The Kitaev-Hubbard chain
(and a parafermion generalization)

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Outline

- Kitaev chain and zero modes
- Kitaev-Hubbard chain
- Overview of the phase diagram
- Strong zero mode?
- Diagnosis using analytic argument and finite temperature edge magnetization
- Parafermion generalization: frustration free line

Kitaev chain

The Kitaev chain: prototypical model with Majorana zero-modes:

$$H_{\text{Kit}} = -J \sum_{j=1}^{L-1} \left(c_j^\dagger c_{j+1}^\dagger + c_j^\dagger c_{j+1} + \text{h.c.} \right) - h \sum_{j=1}^{L} \left(1 - 2 c_j^\dagger c_j \right)$$

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In terms of Pauli matrices (after Jordan-Wigner):

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H_{Kit} is a free-fermion model, solvable by a canonical transformation on the fermion operators:

$$H_{\text{Kit}} = \sum_{j=1}^{L} \epsilon_j \left(2 f_j^\dagger f_j - 1 \right)$$

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Operator connecting the pairs: σ_1^x

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More non-trivial example: XYZ chain (Fendley)
Question: is the strong zero mode due to integrability or can non-integrable models have a strong zero mode?

Kitaev - Hubbard chain

A natural interaction term for the Kitaev chain is of Hubbard type

$$H_{\mathrm{KH}} = -J\sum_{j=1}^{L-1}\left(c_j^\dagger c_{j+1}^\dagger + c_j^\dagger c_{j+1} + \mathrm{h.c.}\right) - h\sum_{j=1}^{L}\left(1 - 2c_j^\dagger c_j\right)$$
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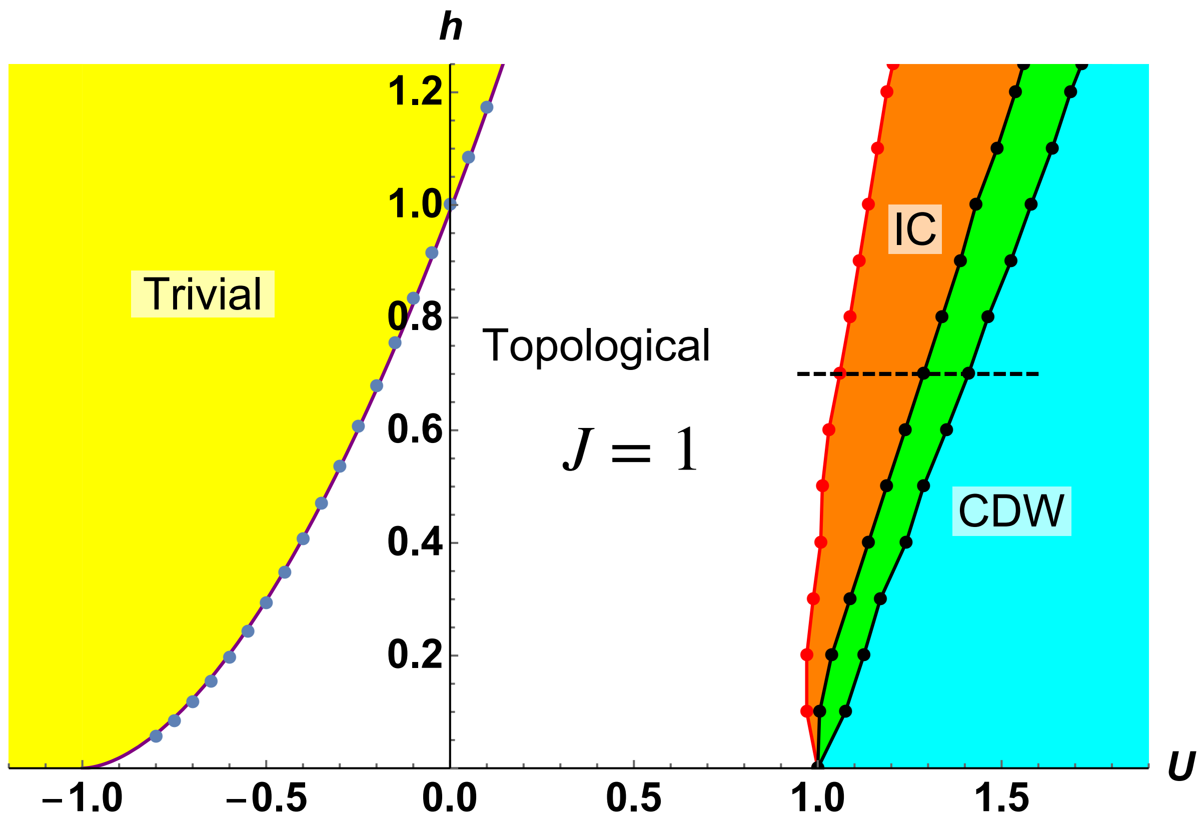
Parity is a good quantum number:

$$P = \prod_{j=1}^{L}\sigma_j^z = \prod_{j=1}^{L}(1-2c_j^\dagger c_j) = \pm 1$$

Phase diagram, $U < 0$ (DMRG)



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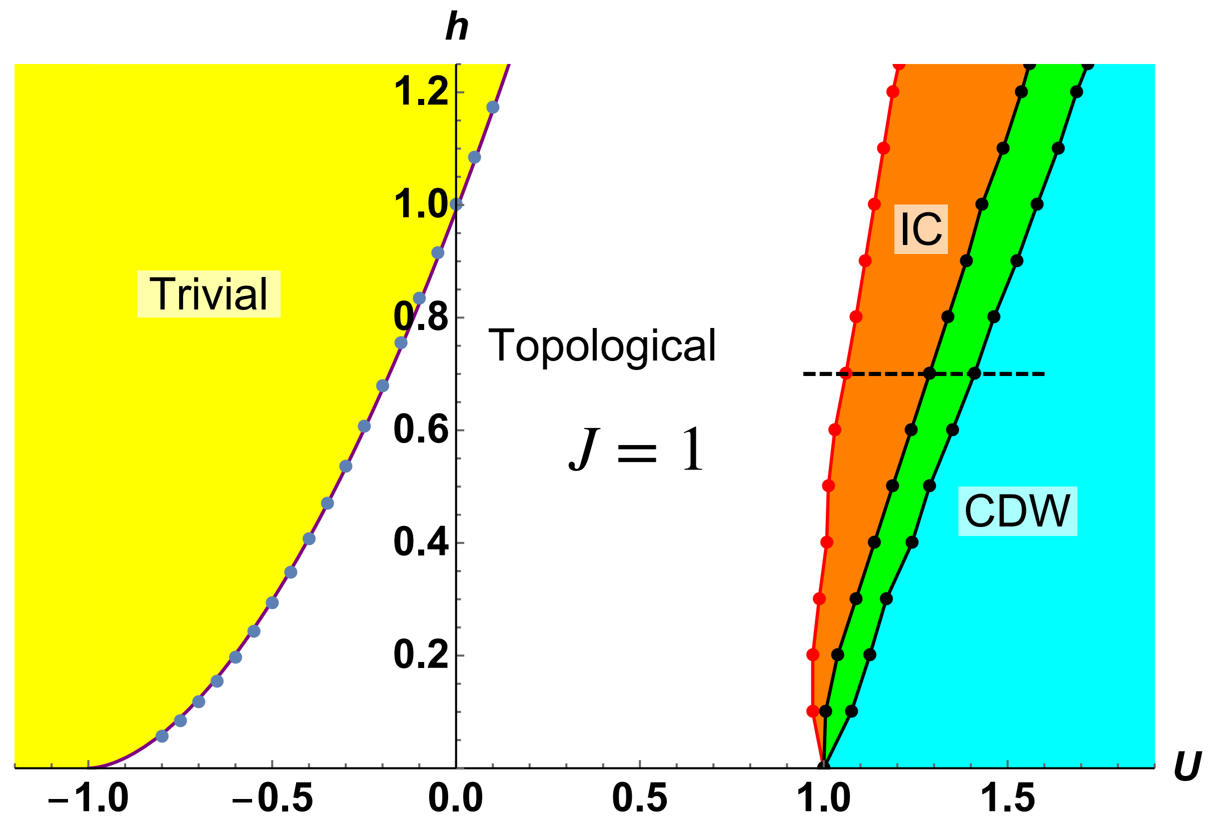

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There is a direct transition to the topological phase, described by Ising CFT ($h > 0$)

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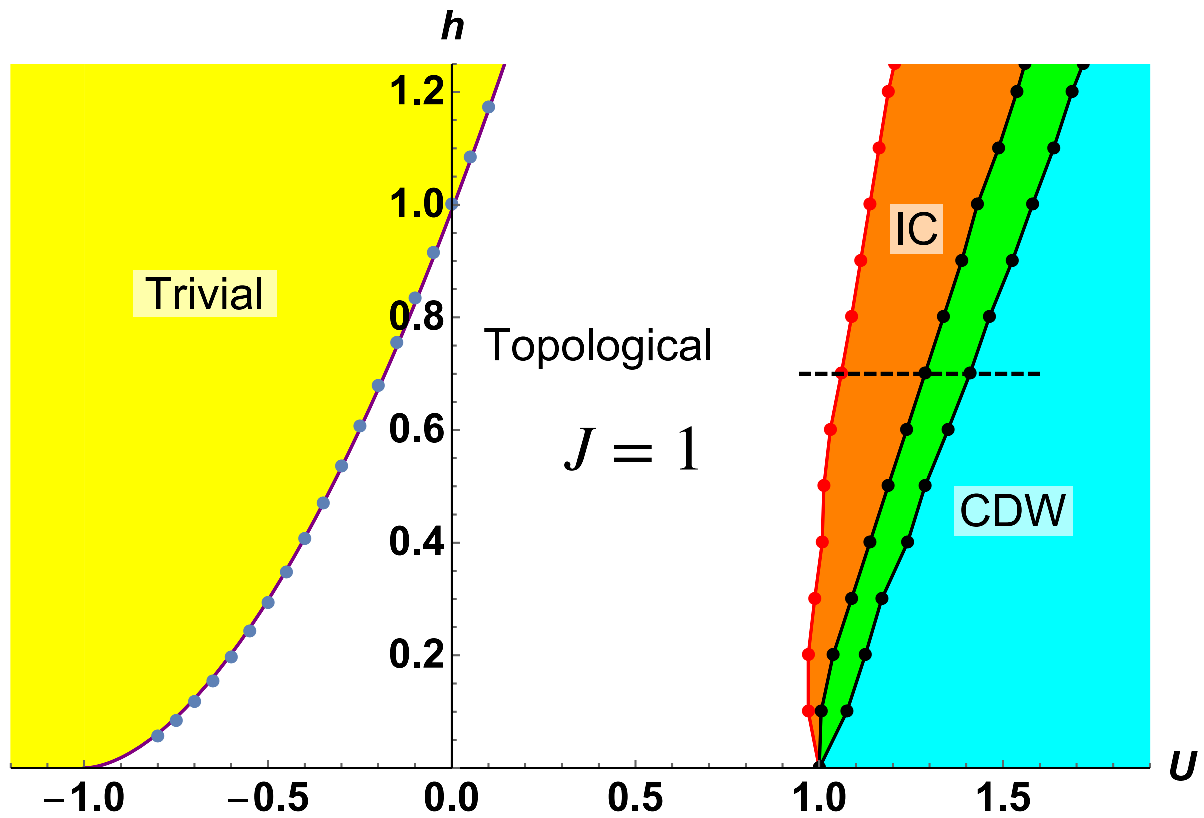

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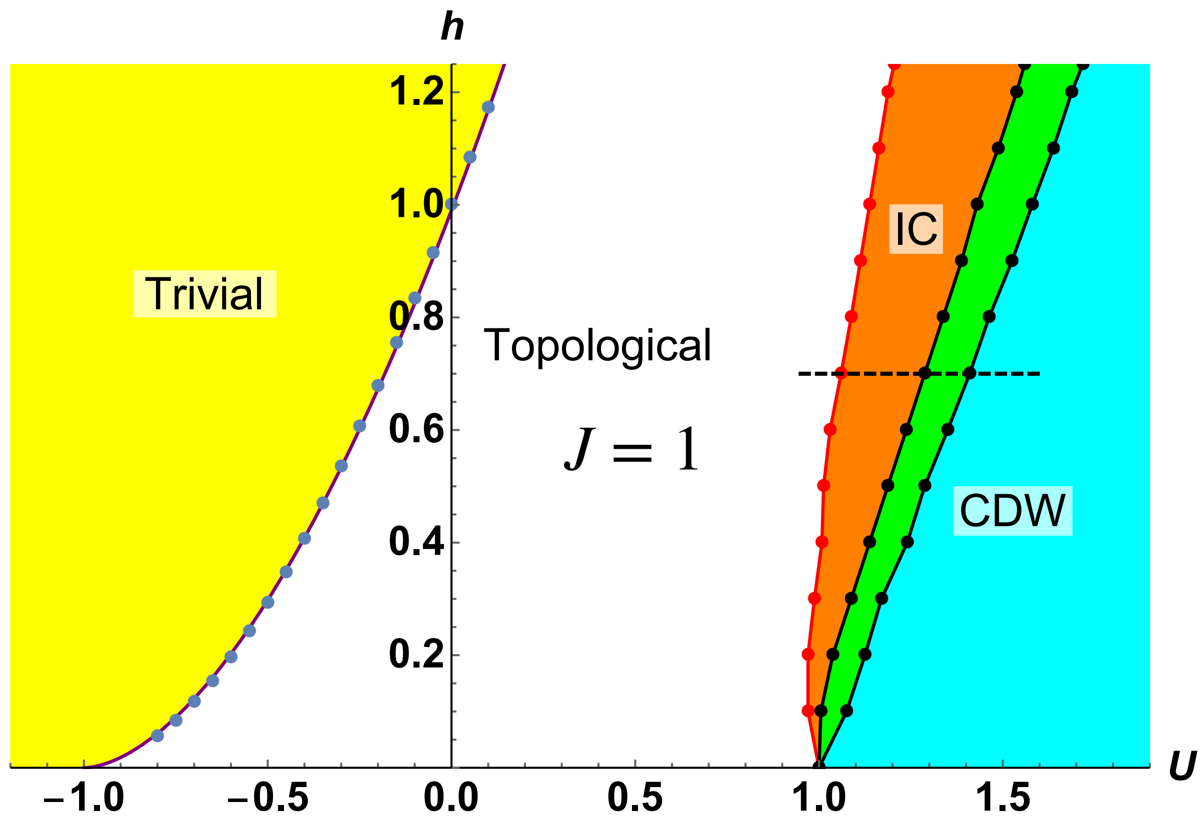

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CDW phase at large U: ground states are $\begin{array}{l}(1,0,1,0,1,\ldots)\\(0,1,0,1,0,\ldots)\end{array}$

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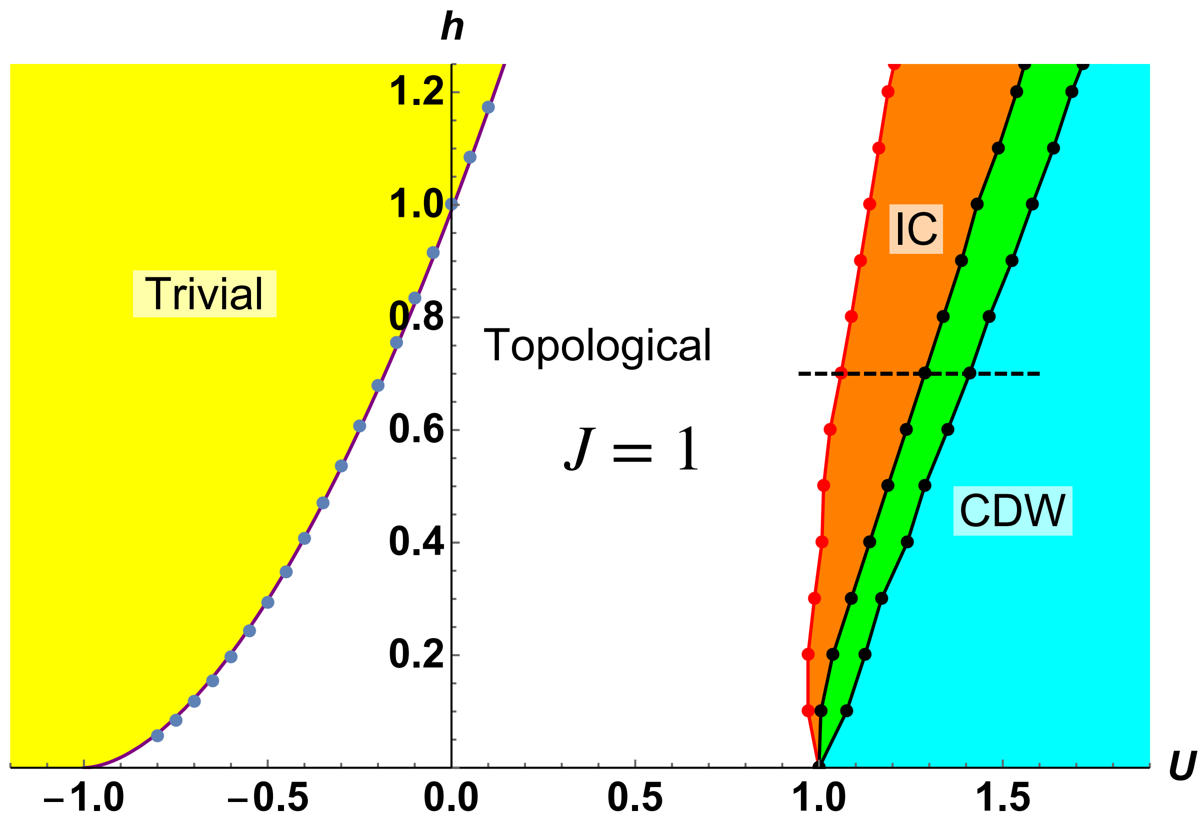

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Topological & CDW phases separated by a gapless incommensurate phase and a ‘exited state CDW’ phase (ground state resembles an exited state of the CDW phase)

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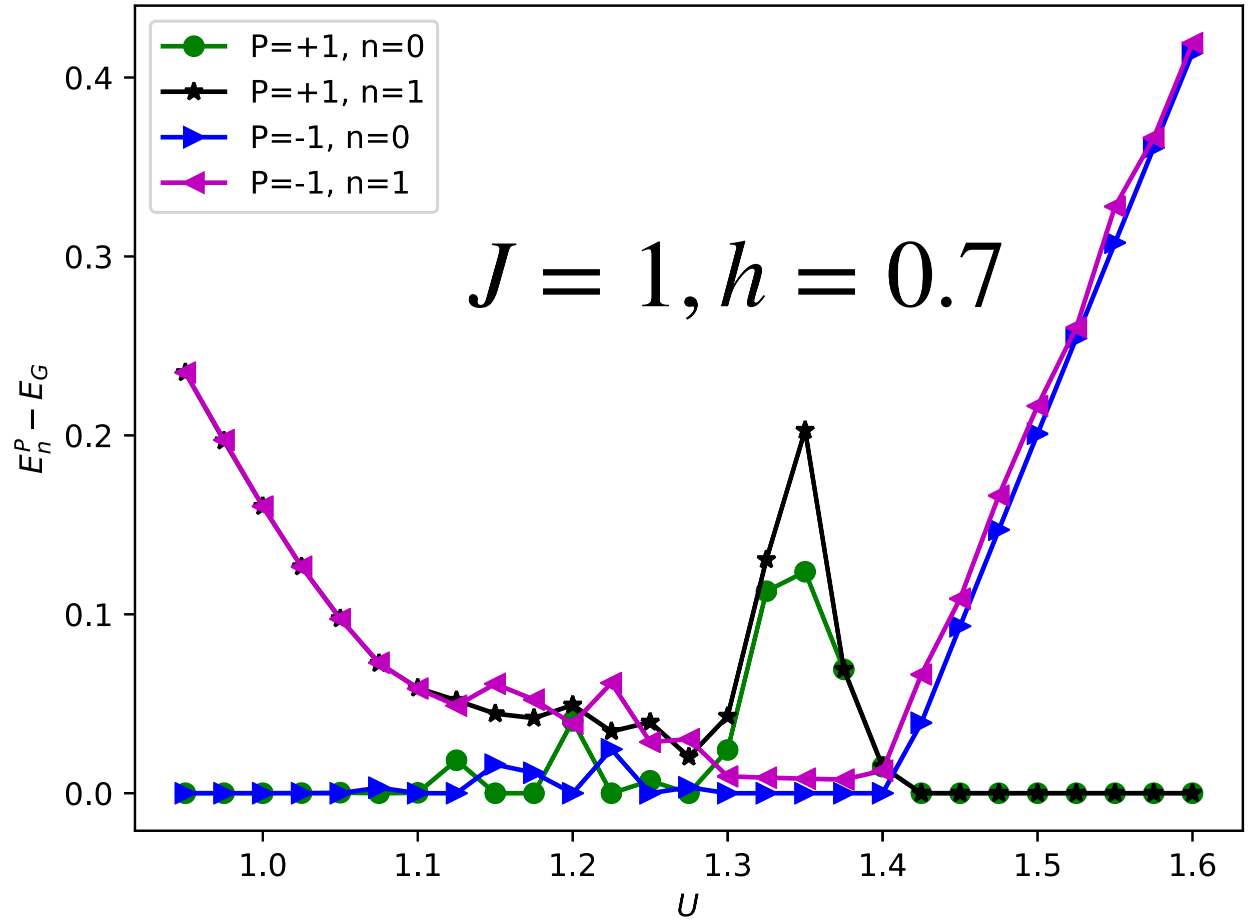

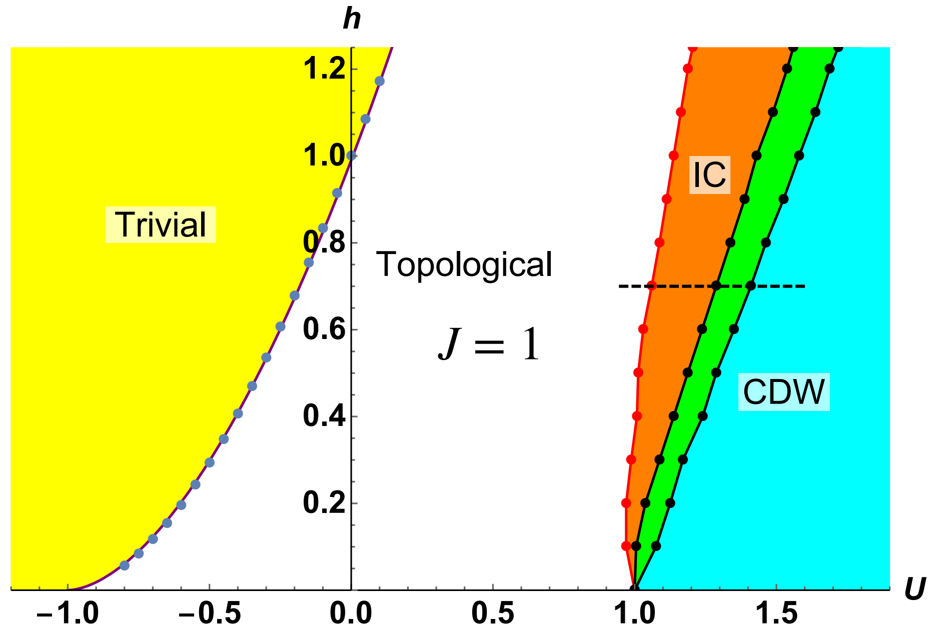

Lowest states in the two parity sectors, at $h = 0.7$ (L=240) clearly show the four phases.

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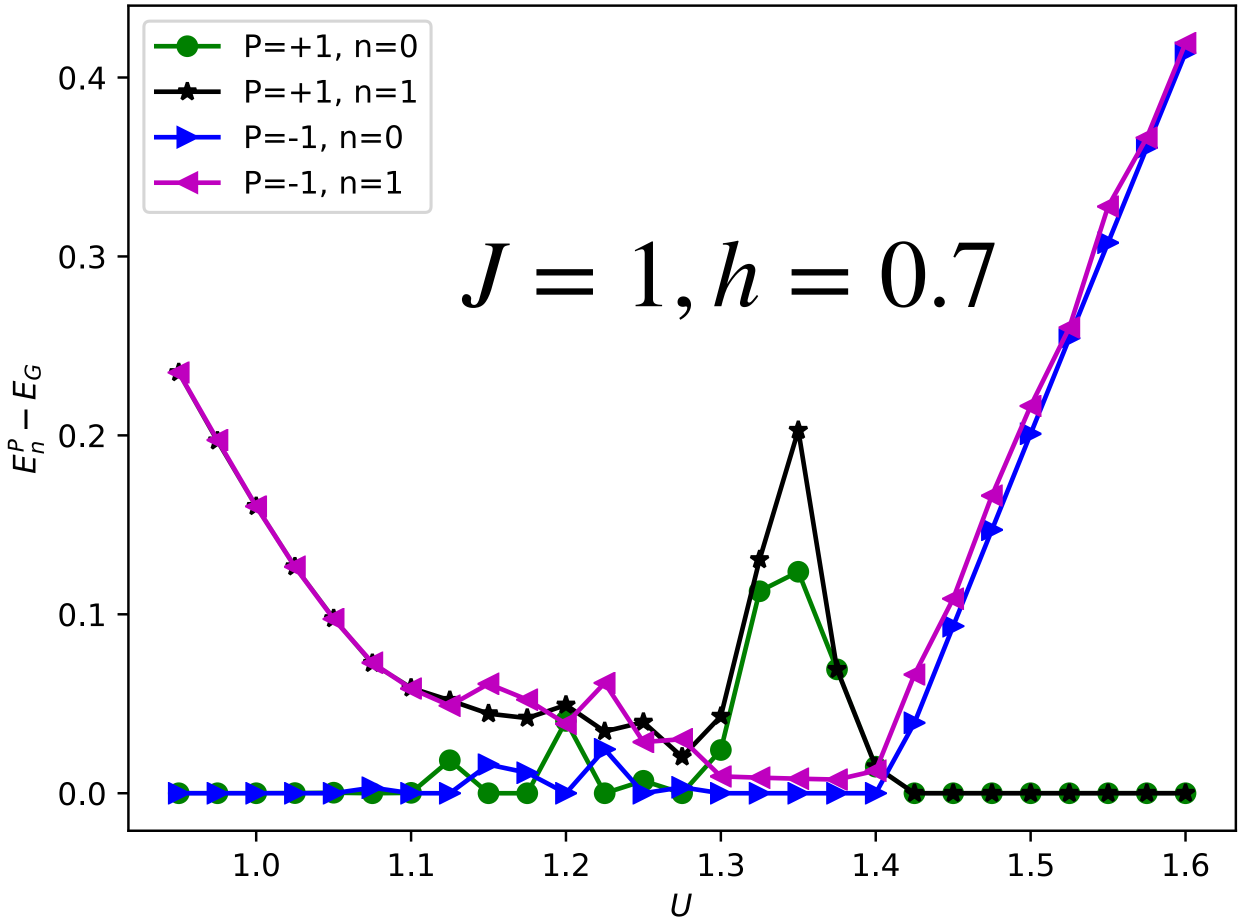

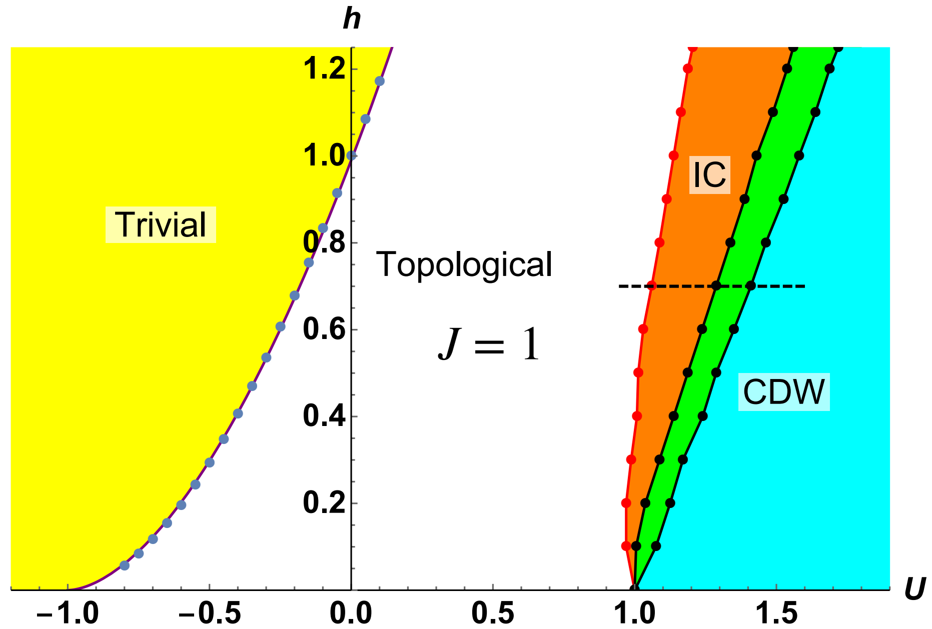

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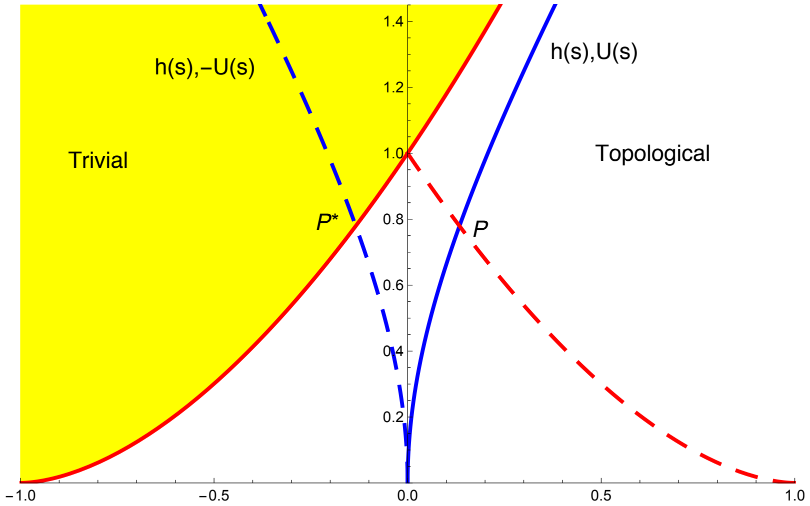
Other observables used: entanglement entropy, site occupation, scaling of the gap.

Strong zero mode?

We can exclude a strong zero mode in a large part of the topological phase.



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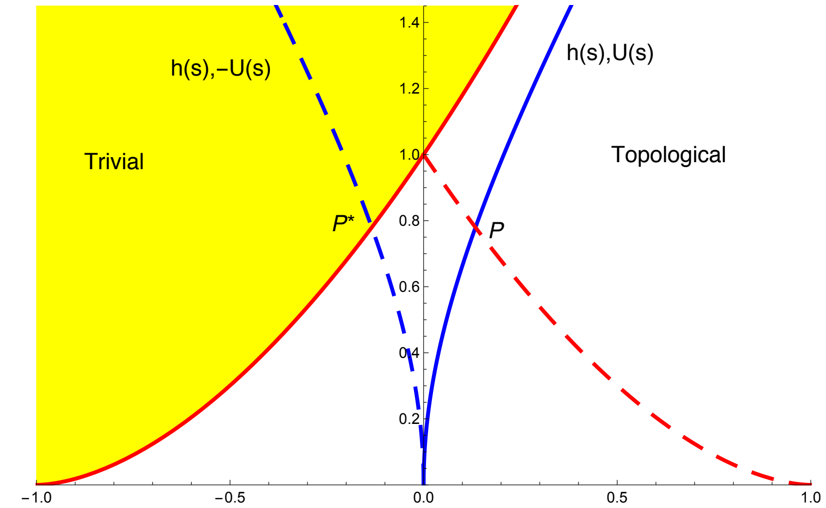

Consider the line $(h(s), U(s))$ in the topological phase and look at $-H$:

$$-H(h, U) = \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x + h \sum_{j=1}^{L} \sigma_j^z - U \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z \ .$$

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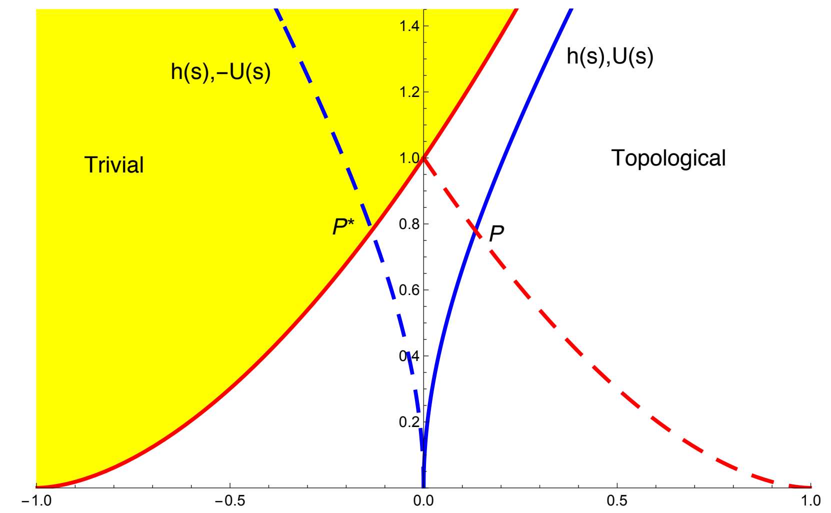

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Performing the rotation $\sigma_j^x \to (-1)^j \sigma_j^x \quad \sigma_j^z \to -\sigma_j^z$ gives

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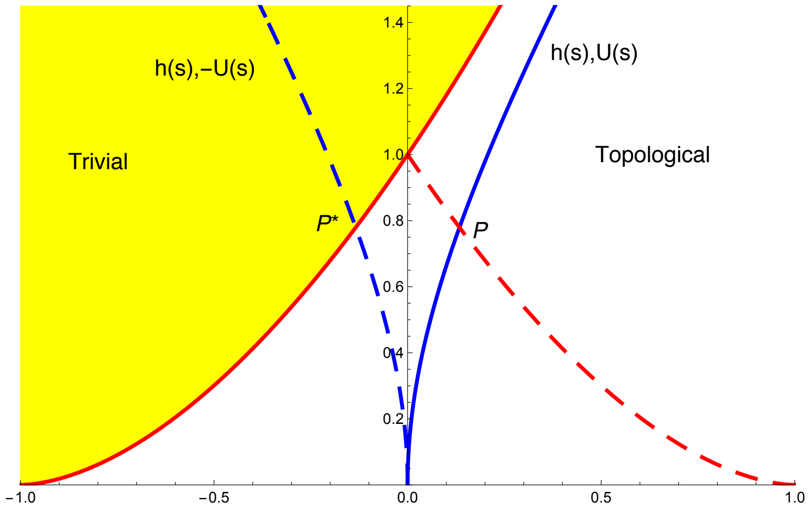

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Highest excited state of $H(h,U)$ is the ground state of $H(h,-U)$!

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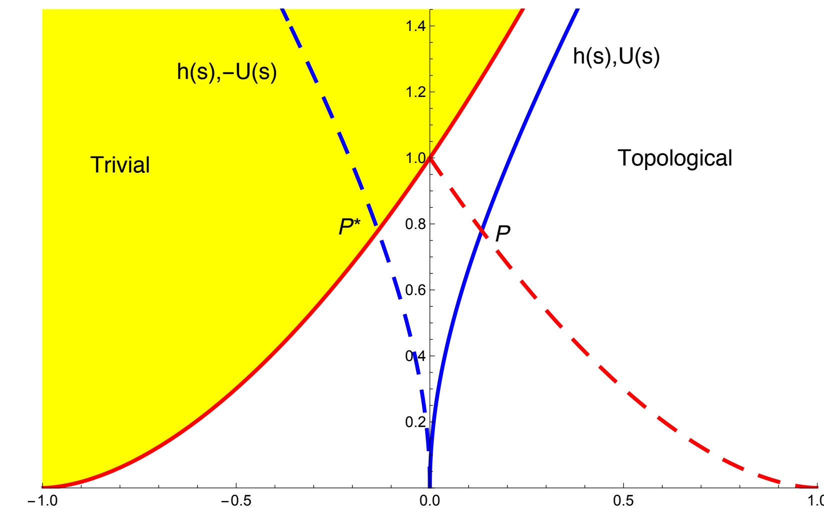

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Ground state in the trivial phase is non-degenerate, so we can exclude a strong zero mode above the dashed line.

Strong zero mode?

In the remainder, we study the strong zero mode via the edge-magnetization (Fendley et al.)

Spin-auto correlation function is defined as
$A_j(t) = \langle j \,|\, \sigma_1^x(t)\sigma_1^x(0) \,|\, j \rangle$

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States with $E_{j_1} \approx E_{j_2}\ , \quad \langle j_2 \,|\, \sigma_1^x \,|\, j_1 \rangle \neq 0$ give finite contribution to edge magnetization at large times.
When $E_{j_1} \neq E_{j_2}$ one gets a sum of incoherent oscillations.

Strong zero mode?

We will consider the edge magnetization at $T > 0$:

$$A(t,T) = \frac{1}{Z}\sum_{j_1} e^{-\epsilon_{j_1}/(kT)} A_{j_1}(t) = \frac{1}{Z}\sum_{j_1,j_2} e^{-\epsilon_{j_1}/(kT)} e^{i(\epsilon_{j_1}-\epsilon_{j_2})t} \left| \left\langle j_2 \middle| \sigma_1^x \middle| j_1 \right\rangle \right|^2$$

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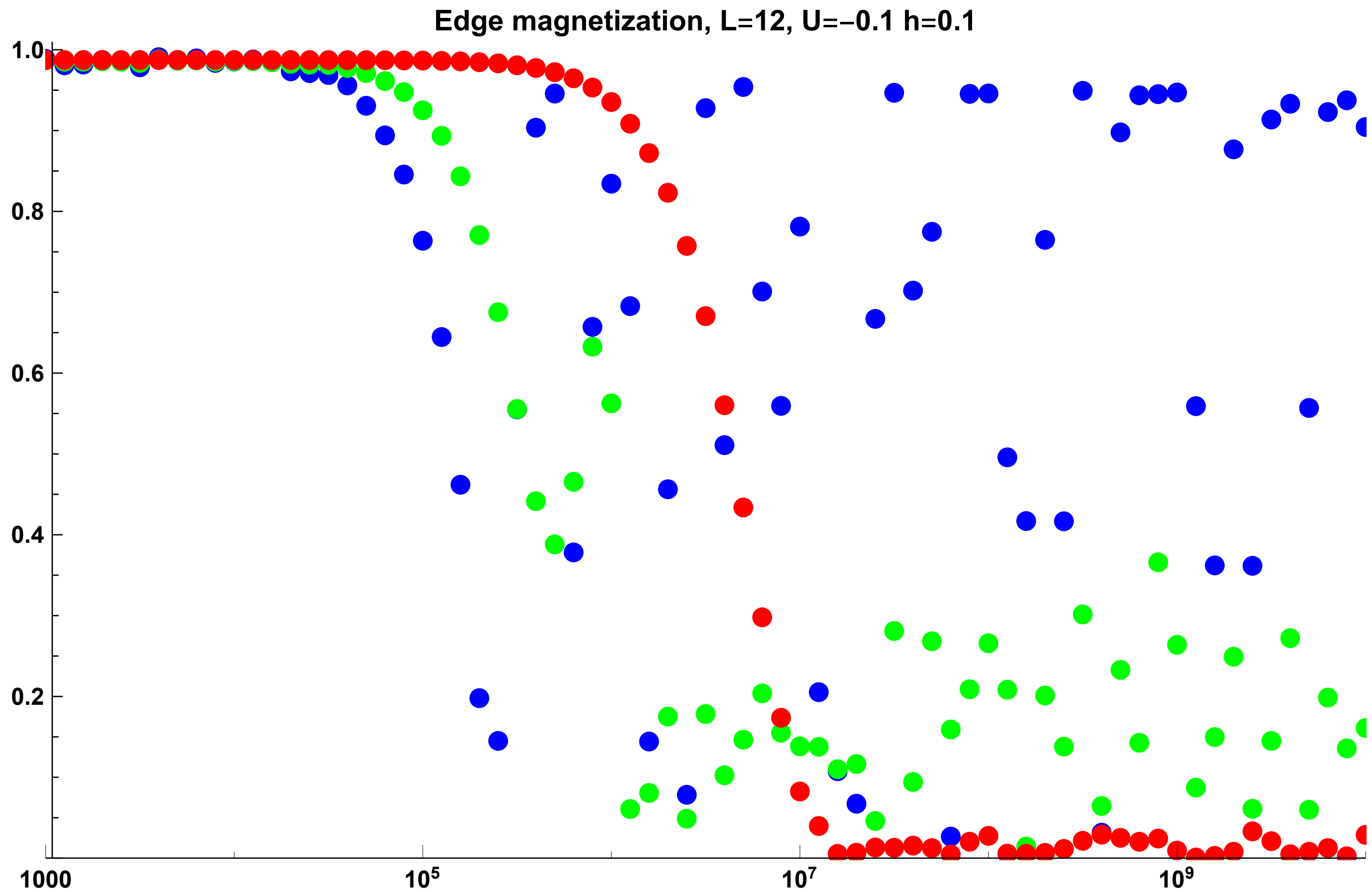
We use exact diagonalization, to obtain *all* the eigenstates, for L up to 16.

Edge magnetization

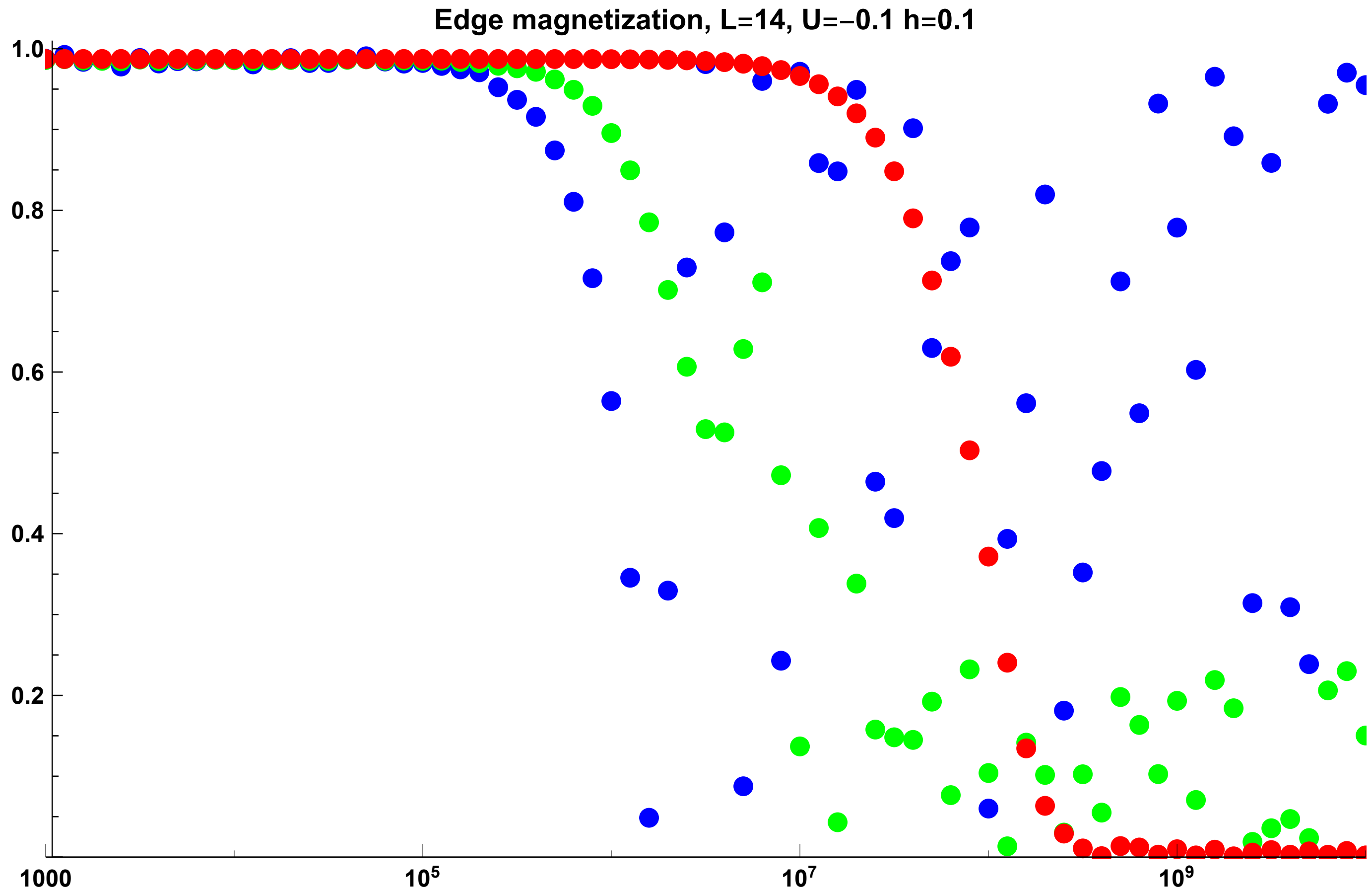
Compare different temperatures (T=0.001, 1, 1000) for h = 0.1, U = -0.1



L=12

Edge magnetization

Compare different temperatures (T=0.001, 1, 1000) for h = 0.1, U = -0.1



L=14

Edge magnetization

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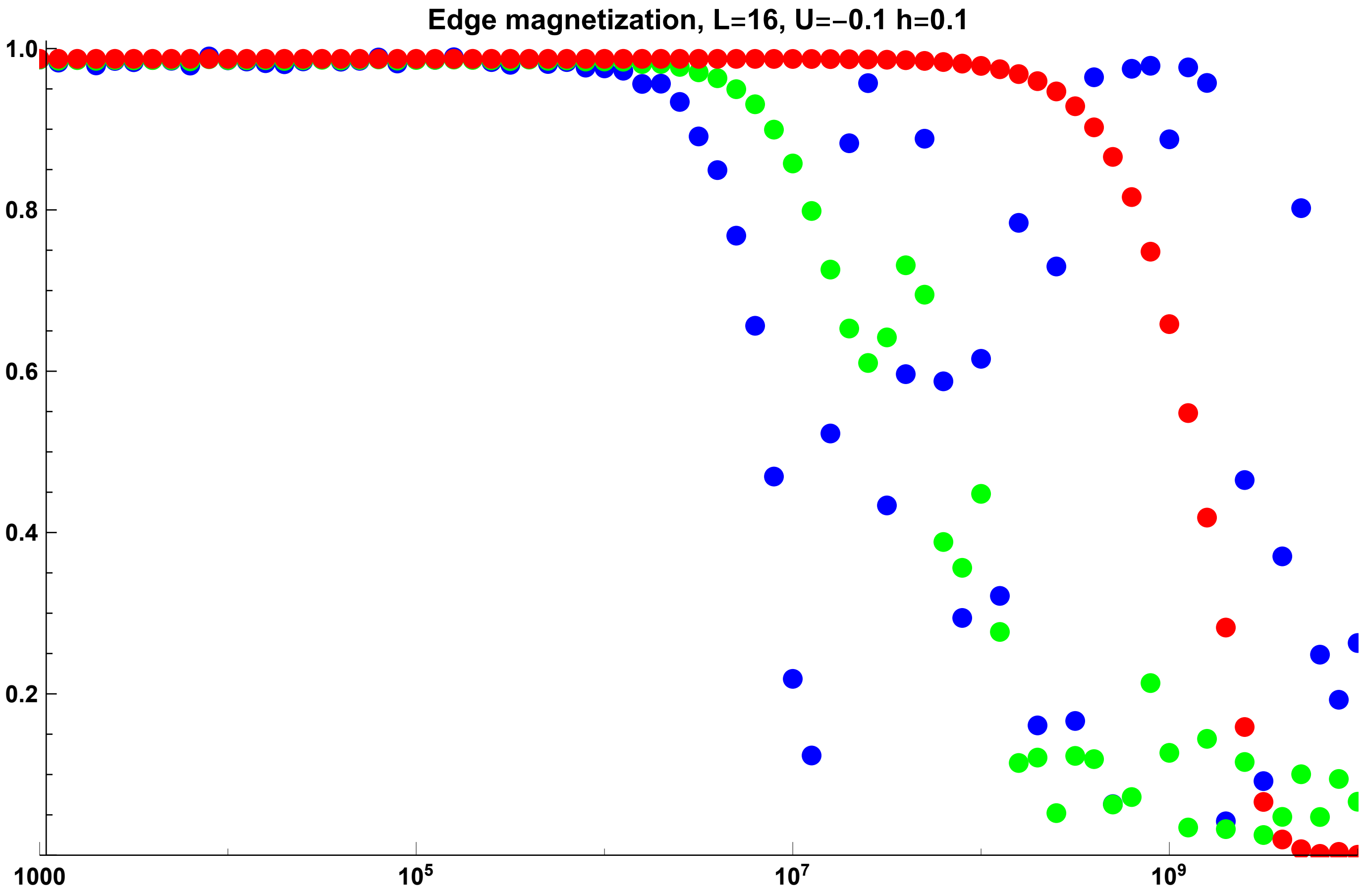
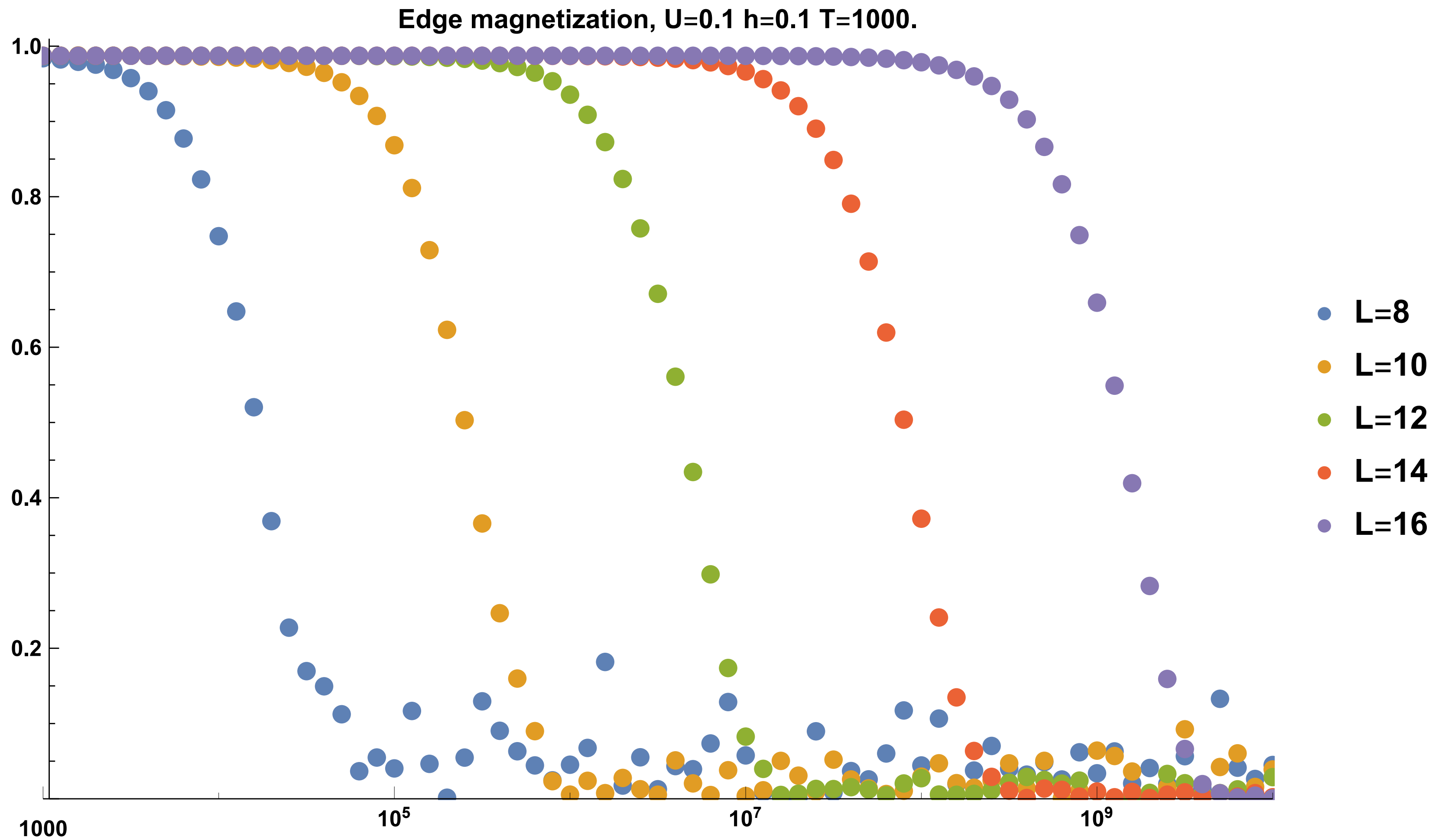


L=16

Edge magnetization

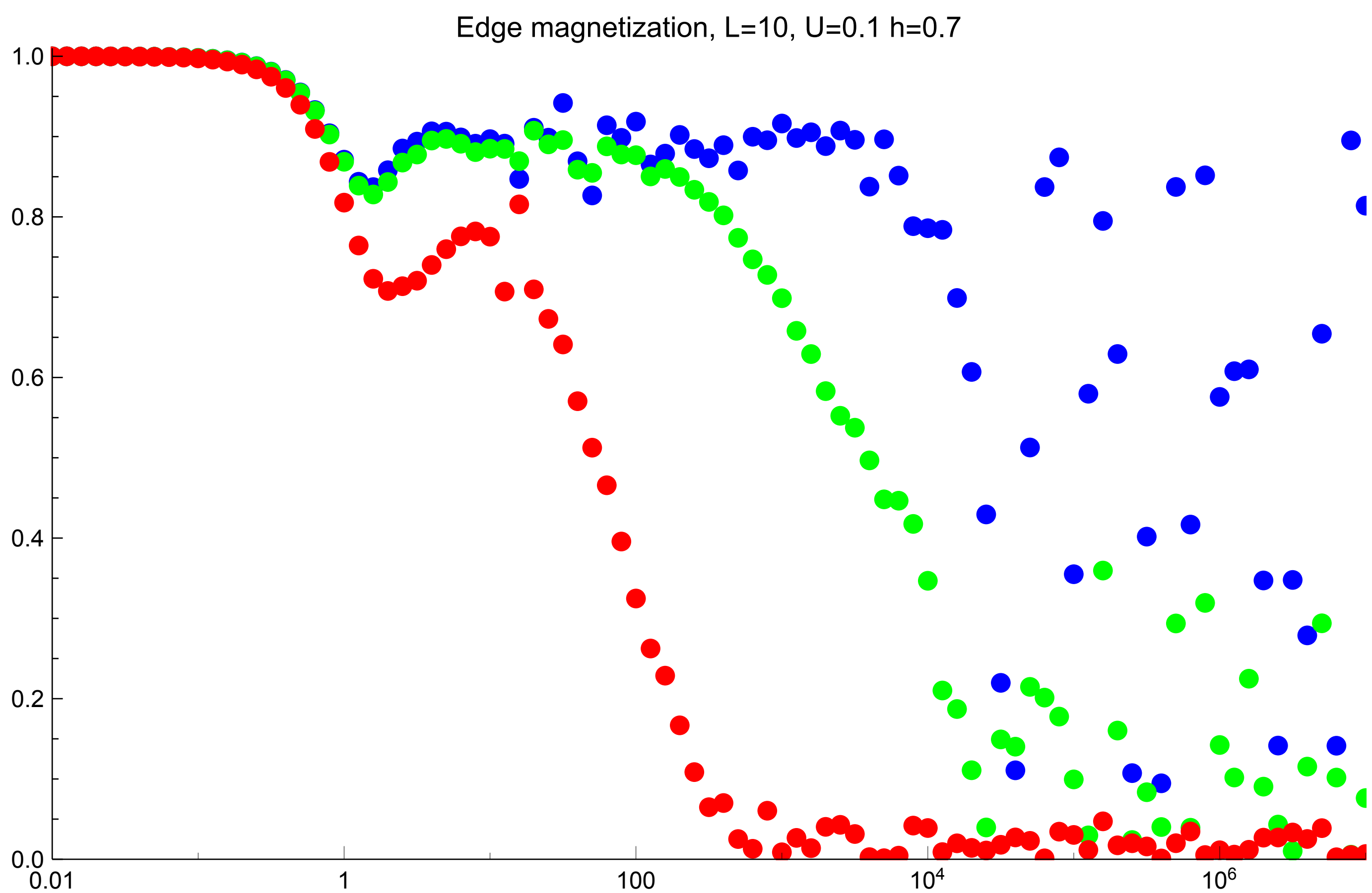
Compare different system sizes for h=0.1, U = 0.1, and T=1000



Correlation time increases exponentially with system size, consistent with a strong zero mode, but unclear if this persists to larger sizes.

Edge magnetization

For U = 0.1, h=0.7, edge magentization at T=1000, shows little dependance on system size



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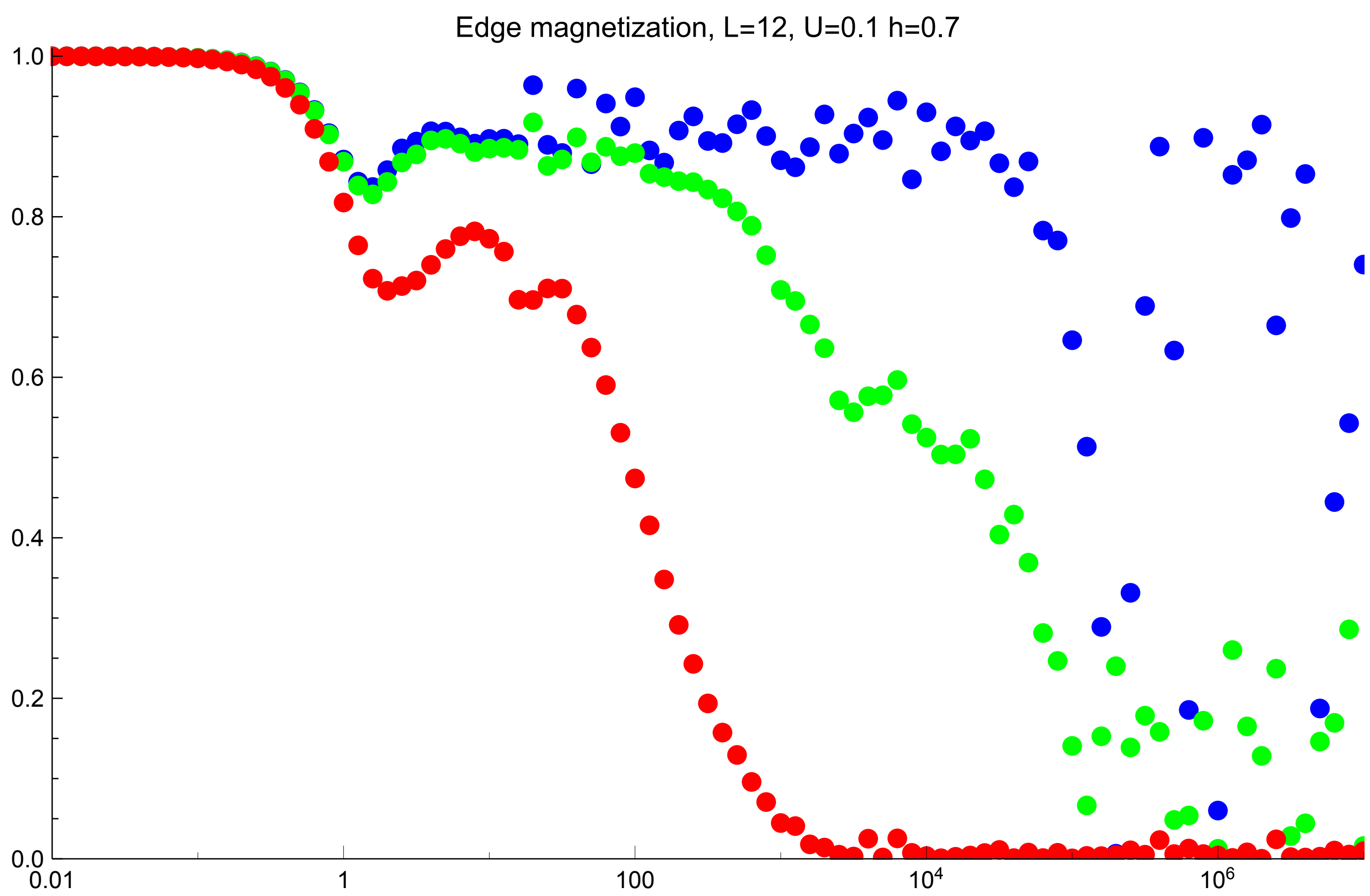

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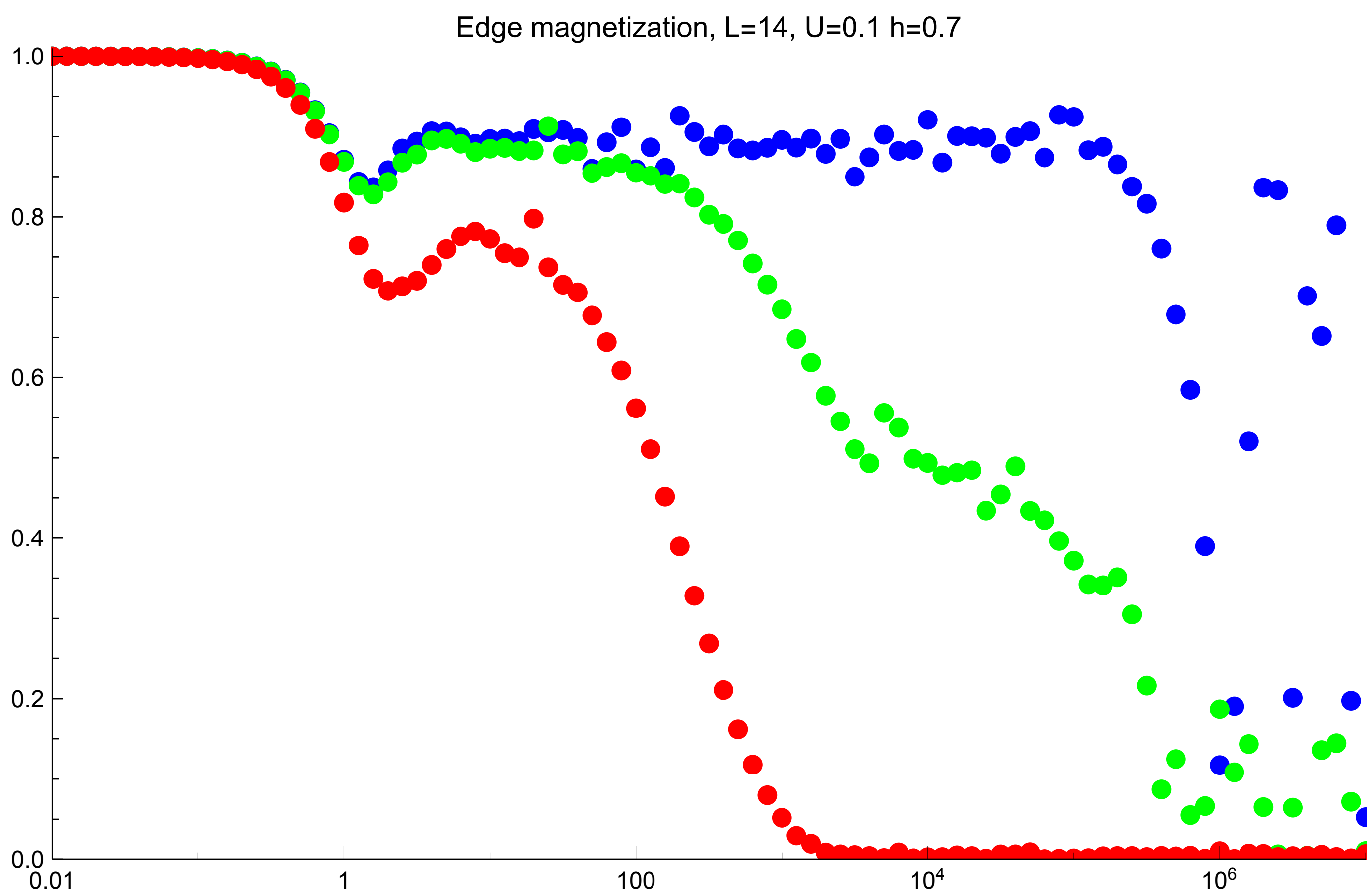

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Frustration free line

Emery-Peschel constructed a ‘frustration free’ line in the topological phase:

$$H = \sum_j h^{\mathrm{PE}}_{j,j+1} \qquad\qquad U(l) = (\cosh(l) - 1)/2 \quad h(l) = \sinh(l)$$

$$h^{\mathrm{PE}}_{j,j+1}(l) = -\sigma^x_j \sigma^x_{j+1} + \frac{h(l)}{2}(\sigma^z_j + \sigma^z_{j+1}) + U(l)\sigma^z_j \sigma^z_{j+1} + (U(l) + 1)$$

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Model has two *exactly degenerate ground states* (E=0):

$$|\psi_1(l)\rangle = (|\uparrow\rangle + e^{\frac{l}{2}}|\downarrow\rangle)^{\otimes L} \qquad |\psi_2(l)\rangle = (|\uparrow\rangle - e^{\frac{l}{2}}|\downarrow\rangle)^{\otimes L}$$

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Parity eigenstates are *not* product states:

$$|E = 0; \pm\rangle = \mathcal{N}_\pm(l)(|\psi_1(l)\rangle \pm |\psi_2(l)\rangle) \qquad \mathcal{N}_\pm(l) = \left[2(1 + e^l)^L \pm 2(1 - e^l)^L\right]^{-\frac{1}{2}}$$

Frustration free line

One can construct *edge-localized* Majorana operators that permutes the parity eigenstates

$$\Gamma_{\mathrm{L}} = \frac{1}{\sqrt{\sum_{j=0}^{L-1} q^{2j}}} \sum_{j=1}^{L} q^{(j-1)} \gamma_{A,j} \qquad \Gamma_{\mathrm{R}} = \frac{1}{\sqrt{\sum_{j=0}^{L-1} q^{2j}}} \sum_{j=1}^{L} q^{(L-j)} \gamma_{B,j} \qquad q = -\tanh(l/2)$$

$$\gamma_{A,j} = \Big(\prod_{k<j} \sigma_k^z\Big) \sigma_j^x \qquad \gamma_{B,j} = \Big(\prod_{k<j} \sigma_k^z\Big) \sigma_j^y$$

Generalization to parafermions

The frustration free line can be generalized to parafermions (as in Joffe's talk): X and Z satisfy

$$X^3 = Z^3 = \mathbf{1} \qquad X^2 = X^\dagger \qquad Z^2 = Z^\dagger \qquad XZ = \omega ZX \qquad \omega = e^{2\pi i/3}$$

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Frustration free model can be constructed: $H = \sum_{j=1}^{L-1} h_{j,j+1}^{Z_3}(r)$

$$h_{j,j+1}^{Z_3}(r) = \left[-X_j^\dagger X_{j+1} - f(r)(Z_j + Z_{j+1}) - g_1(r) Z_j Z_{j+1} - g_2(r) Z_j Z_{j+1}^\dagger + \mathrm{h.c}\right] + \epsilon(r)$$

$$f(r) = (1+2r)(1-r^3)/(9r^2) \qquad g_1(r) = -2(1-r)^2(1+r+r^2)/(9r^2)$$

$$g_2(r) = (1-r)^2(1-2r-2r^2)/(9r^2) \qquad \epsilon(r) = 2(1+r+r^2)^2/(9r^2)$$

Parafermion frustration free line

$$h^{Z_3}_{j,j+1}(r) = \left[-X_j^\dagger X_{j+1} - f(r)(Z_j + Z_{j+1}) - g_1(r) Z_j Z_{j+1} - g_2(r) Z_j Z_{j+1}^\dagger + \mathrm{h.c}\right] + \epsilon(r)$$

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The E=0 ground states are product states:

$$|G_0(r)\rangle = \big(|0\rangle + r|1\rangle + r|2\rangle\big)^{\otimes L} \qquad |G_1(r)\rangle = \big(|0\rangle + r\omega|1\rangle + r\bar{\omega}|2\rangle\big)^{\otimes L} \qquad |G_2(r)\rangle = \big(|0\rangle + r\bar{\omega}|1\rangle + r\omega|2\rangle\big)^{\otimes L}$$

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Use them to construct symmetry eigenstates

$$|E=0;1\rangle = \mathcal{N}_1(|G_0(r)\rangle + |G_1(r)\rangle + |G_2(r)\rangle)$$
$$|E=0;\omega\rangle = \mathcal{N}_\omega(|G_0(r)\rangle + \bar{\omega}|G_1(r)\rangle + \omega|G_2(r)\rangle)$$
$$|E=0;\bar{\omega}\rangle = \mathcal{N}_{\bar{\omega}}(|G_0(r)\rangle + \omega|G_1(r)\rangle + \bar{\omega}|G_2(r)\rangle)$$

$$\mathcal{N}_1 = \left[3(1+2r^2)^L + 6(1-r^2)^L\right]^{-\frac{1}{2}}$$

$$\mathcal{N}_{\omega,\bar{\omega}} = \left[3(1+2r^2)^L - 3(1-r^2)^L\right]^{-\frac{1}{2}}$$

Parafermion frustration free line

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Unfortunately, constructing an *edge-localized* parafermion operator that permutes the states is hard!

Parafermion frustration free line

The most general 2-site operator is:

$$O(r) = \begin{pmatrix} 0 & b_{2;3} & 0 & b_{1;3} & 0 & 0 & 0 & 0 & b_{3;3} \\ 0 & 0 & c_{1;2} & 0 & c_{3;2} & 0 & c_{2;2} & 0 & 0 \\ a_{3;1} & 0 & 0 & 0 & 0 & a_{2;1} & 0 & a_{1;1} & 0 \\ 0 & 0 & c_{1;1} & 0 & c_{3;1} & 0 & c_{2;1} & 0 & 0 \\ a_{3;3} & 0 & 0 & 0 & 0 & a_{2;3} & 0 & a_{1;3} & 0 \\ 0 & b_{2;2} & 0 & b_{1;2} & 0 & 0 & 0 & 0 & b_{3;2} \\ a_{3;2} & 0 & 0 & 0 & 0 & a_{2;2} & 0 & a_{1;2} & 0 \\ 0 & b_{2;1} & 0 & b_{1;1} & 0 & 0 & 0 & 0 & b_{3;1} \\ 0 & 0 & c_{1;3} & 0 & c_{3;3} & 0 & c_{2;3} & 0 & 0 \end{pmatrix}$$

$$a_{1;1} = \frac{2r^3 + (cd + r^2)\cos(\phi_1) + (-d + cr^2)\sin(\phi_1)}{2cd} \qquad a_{1;2} = \frac{2r^3 + (-cd + r^2)\cos(\phi_1) + (d + cr^2)\sin(\phi_1)}{2cd} \qquad a_{1;3} = -\frac{r(-r^3 + \cos(\phi_1) + c\sin(\phi_1))}{cd}$$

$$a_{2;1} = \frac{2r^3 + (-cd + r^2)\cos(\phi_1) - (d + cr^2)\sin(\phi_1)}{2cd} \qquad a_{2;2} = \frac{2r^3 + (cd + r^2)\cos(\phi_1) + (d - cr^2)\sin(\phi_1)}{2cd} \qquad a_{2;3} = \frac{r(r^3 - \cos(\phi_1) + c\sin(\phi_1))}{cd}$$

$$a_{3;1} = \frac{r(1 - r^3\cos(\phi_1) + dr\sin(\phi_1))}{cd} \qquad a_{3;2} = -\frac{r(-1 + r^3\cos(\phi_1) + dr\sin(\phi_1))}{cd} \qquad a_{3;3} = \frac{r^2(1 + 2r\cos(\phi_1))}{cd}$$

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$$c_{1;1} = \frac{r^2(1 + (1 + r^2)\cos(\phi_3))}{d^2} \qquad c_{1;2} = \frac{r^2(1 - \cos(\phi_3) + d\sin(\phi_3))}{d^2} \qquad c_{1;3} = \frac{r(r^2 - r^2\cos(\phi_3) - d\sin(\phi_3))}{d^2}$$

$$c_{2;1} = -\frac{r^2(-1 + \cos(\phi_3) + d\sin(\phi_3))}{d^2} \qquad c_{2;2} = \frac{r^2(1 + (1 + r^2)\cos(\phi_3))}{d^2} \qquad c_{2;3} = \frac{r(r^2 - r^2\cos(\phi_3) + d\sin(\phi_3))}{d^2}$$

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$$c = \sqrt{1 + 2r^4} \quad d = \sqrt{2r^2 + r^4} \quad \phi_1 + \phi_2 + \phi_3 = 0$$

Not easy to find localized operator for arbitrary system size

Summary

- Topological phase survives Hubbard interactions
- Interesting regime in phase diagram for repulsive interaction
- Can exclude strong zero mode for large part of the topological phase
- Finite temperature behaviour of edge magnetization not fully understood
- Generalization to parafermion case, phase diagram not studied in detail
- Frustration free line, but no localized operators (yet?)