. On the 2-torus, on the other hand, we can form two distinct and independent loops (see Figure 5.1).

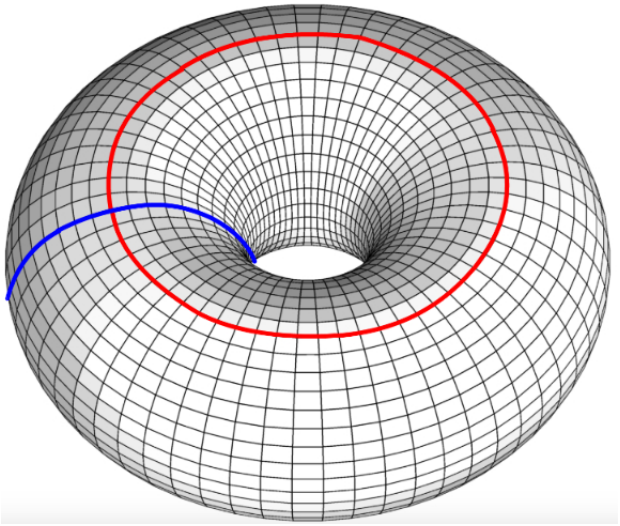


Figure 5.1: *2-torus with genus $g = 1$ on which the two independent non-contractible loops are marked out in blue and red.*

These have conserved numbers defined by the Wilson loop function

$$w_l = \prod_{j \in l} s_j, \quad l = l_1, l_2 \tag{5.1}$$

where $s_j = \pm 1$ labels the σ^z basis [20]. Therefore, since we can create a particle-antiparticle pair[2], and carry one of the particles along two independent, and topologically inequivalent loops, having either $w_l = 1$ or $w_l = -1$, and then annihilate the particles, the ground state degeneracy on a 2-torus (with genus $g = 1$) must be 4 [20, 39]. These 4 states correspond to: no loops, a red or a blue loop and both a red and a blue loop. On a general Riemann surface, however, with arbitrary genus g, the ground state degeneracy is given by $(2 \cdot 2)^g = 4^g$ [20, 19, 39].

5.1.1 Classification of topological phases

The nature of topological order is rather different from classical order so it may not be that surprising that it might require a different and more subtle mathematical framework than Landau's theory in order to classify different topological phases. We know that classical phases are characterized by an order parameter which value depends on whether the symmetry of the system is broken or not. One can thus say that symmetry breaking acts as a signature when a phase transition takes place. However, it was later discovered that some systems can be in different phases but still exhibit the same symmetry, which pointed towards the existence of a new type of order [37, 36].

[2]How to do this is discussed in Section 5.2.2.

We will now mainly follow Ref. [37] and [36] and provide an example of how different topological phases of a strongly interacting spin-$\frac{1}{2}$ system can be classified. For simple fermionic systems the ground state wave function can usually be formulated by means of a Slater determinant, but for more complex ones, such as spin liquids, it is much more complicated. Hence, instead we will use a slave-boson approach to find an approximate wave function [36]. Let us start by introducing a mean-field Hamiltonian ansatz with nearest neighbor interactions [37]

$$H_{mean} = \sum_{<i,j>} (\psi^{\dagger}_{I,i} \chi^{IJ}_{ij} \psi_{J,j} + \psi^{\dagger}_{I,i} \eta^{IJ}_{ij} \psi_{J,j} + h.c.), \tag{5.2}$$

where $I, J = 1, 2$ and $\psi_{K,k}$ denotes the fermion operators. The ground state wave function $|\Psi^{(\chi_{ij},\eta_{ij})}_{mean}\rangle$ of this Hamiltonian is much simpler than the original one which makes it possible to obtain it through a Slater determinant. Then if we employ a slave-boson technique and view the system as a boson model in which the up and down states are identified with the presence and the absence of a boson, respectively, a many-body boson wave function can be obtained through the projection [37]

$$\Phi^{(\chi_{ij},\eta_{ij})}(\mathbf{i}_1, \mathbf{i}_2, ..) = \langle 0| \prod_n b(\mathbf{i}_n) \, |\Psi^{(\chi_{ij},\eta_{ij})}_{mean}\rangle \, , \tag{5.3}$$

where $b(\mathbf{i}) = \psi_{1,\mathbf{i}}\psi_{2,\mathbf{i}}$ so the wave function can be interpreted as the bound state of two fermions [36]. We have now written down an approximate wave function of the system which is much simpler to handle than the real wave function. This strongly interacting spin system possesses topological order which means that different phases of the system still can exhibit the same symmetry. So instead of symmetry, we have to look for some other universal property in order to characterize the different phases. It is sufficient to study the mean-field ansatz (χ_{ij}, η_{ij}) of complex matrices, which is a rather simple set of objects, and thus search for operations that leave the ansatz invariant under its action [37]. Such a set of objects is referred to as a *projective symmetry group* (PSG). However, the ansatz does not correspond to a one-to-one labeling since two mean-field ansatzes can be connected by an $SU(2)$ gauge transformation, which gives rise to identical boson wave functions [36]. Therefore, if we consider a spin system, it is sufficient for it to obey translational symmetry up to an $SU(2)$ transformation of the ansatz, for the system to be regarded as translational invariant [36]. Thus we may impose that the ansatz is invariant under a translation followed by a gauge transformation. The difference from classical order now starts to become transparent. If we consider two spin systems which possess the same translational symmetry, it might be the case that their ansatzes remain invariant under the action of a composition of the same translational operation but different gauge transformations [36]. In other words, even if two systems possess the same symmetry, their respective PSG can still be different! The conclusion we can draw from this is that the PSG provides a more powerful tool to distinguish more subtle structures of the systems we are studying, which allows us to detect more exotic types of order, such as topological order. Moreover, different PSG's cannot be transformed into one another so phases characterized by different PSG's really are different states of matter even if they possess the same symmetry.

5.2 The Toric Code on a $\mathbb{Z}_2$ spin lattice

In Chapter 4 we studied the quantum double and discussed the excitations of the model in terms of fusion and braiding. The Toric-code is an example of a well studied quantum double model (a quantum double of $\mathbb{Z}_2$) which possesses topological order. This model, which first was introduced and studied by Kitaev (1997) [5], is a 2-dimensional $\mathbb{Z}_2$ lattice model embedded on a 2-torus with genus[3] $g = 1$. The Hamiltonian of the model reads [18, 20, 19]

$$H = -J_e \sum_s A_s - J_m \sum_p B_p, \tag{5.4}$$

which consists of four-body star and plaquette interaction terms with coupling strength J_e and J_m respectively (see Figure 5.2), defined as

$$A_s = \prod_{j \in star(s)} \sigma_j^x \tag{5.5}$$

and

$$B_p = \prod_{j \in \partial p} \sigma_j^z \tag{5.6}$$

where σ^x and σ^z represent the Pauli matrices defined in Equation (2.14) which act on the spins living on the edges of the lattice.

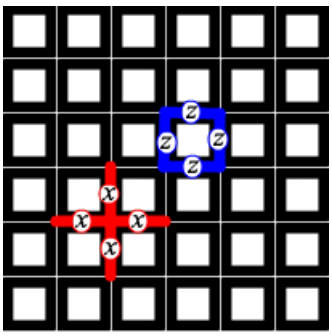


Figure 5.2: *The ordinary Toric-code lattice with a star (red) operator and a plaquette (blue) operator marked out.*

However, instead of studying the original formulation defined by the Hamiltonian in Equation (5.4) we will, for convenience, use a more symmetric model due to Wen[4] in which the Hamiltonian is defined by [37, 5]

$$H = -J \sum_k A_k, \tag{5.7}$$

where $A_k = \sigma_k^x \sigma_{k+\mathbf{i}}^z \sigma_{k+\mathbf{i}+\mathbf{j}}^x \sigma_{k+\mathbf{j}}^z$ and k denotes the lower left corner of each plaquette and $\mathbf{i}, \mathbf{j}$ are unit vectors. As one can see, this Hamiltonian only contains plaquette operators and describes the four-body interactions among the sites belonging to each plaquette in a "chessboard" lattice (see Figure 5.3).

[3] We will from now on assume that the torus in question always is a 2-torus with $g = 1$, and thus just refer to it as a torus.

[4] See Ref. [37].

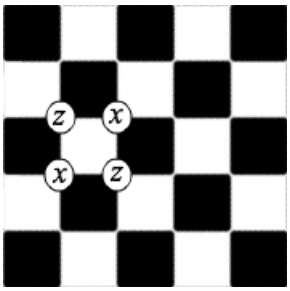


Figure 5.3: *Wen's chessboard lattice with a plaquette operator marked out.*

The reason why the lattice looks like this will become clear in the next Subsection where the excitations will be discussed. Another important difference between this lattice and the original one due to Kitaev is that the degrees of freedom lives on the vertices in this, whereas they live on the edges in the original one. If one consider Figure 5.3, one can see that every plaquette share two sites with each one of its adjacent neighbors. More specifically, two adjacent plaquattes will act with different Pauli operators on the sites shared, which will lead to a total factor of $(-1)^2 = 1$ due to the anti-commutation relation among the Pauli matrices when swapping the order of the operators. Hence every plaquette will commute with all of the adjacent plaquettes. In addition, all plaquettes will trivially commute with themselves and of course also with all non-adjacent ones, so in conclusion we have that all plaquette operators in the Hamiltonian must commute with one another. This further entails that all terms in the Hamiltonian can be simultaneously diagonalized which allows us to solve it term wise since all terms share eigenstates.

5.2.1 The ground states of the model

In order to find the ground state of the model, let us consider the Hamiltonian defined in Equation (5.7). What we want is to minimize the energy and due to the design of the Hamiltonian we can easily conclude that the minimal energy is obtained when the eigenvalues of each term in the Hamiltonian is equal to one[5], since the eigenvalue of each plaquette is ± 1 in a $\mathbb{Z}_2$ model, i.e. $A_k |\Psi\rangle = \pm |\Psi\rangle$. Hence the ground state of the model is a *stabilizer state* of the Hamiltonian. Thus, if we let $\mathcal{H}$ denote the full Hilbert space, the basis of the protected subspace $\mathscr{L} \subset \mathcal{H}$ are the stabilizer states

$$\mathscr{L} = \{|\phi\rangle \in \mathcal{H} \mid A_k |\phi\rangle = |\phi\rangle \,,\ \forall\ k\}, \tag{5.8}$$

so the ground state energy must thus be given by $E_0 = -JN_p$, where N_p is the number of plaquettes in the lattice. Moreover, the ground state manifold of the Toric-code possesses two important features [41]:

1. It is 4-fold degenerate.

2. The degeneracy of the ground state is protected by topology.

The first one of these was already discussed in Section 5.1 where we introduced Wilson loops (see Equation 5.1) and argued that all such non-contractible loops

[5]Assuming that $J > 0$.

will have eigenvalue ± 1, and since there always are two independent loops per hole, the ground state degeneracy for a general Riemann surface is $(2 \cdot 2)^g = 4^g$ where g denotes the genus of the surface. Thus, for a standard torus with $g = 1$, we have a ground state degeneracy equal to $4^1 = 4$. In other words, the ground state can be interpreted pictorially as a superpostion of all closed loop configurations which form a string-net condensate. The second point is a little bit more complicated to show and the main argument that can be used to prove it is that any perturbation can be written as a sequence of Pauli matrices. One can then show that such an arbitrary perturbation acts as a map from the ground state manifold back to itself[6]. Now, let us write down the ground state explicitly which is a linear combination of all basis vectors of $\mathscr{L}$, i.e.

$$|\Psi_0\rangle = \sum_{|\phi\rangle : A_k|\phi\rangle = |\phi\rangle \ \forall k} c_k |\phi\rangle , \tag{5.9}$$

where c_k are equal weight normalization constants such that $\sum_k |c_k|^2 = 1$ [20]. Finally, let us point out that the topological order of the system will be different if the sign of the coupling constant J is changed. We have in the above solution assumed that $J > 0$, so if we let $J < 0$ the underlying PSG will be different [37]. This can be shown by regarding the system as a hard-core boson model and introduce a mean-field Hamiltonian similar to the one in Equation (5.2). A detailed analysis of this is carried out in Ref. [37].

5.2.2 The excitations of the model

Let us now discuss the excitations, or *syndromes*, of the model and how they are created. Since we have already concluded that the ground state corresponds to the positive eigenvalues of the plaquettes, the excitated state must correspond to the negative ones. In other words, we may say that a plaquette holds an excitation if and only if $A_k |\psi\rangle = |\psi\rangle$ is violated [19]. As already mentioned, the Toric-code is a quantum double of $\mathbb{Z}_2$ [5], which means that it has two elementary quasi-particle excitations[7], namely e and m. These two can further be fused together to create a dyon ϵ, and as we shall see in the next Subsection, they produce a vacuum $\mathbb{1}$ when they are fused with themselves. In the original Toric-code lattice the electric charge e lives on the vertices of the lattice whereas the magnetic flux m lives on the plaquettes [20, 19]. However, in the lattice we are working with, we will assign the white plaquettes to the e particles and the black ones to the m particles. Note that this choice is completely arbitrary and one could just as well have assigned the plaquettes to the particles the other way around. As we discussed in the previous Section, excitations of topologically ordered systems are always created in pairs. So how can we create them? If we consider the structure of the Hamiltonian, we see that the σ^x and σ^z operators are located in a diagonal fashion in the lattice which means that, for instance, if we act on a σ^x-site with a σ^z-operator, the two plaquettes on the diagonal which share that σ^x-site will end up in an excited state (assuming that both of them initially were in the ground state), since the two operators anti-commute. Or more formally, if we for example act on the σ_k^x-site in the ground state

[6] For a detailed proof of this statement I refer to Ref. [41]

[7] These are in fact abelian anyons.

$|\phi\rangle$ with σ_k^z, we see that if we measure the eigenvalue by the action of the plaquette operator, we get

$$A_k(\sigma_k^z |\Psi_0\rangle) = (A_k\sigma_k^z) |\Psi_0\rangle = (-\sigma_k^z A_k) |\Psi_0\rangle = -\sigma_k^z(A_k |\Psi_0\rangle) = -(\sigma_k^z |\Psi_0\rangle), \quad (5.10)$$

since $A_k = \sigma_k^x\sigma_{k+\mathbf{i}}^z\sigma_{k+\mathbf{i}+\mathbf{j}}^z\sigma_{k+\mathbf{j}}^z$ so that $A_k\sigma_k^z = -\sigma_k^z A_k$ and $A_k |\phi\rangle = |\phi\rangle$. What we have achieved by doing this is to create two topological charges on these two plaquettes and this process can be continued by acting on other sites with appropriate Pauli operators. In this way one can form strings of arbitrary length along which excitations can be moved since every time one excites a plaquette by acting on a site that is shared with another excited plaquette, the exciation on the initially excited plaquette will annihilate and a new excitation will be produced on the new plaquette, due to the $\mathbb{Z}_2$ structure of the lattice. Thus if a quasi-particle pair is created, the error can be corrected by moving one of the particles around in a loop and then annihilate it with the other one. Note that according this construction, excitations on white plaquettes can only be moved to other white plaquettes, and the same holds of course for excitations on black plaquettes, so we are assured to avoid ambiguity. An example of such processes is depicted in Figure 5.4 where two arbitrary strings are created across the white and black plaquettes respectively.

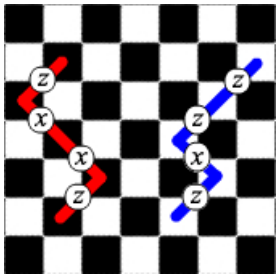


Figure 5.4: *Two examples of string operators which move excitations on black plaquettes (red string) and white plaquettes (blue string).*

An interesting, and for our purpose, important feature of such strings is that they are not sensitive to perturbations. This means that the excitations are independent of the string paths as long as the strings are topologically equivalent. In other words, we can choose any string in a given homotopy class and still get equivalent results as long as the end points of the strings are the same[8].

[8] Assuming that the string is not crossing any excited plaquettes.

To prove this, let us first define a string S_1^e as in Figure 5.5.

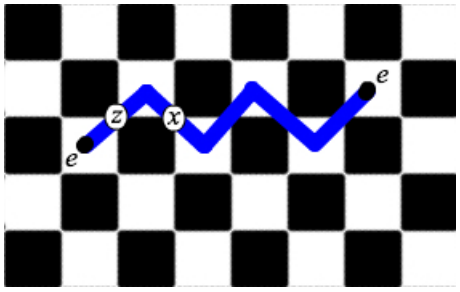


Figure 5.5: *Defining* S_1^e.

Now, if we define another string $S_2^e = A_k S_1^e$, where A_k corresponds to the black plaquette next to the right of the leftmost charge, it should look like the one in Figure 5.6.

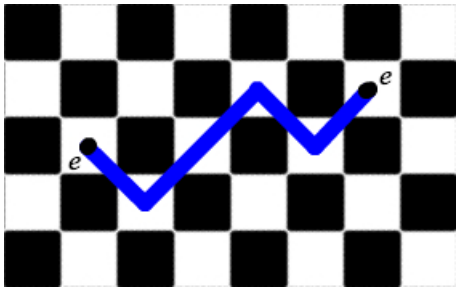


Figure 5.6: *Defining* $S_2^e = A_k S_1^e |\Psi_0\rangle$.

As one can see, the plaquette operator A_k in $S_2^e = A_k S_1^e$ shares two sites with S_1^e, which means that they have to commute. Further, if we assume that the system was in its ground state $|\Psi_0\rangle$ before the charges were created, it holds that $A_k |\Psi_0\rangle = |\Psi_0\rangle$, which thus finally yields that

$$S_2^e |\Psi_0\rangle = A_k S_1^e |\Psi_0\rangle = S_1^e A_k |\Psi_0\rangle = S_1^e |\Psi_0\rangle , \tag{5.11}$$

so in conclusion it must be true that these two topologically equivalent strings yield the same result. Note that S_2^e could have been chosen arbitrarily and that the same technique would still apply to prove the equivalence. The simple choice of S_2^e was nothing but a matter of convenience to illustrate the principle. Also, note that the same holds, of course, for magnetic strings. Another important property possessed by strings in this model is that when a string corresponding to the black plaquettes S_{black} and a string corresponding to the white plaquettes S_{white} cross each other an odd number of times, they will anti-commute, and otherwise commute. In conclusion we can thus formulate two important rules that can be used when strings are to be constructed [5]:

1. Strings are allowed to be continuously deformed into other strings as long they belong to the same homotopoy class.

2. S_{black} and S_{white} anti-commute when they intersect an odd number of times, and commute otherwise.

In addition, if the string that has been constructed form a closed and non-contractible loop, we go from one ground state to another, which of course also is an important feature of the model.

5.2.3 Fusion and braiding

Since we now know how to move the excitations around, the execution of fusion and braiding processes is fairly straight forward. Let us consider two elementary excitations of the same type, say e, separated with some distance. Thus, in order to fuse these together, we have to construct a string that starts on the plaquette which corresponds to the first excitation and ends at the plaquette which corresponds to the second excitation. Note however, as proven in the last Subsection, that the choice of string can be arbitrarily made as long as it belong to the right homotopy class and starts and ends on the right plaquettes. For that reason, let us choose the shortest and simplest path between the excitations as depicted in Figure 5.7. Now, since the first excitation will be moved along this string, a factor -1 will propagate across the plaquettes along which the string is constructed, to the plaquette in the end point which already has eigenvalue -1 since it already is excited. Consequently, the eigenvalue of the operation must be 1 which means that two e excitations will annihilate one another and form a vacuum $\mathbb{1}$, when brought together[9] (see Figure 5.7-5.8). Similarly, after constructing a suitable string operator, the same result when fusing two m particles is obtained. In addition, we already know from the quantum double construction that one e and one m form a dyon ϵ. Hence, in conclusion, we have the following fusion category in the $\mathbb{Z}_2$ Toric-code model:

$$e \times e = m \times m = \epsilon \times \epsilon = \mathbb{1}, \quad e \times m = \epsilon, \quad e \times \epsilon = m, \quad m \times \epsilon = e. \tag{5.12}$$

Also, all charges that are fused with the vacuum is of course left untouched.

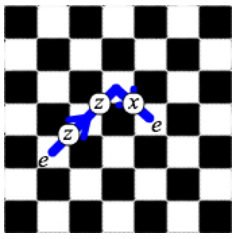


Figure 5.7: *Fusing two charges together.*

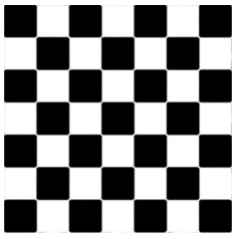


Figure 5.8: *The resulting vacuum.*

If we consider the above result from the point of view of vector spaces, Equation (4.35) tells us that this is an abelian model with trivial quantum dimension. Now, let us discuss the braiding properties. In order to braid one excitation around another, we just have to construct a string operator that realizes this process. Let us for instance move a magnetic flux m around an electric charge e by choosing to construct a simple loop as the one depicted in Figure 5.9. Note that due to the second rule that was stated in the last Subsection[10], the loop can be arbitrarily chosen as long as it corresponds to the right homotopy class.

[9] This is of course expected since the particle pair is created from the vacuum.

[10] See page 46

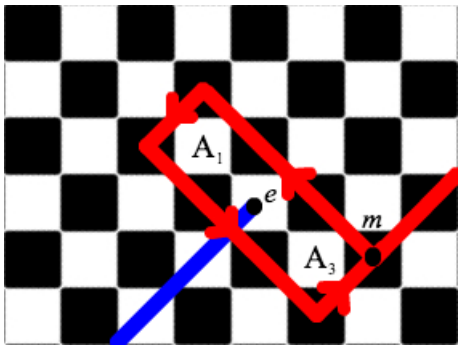


Figure 5.9: *Braiding an m charge around an e charge.*

If we assume that the system was in its ground state $|\Psi_0\rangle$ before the excitations were created, the whole process which corresponds to the creation of two quasi-particle pairs and braiding the m charge around the e charge can be written as $S_{loop,m}(S_m S_e |\Psi_0\rangle)$. Now, if we consider the loop in Figure 5.9 it is easy to see that this operation actually is the same as acting with the plaquette operators A_1, A_2 and A_3 which correspond to the white plaquettes encircled by the loop, since Pauli operators at different sites commute. Thus we have that $S_{loop,m} = A_1 A_2 A_3$ which yields

$$S_{loop,m}(S_m S_e |\Psi_0\rangle) = A_1 A_2 A_3 (S_m S_e |\Psi_0\rangle) = -(S_m S_e |\Psi_0\rangle), \tag{5.13}$$

since one of white plaquettes is excited, so in conclusion we have shown that the state picks up a phase $-1 = e^{i\pi}$ when braiding a magnetic flux around an electric charge. The same holds, of course, the other way around, when braiding an electric charge around a magnetic flux. Also, since the dyon is a composite object, braiding an e or an m around a ϵ will yield a phase equal to -1. Moreover it can be shown by using the same technique that a trivial phase is acquired upon braiding two identical excitations around one another. In conclusion, one can interpret e and m as bosons which obey non-trivial mutual statistics, and the dyon behaves as a fermion. Also note that any loop around e could have been chosen. This is possible since when applying the plaquettes enclosed by the loop, the interior region will split up into smaller loops which all cancel out since each loop segment will have a corresponding counterpart from the adjacent plaquette that goes in the opposite direction, in a similar way as when applying Stokes' theorem to a contour integral. Due to this we are ensured topological protection which is an important feature of the model. A pictorial representation of these rules is given in Figure 5.10-5.12. Hence, excitations of the $\mathbb{Z}_2$ Toric-code model obey mutual abelian statistics with trivial quantum dimension which does not allow for powerful computations [5], so it might thus be regarded as a toy model. Nevertheless, in the next section a strategy will be introduced which allows us to create defects in the lattice which, as we shall see, mimic the behaviour of non-abelian anyons with non-trivial quantum dimension which are more relevant from a computational point of view.

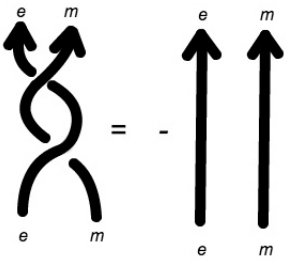


Figure 5.10: *Mutual braiding of e and m.*

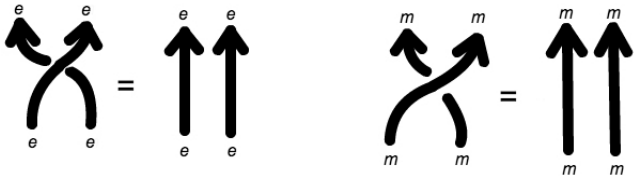


Figure 5.11: *Braiding e and m with themselves.*

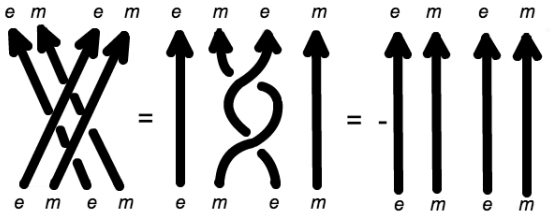


Figure 5.12: *Braiding two dyons as a result of the relations in Figure 5.10 and 5.11.*

5.3 Non-abelian anyons as twist defects

In this Section we will show that exotic excitations can be created in the Toric-code model, which can be regarded as non-abelian objects. In particular, we will, by the guidance of Bombin in Ref. [5], see that such defects referred to as *twists*, can be achieved by introducing dislocations in the lattice.

5.3.1 Introducing twists

We have, at this stage, established the rules for fusion and braiding in the $\mathbb{Z}_2$ model (see Equation (5.12) and Figures 5.10-5.12) and found out that all excitations are abelian objects. The crucial feature of these rules that make the introduction of twist defects possible, is that they possess a certain symmetry. By closer inspection one may notice that they are invariant under the interchange of e and m. This self-duality can be explained by the existence of a self-equivalence of the underlying algebraic operation, or modular tensor category, that interchanges the particles [5]. Due to this symmetry, we are allowed to perform a dislocation in the lattice by an odd number of plaquettes (see Figure 5.13), so that e and m are interchanged across the line along which the dislocation was performed[11], under the action of the symmetry operation.

[11]Note that the location of this line is unphysical.

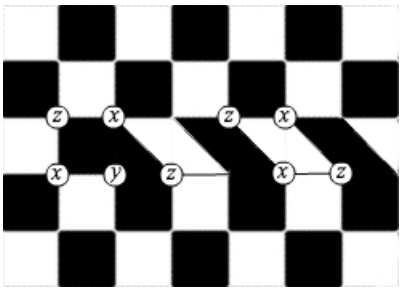


Figure 5.13: *A dislocated lattice.*

It is at the ends of the line, on the pentagonal plaquettes, the twists are located and since the coloring is shifted, any particle that crosses that line cannot form a closed loop, independent of the choice of path. Thus, it becomes clear that twist defects possess a topological nature and due to the shifting, the labeling of e and m becomes ambiguous. Another interesting consequence of twist defects is that if a dyon split into two distinct particles, one e and one m, and then one of the particles are transported in a loop across the line, it will end up on the same plaquette as the other particle and fuse into a vaccum which further means that the dyon and the vaccum cannot be distinguished globally. In conclusion we only have two distinguishable topological charges, instead of four, since $e \leftrightarrow m$ and $\epsilon \leftrightarrow \mathbb{1}$, in the presence of twist defects. In particular, instead of two loops that distinguishes the four charges, we now have one loop that winds twice around a twist [5]. This loop is depicted in Figure 5.14 and as one can see, it self-crosses which gives rise to two new topological charges which we shall label by $\sigma_\pm$ [5, 32], corresponding to the eigenvalues $\pm i$. This result can be derived by squaring the double loop operator Θ which gives us $\Theta^2 = (X_0 Z_0)^2 = X_0 Z_0 X_0 Z_0 = -X_0^2 Z_0^2 = -1$ (where X_0 and Z_0 act on the site where the loop self-crosses), since all other operator will commute and thus square to the identity. Hence Θ^2 must have eigenvalue -1 which means that the eigenvalue of Θ must be $\pm i$. Also note that the plaquette operator that corresponds to the pentagonal plaquette is now modified to be $A_{pentagon} = \sigma^x_k \sigma^y_{k+\mathbf{i}} \sigma^z_{k+2\mathbf{i}} \sigma^x_{k+\mathbf{i}+\mathbf{j}} \sigma^z_{k+\mathbf{j}}$ where the additional Pauli matrix corresponds to the trivalent vertex [33]. This plaquette still commutes with all other plaquettes in the Hamiltonian so the solubility is not destroyed. In conclusion, we thus have the following action of the double loop operator on the ground state

$$\Theta_\pm \left|\Psi_0\right\rangle = \pm i \left|\Psi_0\right\rangle, \tag{5.14}$$

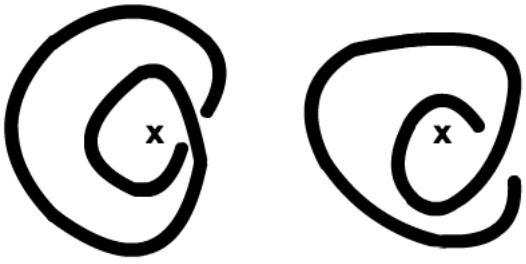


Figure 5.14: *Two double loops around a twist with different orientation.*

In the following Sections we will always choose the same orientation as the left on of these, which has eigenvalue $\pm i$.

5.3.2 Fusion with twists

We will now, from a graphical point of view, discuss and derive the fusion rules that correspond to fusion processes involving twists. Let us start by fusing an e excitation with a $\sigma_{\pm}$ (see Figure 5.15).

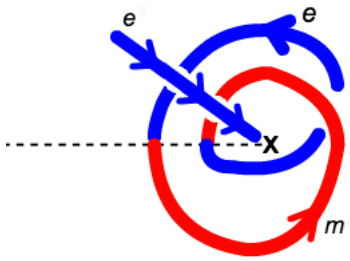


Figure 5.15: *Fusing an e with a $\sigma_{\pm}$. The dashed line illustrates the line through which e and m are exchanged.*

From this figure it is easy to see that the incoming charge (the blue line) will always cross the red part of the loop once, since e and m are exchanged along the dashed line. Therefore, due to rule number two presented in[12] Section 5.2.2, this process will contribute by a factor equal to -1, since it says that two strings that correspond to black and white plaquettes respectively, anti-commute if they cross each other an odd number of times. In addition, we can directly argue that the same process, but with an m charge instead of an e charge, will yield the same result, due to the crossing. Hence, we have that $e \times \sigma_{\pm} = m \times \sigma_{\pm} = -\sigma_{\pm} = \sigma_{\mp}$. Now, let us fuse a dyon with a twist (see Figure 5.16).

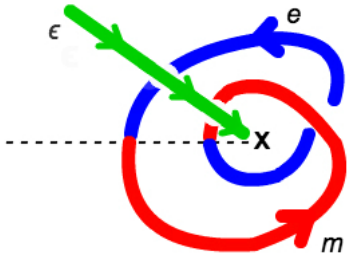


Figure 5.16: *Fusing an ϵ (green line) with a $\sigma_{\pm}$. The dashed line illutrates the line across which e and m are exchanged.*

In this figure the green line represents the dyon, and since a dyon is a composition of an e and an m charge, we know that it has to pick up two minus signs, one for each crossing. For that reason it must be the case that the twist is left unchanged, i.e. $\epsilon \times \sigma_{\pm} = \sigma_{\pm}$. Finally, in order to derive the last two fusion rules between the twists, an additional rule need to be introduced, which states how different strings can be attached/detached to each other. This rule is represented pictorially in Figure 5.17.

[12] See page 46.

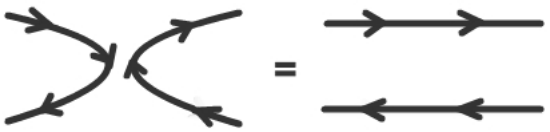
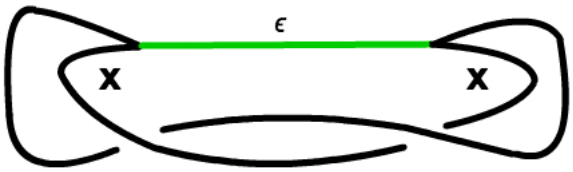
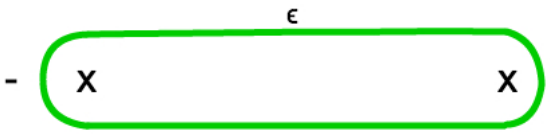


Figure 5.17: *A third rule which shows how strings can be attached/detached to each other. The equality can be proven by deforming one of the strings by the action of an appropriate plaquette, similar to Figure 5.6.*

Now, let us introduce two twists of the same kind as the one to the left in Figure 5.14. By using the deformation rule, in addition to the third rule described in Figure 5.17, we can, in a suitable manner, deform the loops and attach the inner loops and outer loops to each other, respectively, so that we end up with[13]:



where the green line represents the dyonic string. As one can see, the strings are entangled, but if we change the order by swapping the operators at one of the crossings, the strings will "unlock" which makes it possible to deform the loops completely into one another, so that the resulting loop is completely dyonic:



where the minus sign appears due to the anti-commutation relation. Now, since the dyonic loop corresponds to one loop which goes across the white plaquettes and one which goes across the black plaquettes, the resulting operator can be written as $S_{white}S_{black}$, and since we know that any closed loop can be written as a product of the plaquettes encircled by the loop[14], S_{white}/S_{black} has eigenvalue -1 if there is an m/e particle in the region, and 1 otherwise. Thus, since the loops around the twists have eigenvalue $S_{\sigma_\pm} = \pm i$, we have the following equality: $S_{black}S_{white} = -S_{\sigma_\pm}S_{\sigma_\pm} = -(i)^2 = 1$. This means that the fusion between two equal twists should lead to a vacuum or a dyon which results in the fusion rule $\sigma_\pm \times \sigma_\pm = \mathbb{1} + \epsilon$, since if both S_{white} and S_{black} are 1 the result will be a vacuum whereas if both are -1, we both have an e and an m which yields a dyon. This result is also consistent with the rule $\epsilon \times \sigma_\pm = \sigma_\pm$ we derived earlier. One can also interpret this in terms of representations, as in Equation (4.19), where the tensor product between the irreducible representations of the twists decomposes into a direct sum over the irreducible representations corresponding to the vacuum and the dyon. Furthermore,

[13]Note that the specific string colors for e and m are not marked out, only the dyon which is marked out with green.

[14]See Figure 5.9. In this figure the loop operator that carries the m particle around the e particle is denoted by $S_{loop,m}$

if we fuse two twists of different species together the only difference is that the sign will change so that $S_{black}S_{white} = -S_{\sigma_\pm}S_{\sigma_\mp} = -(i)\cdot(-i) = -1$, which implies that the possible outcome can either be an e or an m, or formally $\sigma_\pm \times \sigma_\mp = e + m$, since if S_{white} is -1 we must have that S_{black} is 1, and vice versa. Hence, the full collection of fusion rules in the model is given by:

$$e \times \sigma_\pm = m \times \sigma_\pm = \sigma_\mp, \quad \epsilon \times \sigma_\pm = \sigma_\pm, \quad \sigma_\pm \times \sigma_\pm = \mathbb{1} + \epsilon, \quad \sigma_\pm \times \sigma_\mp = e + m. \tag{5.15}$$

Now, if we consider these relations from the point of view of vector spaces we know according to Equation (4.35), that this is a non-abelian model with non-trivial quantum dimension. The quantum dimension d_x of an anyon follows naturally from Equation (4.34) and is defined as [33, 6]

$$d_a d_b = \sum_c N_{ab}^c d_c, \tag{5.16}$$

which tells us that the quantum dimension of twists is $\sqrt{2}$ which is an irrational number, whereas e, m and ϵ have unit quantum dimension. This result signals an existence of non-abelian braiding statistics for twists which will be discussed in Section 5.3.4. Moreover the total quantum dimension of an anyon model is given by [33, 6]

$$\mathcal{D} = \sqrt{\sum_x d_x^2} \tag{5.17}$$

which for the general $\mathbb{Z}_2$ model with charges $\{e, m, \epsilon, \mathbb{1}\}$ yields $\mathcal{D}_{\mathbb{Z}_2} = 2$, whereas for the twist model with charges $\{e, m, \epsilon, \mathbb{1}, \sigma_\pm\}$ we have $\mathcal{D}_{twist} = \sqrt{8}$. We can also conclude that since the outcome when fusing two twists is non-unique, the corresponding fusion states can be used to represent qubits which allow for protected quantum information to be encoded. Worth to point out though is that twists are in fact not intrinsic non-abelian anyons since they are merely extrinsic defects and not elementary excitations of the model [43]. However, as we shall see, they do indeed carry non-trivial statistics but as a projective representation of the braid group.

5.3.3 Realizing fusion with Majorana operators

An interesting observation of the above rules is that the set $\{\mathbb{1}, \sigma_\pm, \epsilon\}$ exactly mimics the behaviour of Ising anyons under fusion [5]. The Ising anyon model has three charges $\mathbb{1}, \sigma$ and ψ [5], and the behaviour of these is exactly similar to that of twists if we let $\sigma_\pm \leftrightarrow \sigma$ and $\epsilon \leftrightarrow \psi$. Furthermore, it turns out that the σ particle in the Ising anyon model can be characterized by Majorana operators [5]. These similarities reflect the fact that Ising anyons as well as twists have a quantum dimension that coincide with that of Majorana fermions [43]. Also, it may not be that surprising that the system can be interpreted in terms of fermions, since we are dealing with a spin-$\frac{1}{2}$ model in which we simply can identify the up and down-spin with the presence and absence of a fermion [20]. Hence we can perform a so called Jordan-Wigner transformation[15] of the system where we go from a spin representation to a

[15] See Appendix C for a detailed discussion.

fermion representation [20, 43]. These two representations are related as

$$c_{2j-1} = a_j + a_j^\dagger, \quad c_{2j} = \frac{a_j - a_j^\dagger}{i} \tag{5.18}$$

where $a_i^\dagger$ and a_i are creation and annihilation operators in the spin represenattion. The Majorana operators c_i are Hermitian and satisfy $\{c_j, c_k\} = 2\delta_{jk}$. In particular, by defining the operator $-ic_jc_{j+1}$, one obtains the total charge when two σ's are being fused together, where the value 1 corresponds to the vacuum and -1 to an ϵ particle [5]. Moreover, if the Jordan-Wigner transformation is applied to a lattice with twists defects, it can be shown that pairs of twists will produce unpaired Majorana zero modes (for example, see Ref. [43]) due to the trivalent lattice points. Because of this, it might be natural to assume that fusion among twists also can be realized with Majorana operators. This is indeed the case, which can be proven by choosing suitable strings to represent Majorana operators [5]. Assuming that we have two σ_+ twists, and following Bombin's suggestion in Ref. [5], we may define the two strings S_j and S_{j+1} as in Figure 5.18.

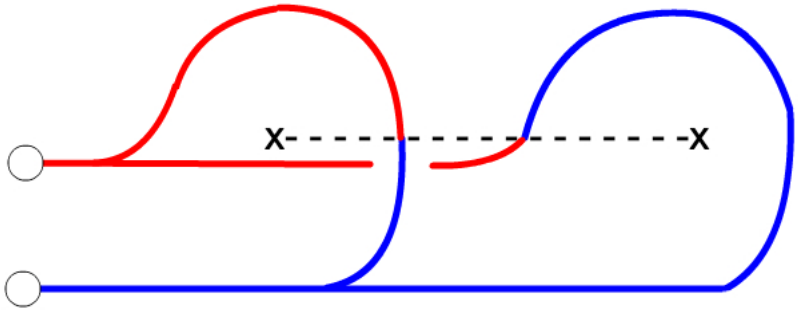
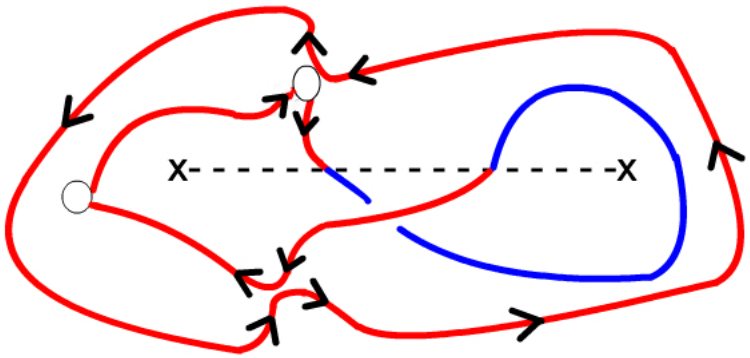
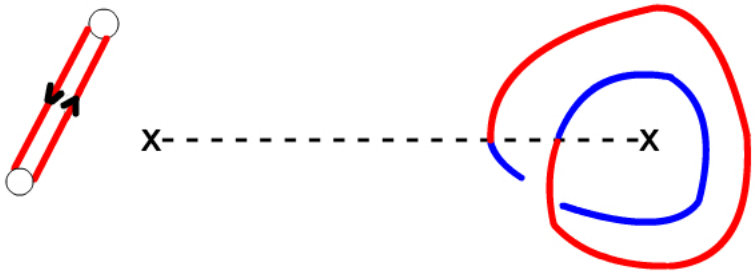


Figure 5.18: *Defining the two string operators S_j and S_{j+1} which realize c_j and c_{j+1}.*

Now, by adding an extra loop, say an S_{black} (a red loop), around this configuration, which has eigenvalue 1 if it encircles a vacuum and -1 if it encircles an e particle, and deform the initial strings, we end up with:



so if we further use the rule depicted in Figure 5.17 we can glue the strings together which finally gives us:



However, if we were to add a loop S_{white} (a blue loop) instead we would end up with a similar configuration only that the double loop would be of the other orientation, which simply differs from this one by a factor -1 due to the anti-commutation relation. Hence, since the double red line will fuse into a vacuum, only the loop around the twist will contribute by $\pm i$, which yields that $-ic_jc_{j+1} = -iS_jS_{j+1} = -i \cdot (\pm i) = \pm 1$, i.e. a vacuum or a dyon. In conclusion, we have shown by choosing the strings as in Figure 5.18, that twists realize Majorana operators where the eigenvalue -1 corresponds to a dyon and the eigenvalue 1 corresponds to a vacuum. In the next Subsection we will see that twists also obey the same braiding statistics as Majorana operators, which confirms the analogy with Ising anyons.

5.3.4 Braiding twists with Majorana operators

In order to braid the twist defects we may transform the "geometry" of the Hamiltonian adiabatically [5], so that the twists are moving around one another. The strings S_j introduced in Figure 5.18 anti-commute so that $\{S_j, S_k\} = 0$ if $j \neq k$ just as Majorana operators, which obey the following set of braiding rules [5]

$$c_j \to c_{j+1}, \quad c_{j+1} \to -c_j, \quad c_k \to c_k. \tag{5.19}$$

We will now prove that twists follow these rules as well. Assume now that we have two σ_+ twists in Figure 5.18 and lets drag the right one anti-clockwise over the left one. This will give us the configuration depicted in Figure 5.19.

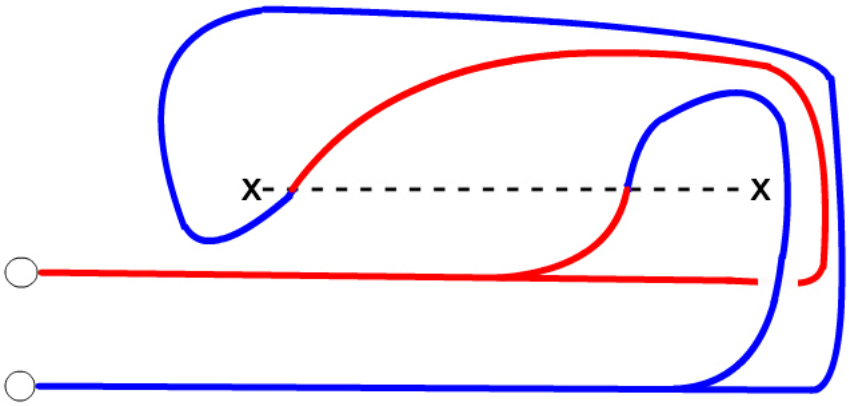
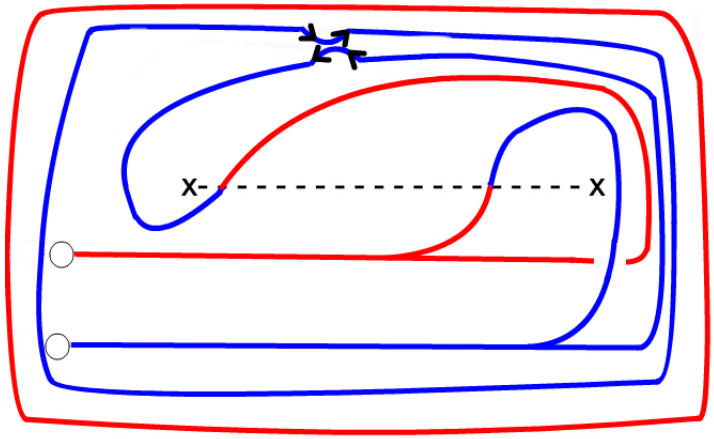
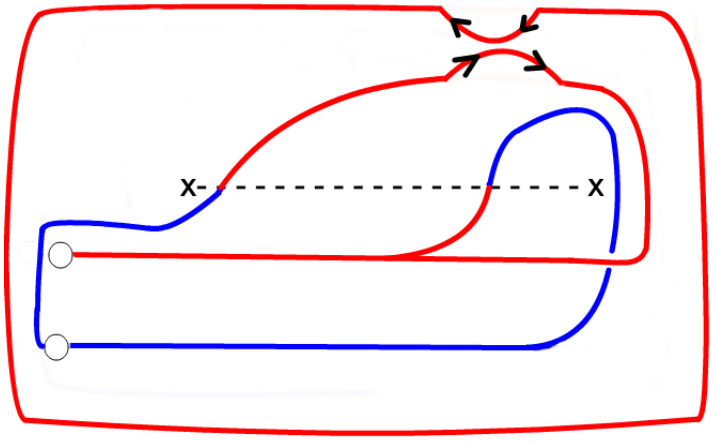


Figure 5.19: *The starting point when deriving the braiding rules.*

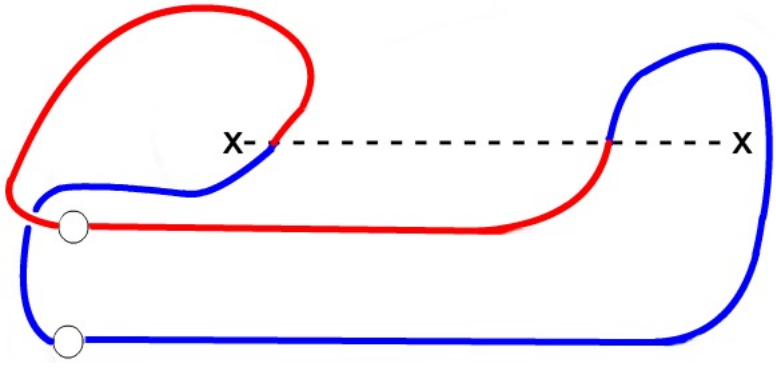
Our goal is to manipulate this configuration so that the end result looks like the configuration in Figure 5.18. Let us start by adding two loops, one S_{black} and one S_{white}, which have eigenvalue 1 in absence of any excitations, around the configuration:



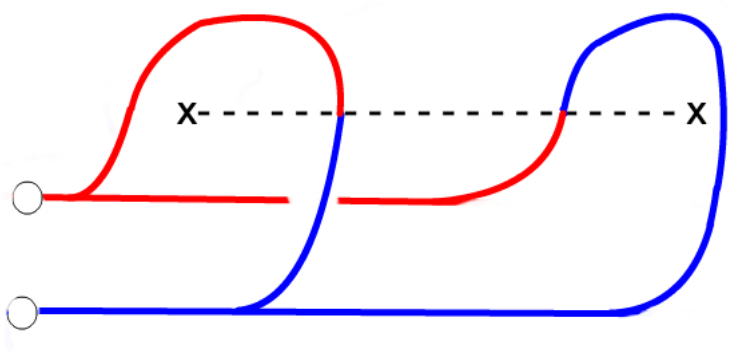
and now we can use the rule defined in Figure 5.17 and deform the blue loop so that:



and doing the same thing with the red loop gives us:



We are now very close to reach the desired end result, and the only things that we have to do are to deform the loop once again, and more crucially, we have to change the order where the loops intersect. The reason for this is simply that if we regard Figure 5.18, and let the right string be S_{i+1} and the left one be S_i, we see that the operator in Figure 5.18 is S_iS_{i+1} whereas we now have he operator $S_{i+1}S_i$, which means that we must change the order which yields a factor -1 due to the anti-commutation relation. By doing this we thus get:



Therefore, due to the appearance of this minus sign we have shown that braiding of twists follow the same rules as the Majorana operators defined in Equation 5.19, so in conclusion, according to the results obtained, twists mimic the behaviour of Ising anyons exactly in terms of fusion and braiding.

5.4 Generalizing to a $\mathbb{Z}_N$ model

At this point we have discussed and solved the original Toric-code and we saw that the excitations of the model do not possess non-abelian braiding statistics which is required for powerful quantum computations to be realized. However, by introducing pentagonal plaquettes in the model, or twists, we saw that these do in fact carry a projective non-abelian representation of the braid group and that they could be interpreted as unpaired Majorana zero modes. A natural attempt may thus be to introduce twists in a generalized lattice with $\mathbb{Z}_N$ symmetry. In such a lattice each lattice point comprises N degrees of freedom instead of 2 as in the $\mathbb{Z}_2$ case. Note that the Hilbert space in this model will expand from being a $(\mathbb{C}^2)^{\otimes m}$ space to a $(\mathbb{C}^N)^{\otimes m}$ space where m denotes the number of sites in the lattice. Therefore, the first thing we need to do is to redesign the Hamiltonian which describes the interactions of the model. To do this we simply have to generalize the Hamiltonian in Equation (5.7), where we instead of the standard Pauli matrices use the generalized Pauli matrices of arbitrary order defined as [42]

$$Z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{5.20}$$

These matrices follow the following algebra

$$X^N = Z^N = I, \quad Z^\dagger = Z^{N-1}, \quad X^\dagger = X^{N-1}, \quad XZ = \omega ZX, \tag{5.21}$$

where $\omega = e^{i\frac{2\pi}{N}}$ and I is the identity matrix. Moreover, for any state $|q\rangle$, we have that

$$Z\,|q\rangle = \omega^q\,|q\rangle\,, \quad X\,|q\rangle = |q-1\ (mod\ N)\rangle\,, \tag{5.22}$$

where $q \in \mathbb{Z}_N$. However, the crucial difference when $N > 2$ is that they are no longer Hermitian. This means that in order for the Hamiltonian to be an observable, we cannot simply exchange σ^x with X and σ^z with Z in Wen's Hamiltonian defined in Equation (5.7). What we can do though is to add the Hermitian conjugate of each term to the Hamiltonian, since it holds for any operator O that $(O + O^\dagger)^\dagger = (O^\dagger + (O^\dagger)^\dagger) = (O^\dagger + O)$, so that the Hamiltonian can be modelled as

$$H_{\mathbb{Z}_N} = -J \sum_k (A_k + A_k^\dagger), \tag{5.23}$$

where $A_k = X_k Z_{k+\mathbf{i}} X_{k+\mathbf{i}+\mathbf{j}} Z_{k+\mathbf{j}}$ and has eigenvalue ω^q which further means that the eigenvalue of $A_k + A_k^\dagger$ is $\omega^q + \omega^{-q} = 2\cos(\frac{2\pi q}{N})$, since $\omega^q = e^{i\frac{2\pi q}{N}}$ and the sine function in the complex exponential is an odd function. Also because all plaquettes share two sites with each one of the adjacent plaquettes, the relation $XZ = \omega ZX$ implies that $[A_k, A_{k'}] = [A_k, A_{k'}^\dagger] = 0$ for all k and k', so it's still possible to solve the model term wise and the ground state is thus given when all eigenvalues are equal to 2, i.e. $(A_k + A_k^\dagger)\,|\Psi_0\rangle = 2\,|\Psi_0\rangle \ \ \forall\ k$.

5.4.1 The excitations

The next thing we are interested in to know is how to create excitations in the model. The elementary excitations of the model are still of the same type only that the energy spectrum is now more diverse. Just as in the $\mathbb{Z}_2$ case, particle-antiparticle pairs can be created by acting with X and Z (or $X^\dagger$ and $Z^\dagger$) operators on sites in a suitable way, due to the "commutation relation" in Equation (5.21). Thus if we, for instance, act on a site k with Z_k, and then measure the energy of the two plaquettes which share that site with X operators, the energy eigenvalue is given by[16]

$$(A_k + A_k^\dagger)(Z_k\,|\Psi_0\rangle) = Z_k(\omega A_k + \omega^{-1} A_k^\dagger)\,|\Psi_0\rangle = Z_k(\omega + \omega^{-1})\,|\Psi_0\rangle = 2\cos(\frac{2\pi}{N})(Z_k\,|\Psi_0\rangle). \tag{5.24}$$

However, this energy corresponds to the first excited state so in order to create a general state, say the n^{th} state, we can just act with the operator Z_k^n which yields

$$(A_k + A_k^\dagger)(Z_k^n\,|\Psi_0\rangle) = 2\cos(\frac{2\pi n}{N})(Z_k^n\,|\Psi_0\rangle), \tag{5.25}$$

so the energy eigenvalue corresponding to the n^{th} state is thus given by $E_n = -J(\omega^n + \omega^{-n}) = -J(e^{i\frac{2n\pi}{N}} + e^{-i\frac{2n\pi}{N}}) = -2J\cos\frac{2n\pi}{N}$, and the ground state energy E_0 of each plaquette is thus obtained when $n = 0$ so that $E_0 = -2J$. However, since the energy is given by a cosine, which is a periodic and even function, some energy

[16] Note that we just as well could have acted with an X operator on a Z site.

states should be degenerate. Through some simple geometrical considerations of the unit circle, it is easy to derive that there are $\frac{N}{2}$ distinct excited states, among which only one of them is non-degenerate, if N is even, and $\frac{N-1}{2}$ distinct excited states among which all are degenerate, if N is odd. This is because all reflection points in the x-axis yield the same cosine value since it's an even function. In conclusion, we thus have with $\epsilon^{(i,j)} = e^i m^j$, the following set of topological charges[17]

$$\{\mathbb{1}, \epsilon^{(1,0)}, .., \epsilon^{(N-1,0)}, .., \epsilon^{(0,1)}, \epsilon^{(1,1)}, .., \epsilon^{(N-1,1)}, .., \epsilon^{(0,N-1)}, \epsilon^{(1,N-1)}, .., \epsilon^{(N-1,N-1)}\} \tag{5.26}$$

where only $\frac{N}{2}$ of the eigenvalues of the pure e's and m's are distinct, and thus $(\frac{N}{2})^2$ of eigenvalues corresponding to the ϵ's, if N is even, whereas if N is odd, we have that $\frac{N-1}{2}$ of the eigenvalues of the pure e's and m's are distinct, and thus $(\frac{N-1}{2})^2$ of ones corresponding to the ϵ's.

5.4.2 Construction of strings

We have already discussed and concluded how to create excitations in the model so it should be fairly straight forward to construct arbitrarily long strings along which the excitations can be moved, by applying the same idea. However, since each plaquette can be in more than two states one has to be a little more cautious than in the $\mathbb{Z}_2$ model. What we want to achieve is to find a sequence of operators that move the excitation step wise by annihilate it on the "old" plaquette and create a new on the "new" plaquette. In the $\mathbb{Z}_2$ case this process could easily be implemented just by acting on a site that belongs to an excited plaquette with the "opposite" Pauli matrix (i.e. a σ^z if the corresponding operator in the plaquette was a σ^x, and vice versa), so that the new plaquette got excited and the previous one returned to its ground state due to the $\mathbb{Z}_2$ symmetry. In the $\mathbb{Z}_N$ generalization, however, we do not get away that easy because the excitation on the previous plaquette will not annihilate automatically due to the wider eigenvalue spectrum. Nevertheless, if we consider an excited state ($Z_0 |\Psi_0\rangle$) and create a new state by acting with $Z_1^\dagger$, we have that

$$(A_1 + A_1^\dagger)(Z_1^\dagger Z_0 |\Psi_0\rangle) = Z_1^\dagger Z_0(\omega^{-1}\omega A_k + \omega\omega^{-1}A_k^\dagger) |\Psi_0\rangle = 2(Z_1^\dagger Z_0 |\Psi_0\rangle), \tag{5.27}$$

so the excitation will be annihilated on the old plaquette (plaquette 1 in Figure 5.20) and since the energy eigenvalue for the new plaquette (plaquette 2 in Figure 5.20) is determined by

$$(A_2 + A_2^\dagger)Z_1^\dagger |\Psi_0\rangle = Z_1^\dagger(\omega^{-1}A_2 + \omega A_2^\dagger) |\Psi_0\rangle = (\omega^{-1} + \omega)(Z_1^\dagger |\Psi_0\rangle), \tag{5.28}$$

the energy of the new excitation will be $-2J\cos\frac{2\pi}{N}$ so the energy is preserved. What happens is that when acting with $Z_1^\dagger$ the anti-particle of the initial one will be created on plaquette 1 which leads to annihilation, and a particle, similar to the one that inhabited plaquette 1, will be created on plaquette 2, so that the excitation is transported from plaquette 1 to 2. Further, if we continue the to build on the string

[17]Note that the pure charges corresponds to $e^i = \epsilon^{(i,0)}$ and $m^j = \epsilon^{(0,j)}$.

by adding an operator Z_2, the process will be repeated so that the excitation ends up on the next plaquette (plaquette 3 in Figure 5.20). The same holds of course for the X operators so we have thus shown that excitations can be moved along straight strings which are constructed such that Z is alternated with $Z^\dagger$ and X with $X^\dagger$. Let us exemplify this by creating an excitation on plaquette 1 in the state $|q\rangle$ by acting with Z_0^q, and then move the excitation to plaquette 3. This process can be formulated as $Z_2^q(Z_1^\dagger)^q(Z_0^q|\Psi_0\rangle)$, and the energy of plaquette 3 can thus be computed as

$$(A_3 + A_3^\dagger)(Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle) = (\omega^{-q} + \omega^q)(Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle), \tag{5.29}$$

whereas if we measure the energy of plaquette i where $i \in \{1, 2\}$ we simply get

$$\begin{aligned}(A_i + A_i^\dagger)(Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle) &= (\omega^{-q}\omega^q + \omega^q\omega^{-q})(Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle) \\ &= 2(Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle),\end{aligned} \tag{5.30}$$

which corresponds to the ground state energy.

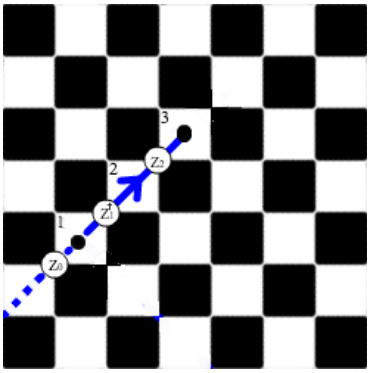


Figure 5.20: *An excitation with energy* $-2J\cos\frac{2\pi q}{N}$ is created on plaquette 1 by the incomming dashed string and moved to plaquette 3 by the solid string.

However, if we wish to change the direction, say from the x-direction to the z-direction (on the diagonal), we have to act with an $Z^\dagger$ operator if the last plaquette was excited by the action of the a $X^\dagger$ operator, and hence an Z operator if the last plaquette was excited by an ordinary X operator. Thus, if we want to move the excitation on plaquette 3 in Figure 5.20 to plaquette 4 (see Figure 5.21) we can act on the lower right site in plaquette 3 with the operator X_3^q, since the plauette was excited by the action of Z_2^q. The full string operator that realizes the creation of an excitation $Z_0^q|\Psi_0\rangle$ with energy $-2J\cos\frac{2\pi q}{N}$ on plaquette 1 and transportation of it to plaquette 4 can thus be formulated as $X_3^q Z_2^q(Z_1^\dagger)^q(Z_0^q|\Psi_0\rangle)$ since

$$\begin{aligned}&(A_3 + A_3^\dagger)((X_3^q Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle) \\ &= (\omega^{-q}\omega^q + \omega^q\omega^{-q})((X_3^q Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle) = 2(X_3^q Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle)\end{aligned} \tag{5.31}$$

and

$$\begin{aligned}&(A_4 + A_4^\dagger)(X_3^q Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle) \\ &= (\omega^q + \omega^{-q})(X_3^q Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle) = 2\cos\frac{2\pi q}{N}(X_3^q Z_2^q(Z_1^\dagger)^q Z_0^q|\Psi_0\rangle),\end{aligned} \tag{5.32}$$

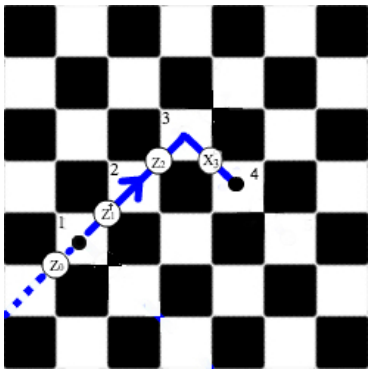


Figure 5.21: *An excitation with energy* $-2J\cos\frac{2\pi q}{N}$ *is created on plaquette 1 by the incoming dashed string and moved to plaquette 4 by the solid string.*

so the excitation is moved from plaquette 3 to plaquette 4.

To finish this Subsection, let us write down the following two rules for constructing strings in the $\mathbb{Z}_N$ model:

1. For straight strings, X^k (Z^k) has to be alternated with $(X^\dagger)^k$ $((Z^\dagger)^k)$.
2. In order to change the direction if the last plaquette was excited by an $X^k/(X^\dagger)^k$ $(Z^k/(Z^\dagger)^k)$ operator, one has to act with a $Z^k/(Z^\dagger)^k$ $(X^k/(X^\dagger)^k)$ operator.

Also note that if one considers Figure 5.5 and 5.6, it is easy to verify, by commuting the string and the plaquette, that strings in the $\mathbb{Z}_N$ model are insensitive to perturbations as well, which of course is a very important feature of the model.

5.4.3 Fusion and braiding in the $\mathbb{Z}_N$ model

Now that we know how excitations can be moved around, the next step is to discuss fusion and braiding in the model. As you may already have guessed, the diversity of fusion channels in a $\mathbb{Z}_N$ model gets increasingly larger as N increases since there are $N-1$ electric and magnetic excited states respectively, and thus $(N-1)^2$ dyonic states. Let us start by considering an electric charge e^q in its q:th state which is fused with another electric charge e^p in its p:th state. Thus, due to the cyclic nature of the model the fusion outcome should be

$$e^q \times e^p = e^{q+p}\ (mod\ N), \tag{5.33}$$

since the phases should add up when one of the excitations is moved to the plaquette inhabited by the other excitation, and the same holds of course for the magnetic charge. For a general dyon, however, we may denote it by $\epsilon^{(k,l)} = e^k m^l$ such that $e^k \times m^l = \epsilon^{(k,l)}$. Hence, if we fuse an electric charge e^i with $\epsilon^{(k,l)}$ we get

$$e^i \times \epsilon^{(k,l)} = e^i \times (e^k \times m^l) = (e^i \times e^k) \times m^l = e^{i+k} \times m^l = \epsilon^{(i+k,l)}\ (mod\ N), \tag{5.34}$$

and similarly a magnetic flux m^j gives us

$$m^j \times \epsilon^{(k,l)} = m^j \times (e^k \times m^l) = e^k \times (m^j \times m^l) = e^k \times m^{j+l} = \epsilon^{(k,j+l)}\ (mod\ N). \tag{5.35}$$

Finally if we consider two dyons $\epsilon^{(i,j)}$ and $\epsilon^{(k,l)}$ we have, in the same way, that

$$\epsilon^{(i,j)} \times \epsilon^{(k,l)} = (e^i \times e^k) \times (m^j \times m^l) = e^{i+k} \times m^{j+l} = \epsilon^{(i+k,j+l)} \ (mod\ N), \quad (5.36)$$

so in conclusion the following rules determine the fusion outcomes in $\mathbb{Z}_N$:

$$\begin{gathered} e^q \times e^p = e^{q+p}, \quad m^j \times m^l = m^{j+l}, \quad e^k \times m^l = \epsilon^{(k,l)} \\ e^i \times \epsilon^{(k,l)} = \epsilon^{(i+k,l)}, \quad m^j \times \epsilon^{(k,l)} = \epsilon^{(k,j+l)}, \quad \epsilon^{(i,j)} \times \epsilon^{(k,l)} = \epsilon^{(i+k,j+l)} \end{gathered} \quad (5.37)$$

where all exponents are taken modulus N. As one can see, all these rules imply that the fusion outcome always is definite, which means that the corresponding fusion states are not suitable as implementations of qubits.

Now, let us continue our discussion and consider braiding in the model. From Figure 5.9 it became clear that a loop operator that takes an m around an e (or similarly an e around an m), could be written as a product of the plaquettes enclosed by the loop. Nevertheless, since the excitations do not terminate automatically one has to be a little bit cautious since the regular plaquette operators must be alternated with Hermitian conjugate ones. If we regard the same configuration as in Figure 5.9 but now with two general charges e^q and m^p (see Figure 5.22), and assume that m^p was created by the action of Z^p, we see that if we use the rules as those defined in the last Subsection, the action of this loop operator gives the same result as if we act with $A_1^p (A_2^\dagger)^p A_3^p$. Formally we can thus write

$$S_{loop,m}(S_m S_e \left|\Psi_0\right\rangle) = (A_1^p)^\dagger A_2^p (A_3^p)^\dagger (S_m S_e \left|\Psi_0\right\rangle) = A_2^p (S_m S_e \left|\Psi_0\right\rangle) = \omega^{qp} (S_m S_e \left|\Psi_0\right\rangle), \quad (5.38)$$

where S_m and S_e are strings that created the m^p and e^q charges respectively, and $S_{loop,m}$ is the loop operator which carries m^p around e^q.

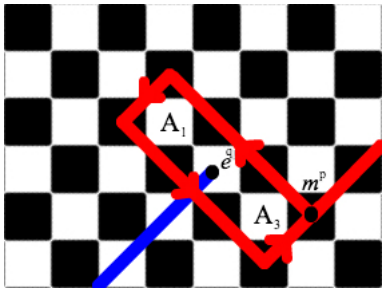


Figure 5.22: *A magnetic flux m^p is carried around an electric charge e^q.*

In the same way one can show that a general phase is acquired upon braiding an e or an m around an ϵ, and that two identical excitations give a trivial phase. In conclusion we thus know that the mutual braiding statistics between e and m is still non-trivial whereas braiding two identical charges do not change the state.

5.5 Introducing twists

Following the same reasoning as in the $\mathbb{Z}_2$ model we can, due to the exchange symmetry of the fusion and braiding rules under $e \leftrightarrow m$, create a dislocation by introducing pentagonal plaquettes in the lattice such that e and m are exchanged under the symmetry operation across the line which connects these plaquettes. So if we let the operator corresponding to the trivalent lattice point be $Y = XZ$, it is easy to check that this plaquette still commutes with all other plaquettes so that the solubility is preserved as desired. However, since we are now dealing with a more diverse particle spectrum, the eigenvalues of the twists will depend upon the state in which the excitation is that is carried around it, and will as a result also have a wider spectrum with N eigenvalues. Also in this lattice we have a Wilson loop Θ that winds twice around the defects and in order to determine the eigenvalues, let us consider the N^{th} power of the double loop operator Θ where we assume that it was created by dragging an e or m in the first excited state twice around it. We know that operators that act on different sites commute so due to this we can write $\Theta^N = (X_0 Z_0^\dagger)^N X_1^N Z_2^N ...$ where $(X_0 Z_0^\dagger)$ corresponds to the crossing for a given orientation. But $X^N = (X^\dagger)^N = Z^N = (Z^\dagger)^N = I$, so the N^{th} power of the double loop operator should simply be $\Theta^N = (X_0 Z_0^\dagger)^N$. What we have to do next is to swap the operators in a way which gives us a phase and operators of the N^{th} power, which yields a phase and identity operators. Lets write it out explicitly:

$$
\begin{aligned}
(X_0 Z_0^\dagger)^N &= \underbrace{X_0 Z_0^\dagger X_0 Z_0^\dagger X_0 \cdots X_0 Z_0^\dagger}_{2N \ factors} \\
&= \omega X_0 X_0 Z_0^\dagger Z_0^\dagger X_0 \cdots X_0 Z_0^\dagger \\
&= \omega\omega^2 X_0 X_0 X_0 Z_0^\dagger Z_0^\dagger \cdots X_0 Z_0^\dagger \\
&= ... = \omega\omega^2 \cdots \omega^{(N-1)} \underbrace{X_0 X_0 \cdots X_0}_{N \ factors} \underbrace{Z_0^\dagger Z_0^\dagger \cdots Z_0^\dagger}_{N \ factors} \\
&= \omega^{1+2+...+(N-1)} I^2 = \omega^{1+2+...+(N-1)} = \omega^{\sum_{s=1}^{N-1} s}
\end{aligned}
\tag{5.39}
$$

but $\sum_{s=0}^{N-1} s = \frac{1}{2}N(N-1)$ so the eigenvalue one obtains when acting with Θ^N should be $\omega^{\frac{1}{2}N(N-1)} = e^{i\pi(N-1)} = \pm 1$ ($+1$ for odd N and -1 for even N), which means that we have $e^{i\frac{\pi}{N}(N-1)}$ for Θ. If we now stop for a moment and and regard this expression one may notice that the eigenvalue quantization depends on whether N is even or odd. For instance, if we choose N to be even, we see that $\frac{N-1}{2}$ must be an odd multiple $2m-1$ of $\frac{1}{2}$ so that the eigenvalues are of the form $e^{i\frac{\pi}{N}(2m-1)}$, whereas if N is odd the eigenvalues $e^{i\frac{2\pi}{N}\frac{2m}{2}} = e^{i\frac{2\pi}{N}m}$. In other words we can conclude that the eigenvalue spectrum of Θ consists of phases which are all multiples of $\frac{\pi}{N}$ if N is even and multiples of $\frac{2\pi}{N}$ if N is odd. It is also easy to verify that the eigenvalues in the $\mathbb{Z}_2$ case is obtained if we let $N = 2$.

5.5.1 Fusing twists in the $\mathbb{Z}_N$ model

Now, let us discuss fusion among twists. If we consider an incoming electric charge e^q in the q^{th} state that is brought together with a twist of type σ_+ as the one with eigenvalue of Θ that is $e^{i\frac{2\pi}{N}\frac{(N-1)}{2}}$ (see Figure 5.23), and use the same argument as in the $\mathbb{Z}_2$ case, we should pick up a phase $e^{i\frac{2\pi}{N}q}$ so that the resulting eigenvalue is $e^{i\frac{2\pi}{N}(q+\frac{(N-1)}{2})}$ since the incoming blue string crosses the red part of the loop once, which yields the rule $e^q \times \sigma_k = \sigma_{q+k}$ $(mod\ N)$. Similarly, by fusing the twist with an m^p in the p:th state, we obtain $m^p \times \sigma_k = \sigma_{p+k}$, and since a dyon $\epsilon^{(i,j)}$ will contribute twice, once for each intersection, the fusion rule involving a dyon reads $\epsilon^{(l,j)} \times \sigma_q = \sigma_{q+l+j}$, since the phase we pick up is $e^{i\frac{2\pi}{N}l}e^{i\frac{2\pi}{N}j} = e^{i\frac{2\pi}{N}(l+j)}$. So by fusing twists with these charges, the twists transform into different states by absorbing (or emitting) a quasi-particle.

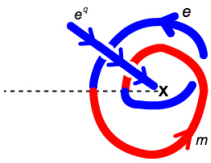


Figure 5.23: *Fusing an e^q with a twist.*

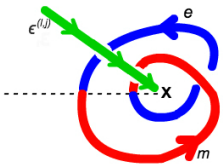
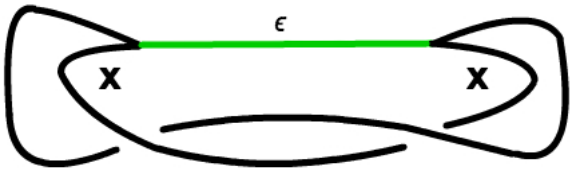


Figure 5.24: *Fusing an $\epsilon^{(l,j)}$ with a twist.*

If we now consider the fusion between two twists σ_s and σ_t, and use the same strategy as in the $\mathbb{Z}_2$ model, we can deform and attach the two double loops so that we end up with the following configuration[18]:



Further by changing the order where they intersect with the given orientation, which should yield a factor $\omega = e^{i\frac{2\pi}{N}}$, and then deform the two loops completely, we get:

so if we let S_{white}/S_{black} be the loops corresponding to the white/black plaquettes[19], and let $S_{\sigma_s}/S_{\sigma_t}$ be the two double loops, we have that

$$\omega S_{white} S_{black} = S_{\sigma_s} S_{\sigma_t},$$

[18] We also choose double loops with the same orientation (the right one in Figure 5.14).

[19] These have eigenvalue ω^k since if we drag an e/m around an m^k/e^k (similarly as in the $\mathbb{Z}_2$ case), we obtain ω^k.

or equivalently

$$S_{white}S_{black} = \omega^{-1}S_{\sigma_s}S_{\sigma_t} = e^{-i\frac{2\pi}{N}}e^{i\frac{2\pi}{N}(s+\frac{(N-1)}{2})}e^{i\frac{2\pi}{N}(t+\frac{(N-1)}{2})} = e^{i\frac{2\pi}{N}(s+t+N-2)} = e^{i\frac{2\pi}{N}(s+t-2)}.$$

Now we are interested in figuring out how many unique possible combinations of S_{white} and S_{black} that yield $S_{white}S_{black} = e^{i\frac{2\pi}{N}(s+t+N-2)}$. Note that, for a given N (odd or even), the factor $e^{i\frac{2\pi}{N}(s+t-2)}$ is the same for all allowed combinations of s and t such that their sum $s+t$ is constant. This should imply that the fusion rules for all such s and t are the same! However, we already know, due to cyclicity, that both S_{white} and S_{black} have N eigenvalues each, and that all correspond to phases that are multiples of $\frac{2\pi}{N}$. This means that if we choose any of the N eigenvalues of S_{white}, we can only choose the eigenvalue of S_{black} in one single way in order to obtain the required phase, which further entails that we have N different combinations which all correspond to a unique dyon. Or more explicitly, if we for example choose S_{white} to be $e^{i\frac{2\pi}{N}(s+t-2)}$, S_{black} must be equal to $e^0 = 1$, and if we choose $e^{i\frac{2\pi}{N}((s+t-2)-1)} = e^{i\frac{2\pi}{N}((s+t-3)}$, we have to choose S_{black} equal to $e^{i\frac{2\pi}{N}}$, and so on. Formally we thus have the fusion rule

$$\sigma_s \times \sigma_t = \epsilon^{(s+t-2,0)} + \epsilon^{(s+t-3,1)} + \epsilon^{(s+t-4,2)} + ... + \epsilon^{(s+t-2-(N-1),N-1)} = \sum_{k=0}^{N-1} \epsilon^{(s+t-2-k,k)}$$

for any N (even and odd), where each specific dyon labeled by $\epsilon^{(s+t-2-k,k)}$ depends upon the value of the sum $s+t$. Note that if we were to choose the other orientation (the rightmost in Figure 5.14), which would give another value for $S_{white}S_{black}$, a phase $\omega^{-1} = e^{-i\frac{2\pi}{N}}$ would be acquired upon changing the order were the loops intersect, which would give us

$$S_{white}S_{black} = \omega S_{\sigma_s}S_{\sigma_t} = e^{i\frac{2\pi}{N}}e^{i\frac{2\pi}{N}(s+\frac{(N-1)}{2})}e^{i\frac{2\pi}{N}(t+\frac{(N-1)}{2})} = e^{i\frac{2\pi}{N}(s+t+N)} = e^{i\frac{2\pi}{N}(s+t)},$$

and hence the rule

$$\sigma_s \times \sigma_t = \epsilon^{(s+t,0)} + \epsilon^{(s+t-1,1)} + \epsilon^{(s+t-2,2)} + ... + \epsilon^{(s+t-(N-1),N-1)} = \sum_{k=0}^{N-1} \epsilon^{(s+t-k,k)},$$

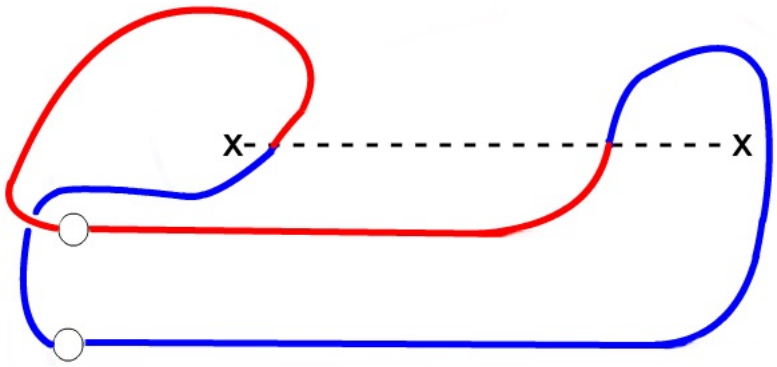
which looks a little bit nicer. So in conclusion, we have shown that we always have N-fold degeneracy (independent of the values of s and t), but that the rules can be categorized into classes, depending on the value of $s+t$, for which the rules are identical. We can also conclude that the dimension of the fusion space which corresponds to the fusion between two twists depends on the number of degrees of freedom per lattice site, due to the N-fold degeneracy. Moreover, if we consider Equation (5.16), the quantum dimension of twists in a $\mathbb{Z}_N$ model is $\sqrt{N}$ since the quantum dimension of each dyon is 1, so in conclusion the $\mathbb{Z}_N$ twist model is non-abelian for any $N > 1$ according to Equation (4.35). Finally, let us write down the collection of fusion rules for the twists:

$$e^q \times \sigma_s = m^q \times \sigma_s = \sigma_{q+s}, \quad \epsilon^{(q,p)} \times \sigma_s = \sigma_{s+q+p}, \quad \sigma_s \times \sigma_t = \sum_{k=0}^{N-1} \epsilon^{(s+t-k,k)}. \quad (5.40)$$

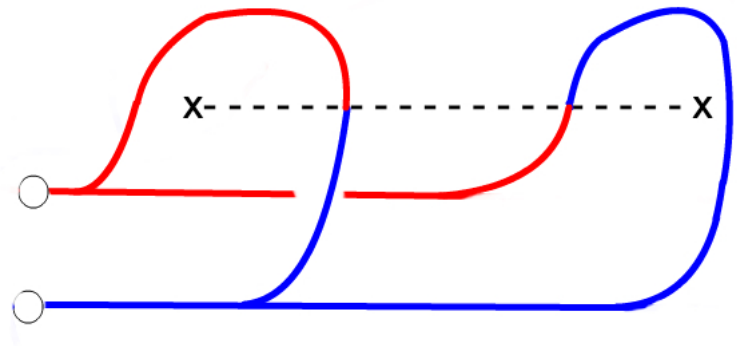
Also note that since the dimensionality of the fusion space depends on N, the amount of information that can be encoded in a fusion state increases drastically with N. In other words, the capacity of the quantum memory will be greatly amplified as N increases.

5.5.2 Braiding twists in the $\mathbb{Z}_N$ model

Now, let us see what happens if we assume that the string operators for general twists can be defined in the same way as in the $\mathbb{Z}_2$ model (see Section 5.3.4). If we then apply the same technique to find the braiding statistics we obtain the following configuration:



by putting a red and a blue loop around it which are then attached and deformed in a suitable way. Further by deforming this, and swapping the order where the strings intersect, we obtain:



So if we assume that the quasi-particles that formed the strings initially were in their k^{th} and m^{th} state, respectively, a phase $e^{i\frac{2\pi}{N}km}$ should be acquired since the operators must be swapped km times, given that the strings are defined in a particular way. However, if we choose to construct the strings differently, we may end up with a phase $e^{-i\frac{2\pi}{N}km}$ instead since the strings may overlap with different operators where they intersect. In the $\mathbb{Z}_2$ model we were always guaranteed a factor -1 since the Pauli operators anti-commute which means that it does notn't matter how the strings overlap, a minus sign will always be acquired when changing the order. Now we obtain the following rules instead:

$$S_i \to S_{i+1}, \quad S_{i+1} \to \omega^{\pm km} S_i \tag{5.41}$$

where the sign depends on how the string operators are defined. In conclusion, assuming that the loops can be defined in a similar way as in the $\mathbb{Z}_2$ model, and if we apply the same strategy when braiding the twists, we see that a general phase is acquired which depends upon the number of degrees of freedom N, and the states of the quasi-particles that formed the strings.

Chapter 6

Conclusions and outlook

The field of topological quantum computing is a very wide and diverse discipline which comprises many different subjects in fundamental physics as well as mathematics. We started out with an introduction to quantum computing in general where we discussed how to represent information by means of binary numbers, and then we argued that quantum states could be used to represent information which, due to the superposition principle, would lead to an enormous capacity. The information, however, need to be processed in order for the computer to perform actual computations. The main idea that the models we analyzed are based upon is to use braiding as an implementation of logical gates. As was discussed in Chapter 3, the special topological features of the configuration space in two dimensions gives rise exotic phenomena such as anyonic exchange statistics that can be used to process the information. First we discussed the already well studied Toric-code model (see Kitaev in Ref. [20]) and found that its ground state was four fold degenerate and separated with a finite energy gap from the excited states. This degeneracy could be deduced from the topology of the manifold on which the lattice was defined, which in this case was a torus, and due to the energy gap to the excited states we argued that the ground state could be used as a suitable representation of a qubit since its degeneracy is protected by topology. We also derived the fusion category and braiding rules of the model and found that it indeed had well defined rules. However, in order to perform powerful quantum computations the braiding statistics need to be non-abelian and this requirement was not met. As was shown in Section 5.2.3, the electric charge and the magnetic flux behaved like bosons whereas their fusion product, the dyon, behaved like a fermion in terms of exchange statistics. Their mutual braiding statistics, however, is non-trivial but still abelian which is not sufficient. Then we followed Bombin in Ref. [5] and argued, due to the symmetry of the fusion and braiding rules under exchange $e \leftrightarrow m$, that we were allowed to create a dislocation in the lattice so that electric charges became magnetic fluxes, and vice verse, as they moved across the line along which the dislocation was performed. These dislocation defects, or twists as we call them, could also be interpreted as particle like objects and we also showed that they had well defined fusion and braiding rules. When we derived the fusion rules we found out that the twist defects must be non-abelian objects since their fusion outcome was indefinite. This was indeed the case which we saw when we derived the braiding rules explicitly.

Another interesting feature that we found about twist defects in the $\mathbb{Z}_2$ model is that they realize Majorana fermions. In Section 5.3.3 and 5.3.4 we followed Bombins

suggestion in Ref. [5] and defined operators which formed closed strings around the twists and showed that the braiding and fusion rules of Majorana fermions were exactly reproduced by twist defects. Moreover, since the twists do obey non-abelian braiding statistics their fusion space might be a suitable place for quantum computations in which the fusion states constitute the qubits. Next we generalized the twist model to a lattice which comprises an arbitrary number N of degrees of freedom per lattice site. The first thing that was done was to derive the excitation spectrum of the model and we found, not very surprisingly, that the width of the spectrum depends on N. More exactly we saw that the number of possible excitations grew rapidly with N since there are $N-1$ distinct pure charges of each type and hence many different combinations of composite charges (dyons). In particular, the eigenvalue quantization of the plaquette operators depends on if whether N is even or odd. Furthermore, in order to perform braiding and fusion, strings along which the excitation can be moved had to be constructed and since the model no longer had $\mathbb{Z}_2$ symmetry, we found that the strings had to be constructed in a specific way to transport the excitations properly. For example, we found that in order to move excitations along straight lines, the X and Z operators had to be alternated with their Hermitian conjugates. Then we computed the braiding rules and just like in the $\mathbb{Z}_2$ model, we saw that the excitations obeyed mutual abelian statistics, but now an arbitrary complex phase could be acquired. The fusion rules were also pretty similar to those of the $\mathbb{Z}_2$ model only that there were more combinations of particles that could be fused.

Moreover, twist defects were introduced in the lattice and according to the fusion rules we derived, the twist model is non-abelian for any $N \geq 2$. In fact, if the lattice sites comprise N degrees of freedom, the fusion state corresponding to two twists will be N fold degenerate. In the $\mathbb{Z}_2$ case the fusion state corresponds to 2 simultaneous bits, which implies that m pairs of twists correspond to 2^m bits, whereas for a general N the fusion state that corresponds to two twists represents a general qudit in which even more information can be stored. For instance, if we have $N = 4$ or $N = 5$, one single qudit corresponds to two qubits, and if we have $N = 6$ or $N = 7$, the qudit corresponds to 3 qubits, and so on. In conclusion we can thus say that the more degrees of freedom, the more information can be encoded. However, as already discussed, braiding need to be performed in order to process the information. In the $\mathbb{Z}_2$ model we defined string operators in a suitable way which resulted in a specific set of brading rules. The same technique was then used to braid the twists in the $\mathbb{Z}_N$ model and we saw that a general phase was picked up, which depended on the states in which the excitations were that created the strings. Hence, according to the results obtained, any $\mathbb{Z}_N$ model can, at least theoretically, be used to make up a topological quantum computer.

All lattice models that were studied in this work were defined as square lattices so another interesting idea would be to see if it's possible to introduce twist defects in lattices of other geometries. The Toric-code but with a hexagonal lattice has already been solved by Kitaev in Ref. [20] so it might be the case that it supports twist defects with projective non-abelian braiding statistics as well. If that is the case, and if it also turns out that the fusion rules are well defined, it could also be a candidate for topological quantum computing.

Appendix A

The fundamental group

The below definitions are taken from Ref. [16]. Let X be a space of some topology, and x_0 an arbitrary point in X. Now, consider two loop functions

$$f, g : \ [0,1] \to X$$

of different homotopy such that $f(1) = g(0) = x_0$. Then, the composition of these loops is defined by the following parametrization

$$f \circ g(s) = \begin{cases} f(2s), \ 0 \leq s \leq \frac{1}{2} \\ g(2s-1), \ \frac{1}{2} \leq s \leq 1 \end{cases}$$

This parametrization can be interpreted as the composition of f and g, which first goes along f from $s = 0$ to $s = \frac{1}{2}$ with twice the pace, and then along g from $s = \frac{1}{2}$ to $s = 1$ with twice the pace. Thus, with the above defined composition rule, the set of homotopy classes $\{[f], [g], ..., [h]\}$ forms a group of loops in X

$$\pi_1(x_0, X) = \{[f], [g], ..., [h]\}$$

which all start and end in x_0.

Appendix B

Hopf algebras

Here we will follow Ref. [30]. A Hopf algebra over a field $\Bbbk$ is an algebraic structure consisting of the tuple $(A, m, \eta, \Delta, \epsilon, \mathcal{S})$, where (A, m, η) forms an algebra over $\Bbbk$ such that

$$A \otimes A \xrightarrow{\mathrm{m}} A, \qquad m(a \otimes b) = ab \quad \forall a, b \in A$$

$$k \xrightarrow{\eta} A, \qquad \eta(1_k) = 1_A \qquad k \in \Bbbk$$

which is associative, i.e.

$$a(bc) = (ab)c \; \forall a, b, c \in A,$$

and (A, Δ, ϵ) is a coalgebra over $\Bbbk$ so that

$$A \xrightarrow{\Delta} A \otimes A, \qquad \Delta(a) = a_{(1)} \otimes a_{(2)}, \quad \forall a_{(i)}, a_{(j)} \in A \; (\textit{coproduct})$$
$$A \xrightarrow{\varepsilon} k, \qquad k \in \Bbbk \; (\textit{counit})$$

which is coassociative

$$a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}$$

$$\varepsilon(a_{(1)})a_{(2)} = a = a_{(1)}\varepsilon(a_{(2)})$$

and Δ and ε are both morphic maps

$$\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \qquad \Delta(1) = 1 \otimes 1$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \qquad \varepsilon(1) = 1.$$

The antipode $\mathcal{S}$ is defined by the morphic map

$$A \xrightarrow{\mathcal{S}} A$$

,

$$\mathcal{S}(ab) = \mathcal{S}(a)\mathcal{S}(b) \; \forall a, b \in A$$

and satisfies

$$a_{(1)}\mathcal{S}(a_{(2)}) = \varepsilon(a)1 = \mathcal{S}(a_{(1)})a_{(2)}.$$

Appendix C

The Jordan-Wigner transformation

The Jordan-Wigner transformation provides a way to go from a spin representation to a particle representation of a lattice model. This is achieved by the identification of a certain number of particles by the projection of a certain spin onto the z-axis, i.e. $|s\rangle \leftrightarrow n = s$, where s is the spin state in the z-direction and n is the number of particles present. Following Ref. [17] the transformation which maps tensor products of generalized Pauli matrices onto tensor products of parafermion operators, is for odd and even lattice sites defined as

$$\alpha_{2i-1} = (\prod_{k<i} X_k)Z_i, \quad \alpha_{2i} = e^{i\frac{2\pi}{N}\frac{(N+1)}{2}}(\prod_{k\leq i} X_k)Z_i,$$

where X and Z are the generalized Pauli matrices for a $\mathbb{Z}_N$ lattice with arbitrary degrees of freedom per lattice site. The crucial feature of this transformation is that the desired algebra is still satisfied so that

$$\alpha_l\alpha_m = e^{i\frac{2\pi}{N}}\alpha_m\alpha_l, \quad \alpha_i^N = I.$$

Moreover the parity operator is defined as

$$\Lambda_i = e^{i\frac{2\pi}{N}\frac{(N+1)}{2}}\alpha_i\alpha_{i+1}^\dagger,$$

where the factor $e^{i\frac{2\pi}{N}}$ is needed so that $\Lambda_i^N = I$ for all N. These operators obey the following relations

$$\begin{cases} \Lambda_i\Lambda_j = \Lambda_j\Lambda_i, \; if \; |i-j| > 1 \\ \Lambda_i\Lambda_j = e^{i\frac{2\pi}{N}sgn(j-i)}\Lambda_j\Lambda_i, \; if \; |i-j| = 1. \end{cases}$$

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