Conformal field theory approach to fractional quantum Hall physics - non-Abelian statistics and quantum computation
Nordita, Stockholm, August 14, 2008

Kitaev's honeycomb lattice model on torus

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Outline

Kitaev honeycomb lattice model

Symmetries on torus

Finite size effects

Vortex dynamics

Effect of magnetic field

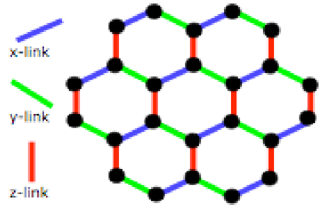
Future research directions

Kitaev honeycomb lattice model

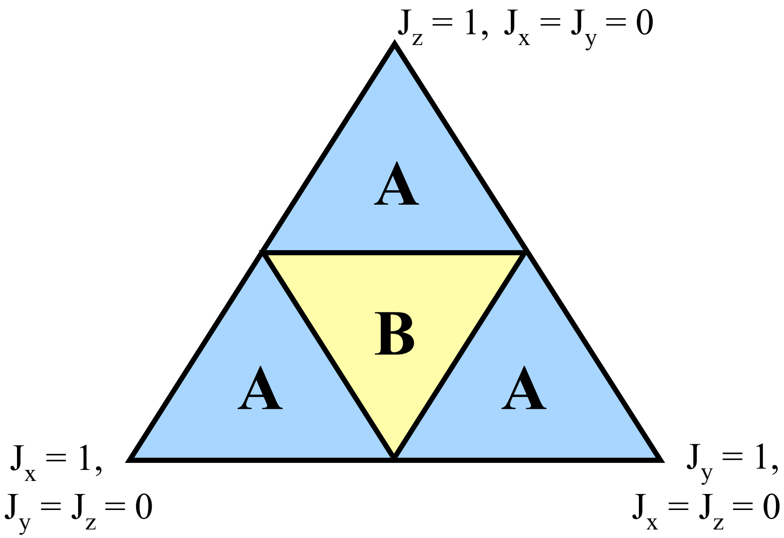
Without external magnetic field:

$$H_0 = J_x \sum_{\substack{i,j \\ x\text{-link}}} \sigma^x_i \sigma^x_j + J_y \sum_{\substack{i,j \\ y\text{-link}}} \sigma^y_i \sigma^y_j + J_z \sum_{\substack{i,j \\ z\text{-link}}} \sigma^z_i \sigma^z_j$$

$$= \Sigma_\alpha J_\alpha \Sigma_{i,j} \sigma^\alpha_i \sigma^\alpha_j = \Sigma_\alpha J_\alpha \Sigma_{i,j} K^\alpha_{ij} \qquad \alpha\text{-link}$$

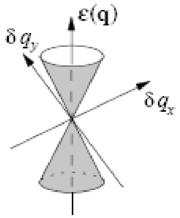


Analytical insights into the model on a plane at the thermodynamic limit:



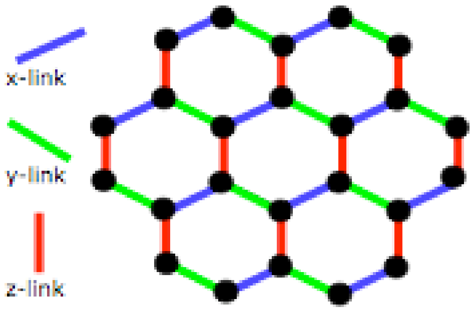
Phase diagram

- phase A - can be mapped perturbatively onto abelian $Z_2 \times Z_2$ topological phase (Toric code)
- phase B - gapless



A.Y.Kitaev, *Anyons in an exactly sovable model and beyond*, Ann. Phys. 321, 2 (2006).

Kitaev honeycomb lattice model



With magnetic field:

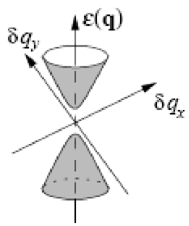
$$H = H_0 + \Sigma_i \Sigma_{\alpha=x,y,z} B_\alpha \sigma_{\alpha,i}$$

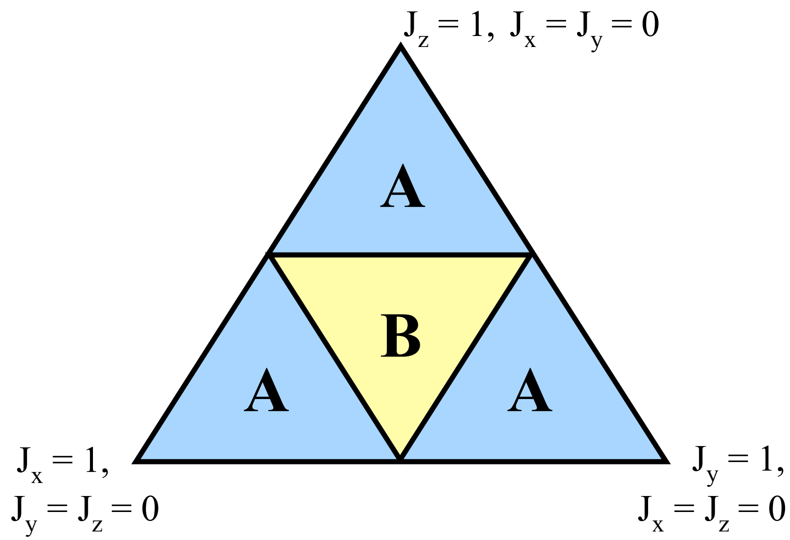
Phase diagram

- phase A - abelian topological phase $Z_2 \times Z_2$;

- phase B is on 3rd order perturbation theory related to non-abelian topological phase $SU(2)_2$ with quasiparticles $(1, \sigma, \varepsilon)$;

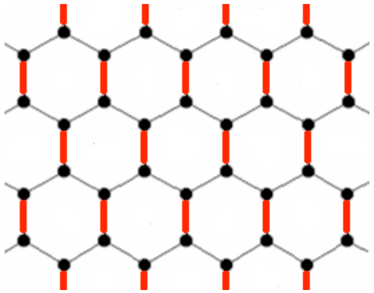
    - FQH state at $\nu=5/2$, p-wave sc, graphene.





A.Y.Kitaev, *Anyons in an exactly sovable model and beyond*, Ann. Phys. 321, 2 (2006).

Mapping to Toric code

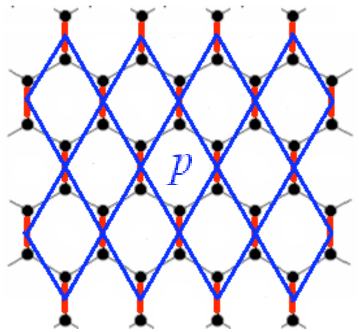


When $J_z >> J_y, J_x$

$$H_D = -J_z \sum_{z-links} \sigma_j^z \sigma_k^z, \quad V = -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y$$

Effective spins

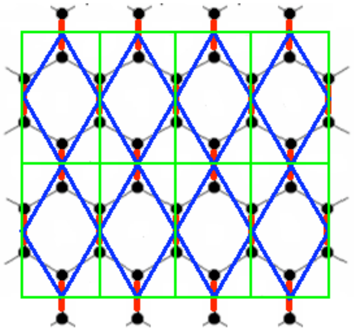
$$|\uparrow\rangle_{eff} = |\uparrow\uparrow\rangle \quad |\downarrow\rangle_{eff} = |\downarrow\downarrow\rangle$$



The first non-constant term of PT occurs on the 4th order

$$H_{\text{eff}} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_p Q_p, \qquad Q_p = \sigma^y_{\text{left}(p)} \sigma^y_{\text{right}(p)} \sigma^z_{\text{up}(p)} \sigma^z_{\text{down}(p)}$$

Q_p is defined on the lattice in which effective spins lie on the vertices.



For the green lattice, H_{eff} can be written as the toric code Hamiltonian.

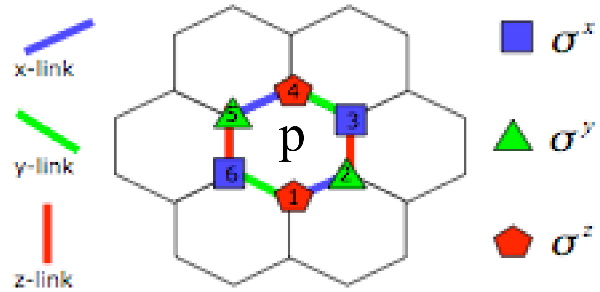
$$H_{\text{eff}} = -J_{\text{eff}} \left(\sum_{\text{vertices}} Q_s + \sum_{\text{plaquettes}} Q_p \right)$$

A.Y.Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. 303, 2 (2003).

Vortex operators

$$W_p = \sigma^z{}_1 \sigma^y{}_2\, \sigma^x{}_3 \sigma^z{}_4\, \sigma^y{}_5 \sigma^x{}_6 =$$

$$= K^x{}_{1,2} K^z{}_{2,3} K^y{}_{3,4} K^x{}_{4,5} K^z{}_{5,6} K^y{}_{6,1}$$



$$[H_0, W_p] = 0 \qquad (K^\alpha{}_{k,k+1})^2 = 1$$

$$K^\beta{}_{k+1,k+2}\, K^\alpha{}_{k,k+1} = -\, K^\alpha{}_{k,k+1} K^\beta{}_{k+1,k+2}$$

$$H_0\, |n\rangle = E_n\, |n\rangle$$

$w_p = \langle n|W_p|n\rangle = +1$ no vortex at plaquette p

$w_p = \langle n|W_p|n\rangle = -1$ a vortex at plaquette p

Each energy eigenstate is characterized by some vortex configuration and the Hilbert space splits into vortex sectors.

$$L = \bigoplus_{w_1,\dots\dots,w_m} L_{w_1,\dots,w_m}$$

Loop symmetries on torus

On torus, there is one nontrivial relation between the plaquette operators

$$\Pi_{\mathrm{p}} W_p = 1$$

which, for a system of N spins on torus (i.e. a system with N/2 plaquettes), implies that there are N/2-1 independent quantum numbers $\{w_1, \ldots, w_{N/2-1}\}$.

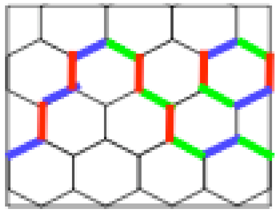
Loops on torus $\quad K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} \ldots K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)}$

$$(K^{\alpha}_{k,k+1})^2 = 1$$

- all homologically trivial loops are generated by plaquette operators.
- two homologically nontrivial loops have to be introduced to generate the full loop symmetry group (the third nontrivial loop is a product of these two).

The full loop symmetry of the torus is the abelian group with N/2+1 independent generators of the order 2 (loop^2=I), i.e. $Z_2^{N/2+1}$.

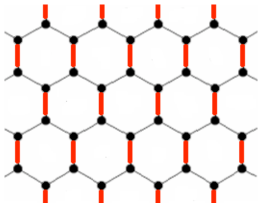
All loop symmetries can be written as



$$C_{(k,l)} = G_k F_l(W_1, W_2, \ldots, W_{N-1})$$

where k is from {0,1,2,3} and $G_0 = I$, and G_1, G_2, G_3 are arbitrarily chosen symmetries from the three nontrivial homology classes, and F_l, with l from $\{1, \ldots, 2^{N/2-1}\}$, run through all monomials in the W_p.

Effective (low energy) Hamiltonian



$$J_z >> J_y, J_x$$

$$H_D = -J_z \sum_{z-links} \sigma_j^z \sigma_k^z,$$

$2^{N/2}$ degenerate ground state = "ground state manifold"

$$V = -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y$$

Projected onto the "ground state manifold", the loop symetries play an important role in the Brillouin-Wigner perturbation theory which allows exact perturbative derivation of the effective (low energy) Hamiltonian on torus in the abelian phase of the model :

$$H_{eff} = \sum_{i=0}^{3} \sum_{j=1}^{2^{N/2-2}} c_{i,j} G_i(z,y) F_j(Q_1, Q_2, \ldots\ldots Q_{N/2-2})$$

trivial
$W_p \longrightarrow Q_p$

nontrivial
- reflects topology

The eigenstates of the **effective** system are the **zeroth** order approximations to those of the **full** system.

The loop symmetries further allow:

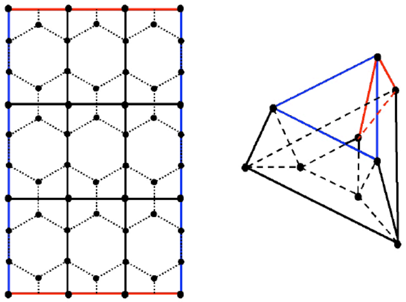
- Classification of all finite size effects.
- Manipulation of vortices in the effective system.

G. Kells, A. T. Bolukbasi, V. Lahtinen, J. K. Slingerland, J. K. Pachos and J. Vala, ***Topological degeneracy and vortex manipulation in the Kitaev honeycomb model****,* *arXiv:0804.2753* (submitted).

Finite-size effects in small systems on torus

Toric code emerges on the 4th order of perturbation theory the low energy sector of H:

$$\sigma(H) - E_0 = \sigma(J_{eff} H_{TC}) + O(J^6) \qquad J_{eff} = \frac{J_x^2 J_y^2}{16|J_z|^3} = \frac{J^4}{16|J_z|^3} \qquad J = J_x = J_y \ll J_z$$

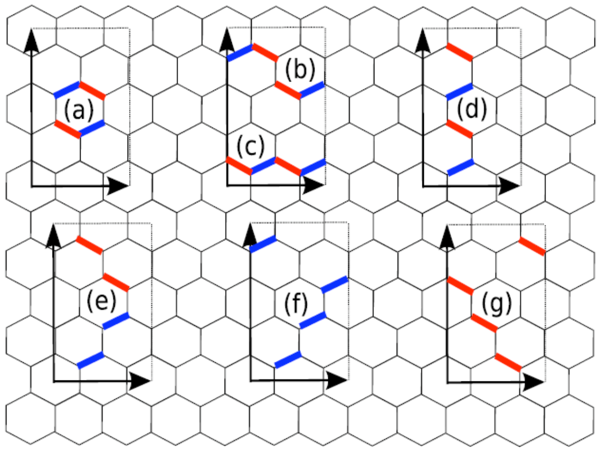


The minimal size of the lattice with no finite size terms on the 4th order is

N = 36

i.e. Toric code on the lattice of 3x3 square plaquettes which properly represents the torus

For smaller systems, the finite size effects are substantial on the 4th order, for example N=16:

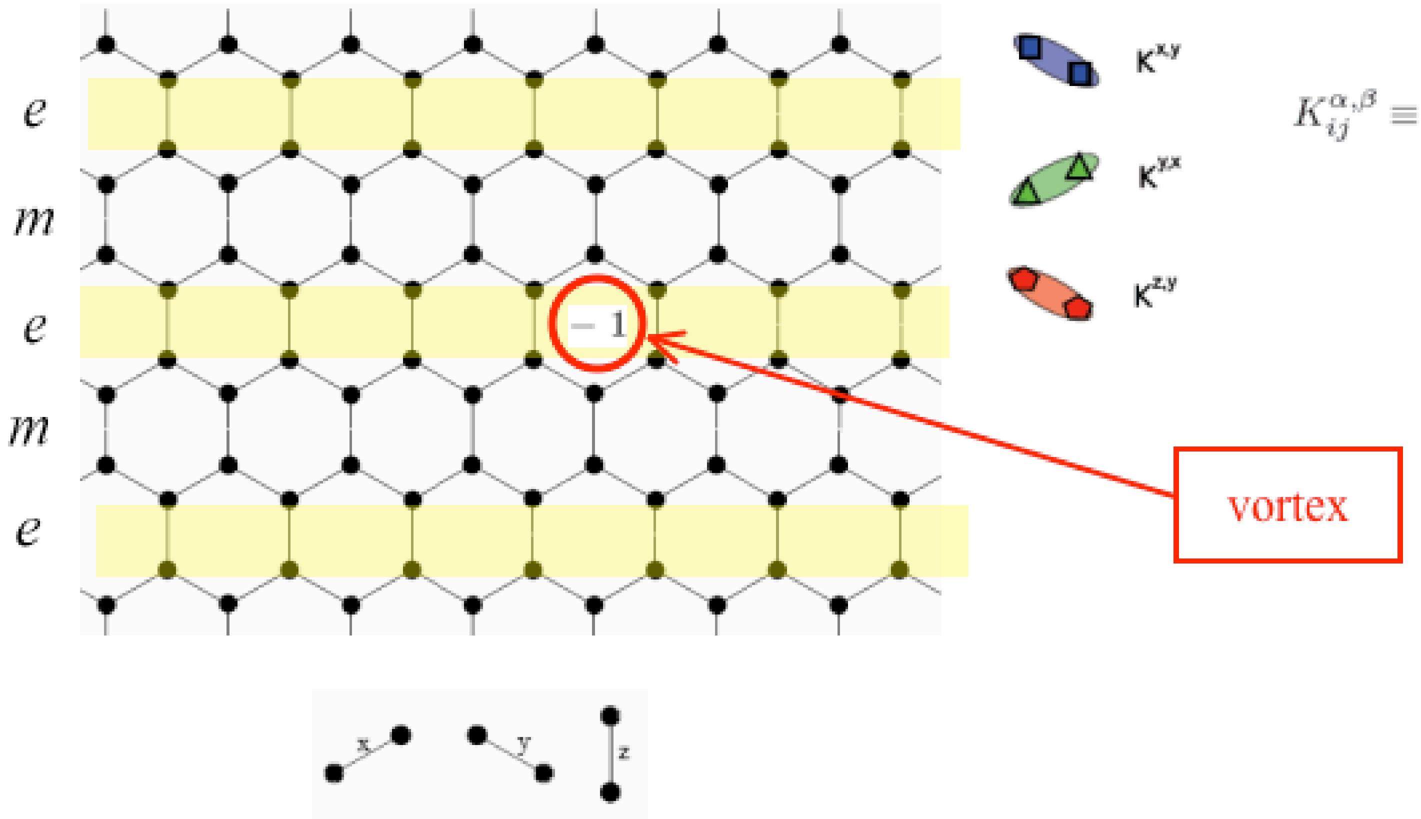


$$\begin{aligned} H^{(4)} &= -\frac{J_x^2 J_y^2}{16|J_z|} \sum^{8} (Q_n + R_n - 5A_n) \\ &\quad - \frac{J_x^2 J_y^2}{16|J_z|} \sum^{4} (Z_n + 5Y_n) \\ &\quad - \frac{5}{16|J_z|} \left(J_x^4 \sum_{n=1}^{2} X_n + J_y^4 \sum_{n=3}^{4} X_n\right) \end{aligned}$$

The toric code spectrum can be reconstructed by extracting the finite size effects from the spectrum of H.

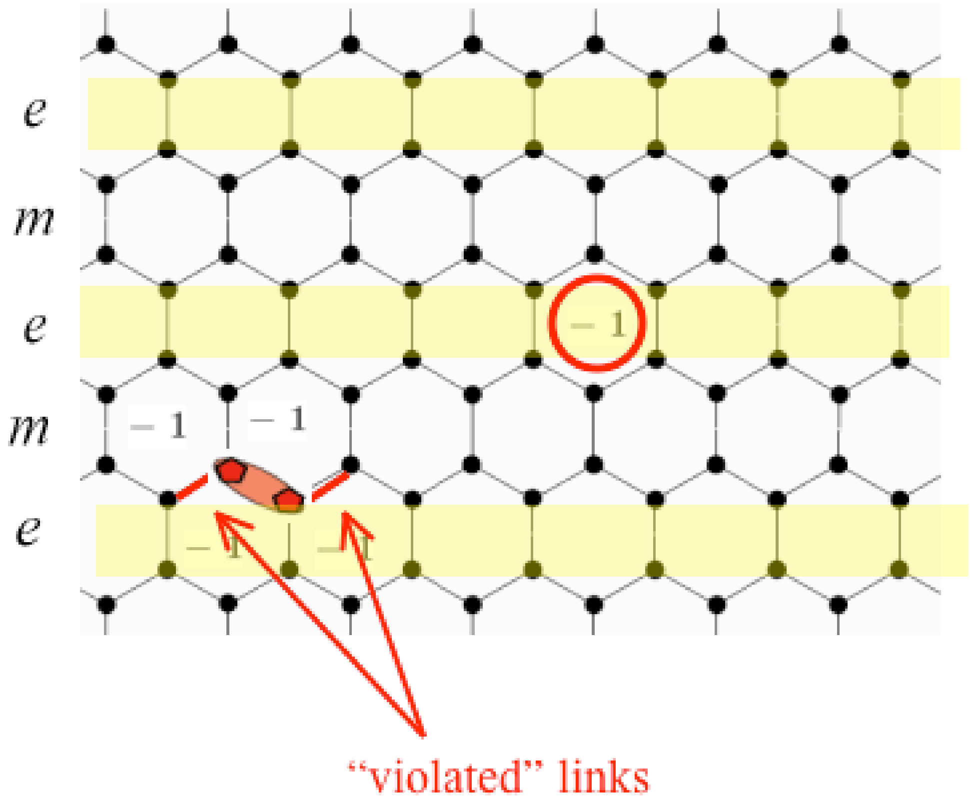
G. Kells, N. Moran, J. Vala,
Finite size corrections in the Kitaev honeycomb lattice model, (to be submitted).

Mutual statistics of fermions and vortices



$K_{ij}^{\alpha,\beta} \equiv \sigma_i^{\alpha} \otimes \sigma_j^{\alpha}$ along β-links.

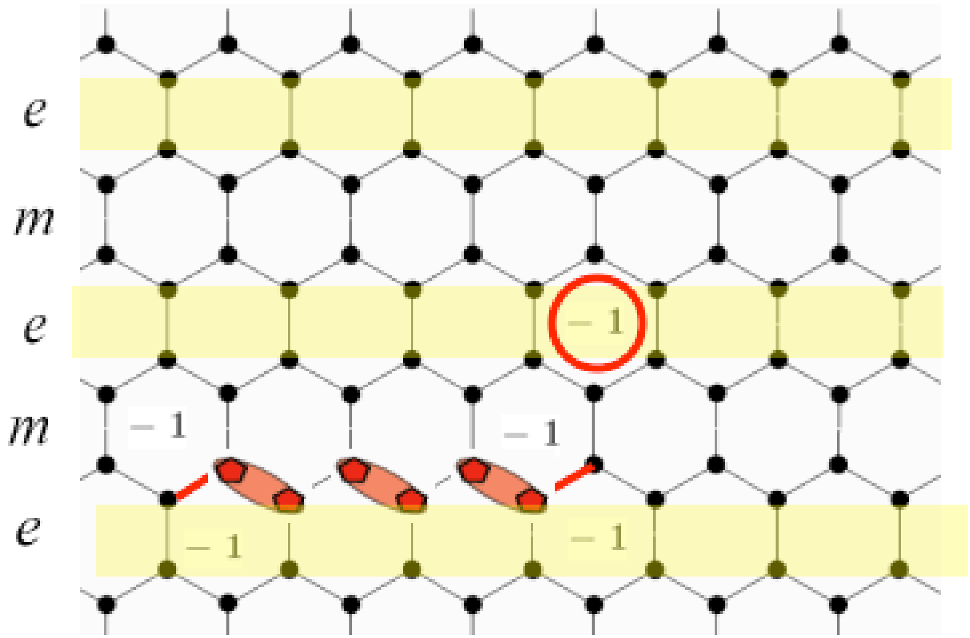
Mutual statistics of fermions and vortices



$K_{ij}^{\alpha,\beta} \equiv \sigma_i^\alpha \otimes \sigma_j^\alpha$ along β-links.

We can excite two pairs of vortices,
i.e. two fermions.

Mutual statistics of fermions and vortices



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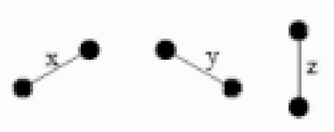
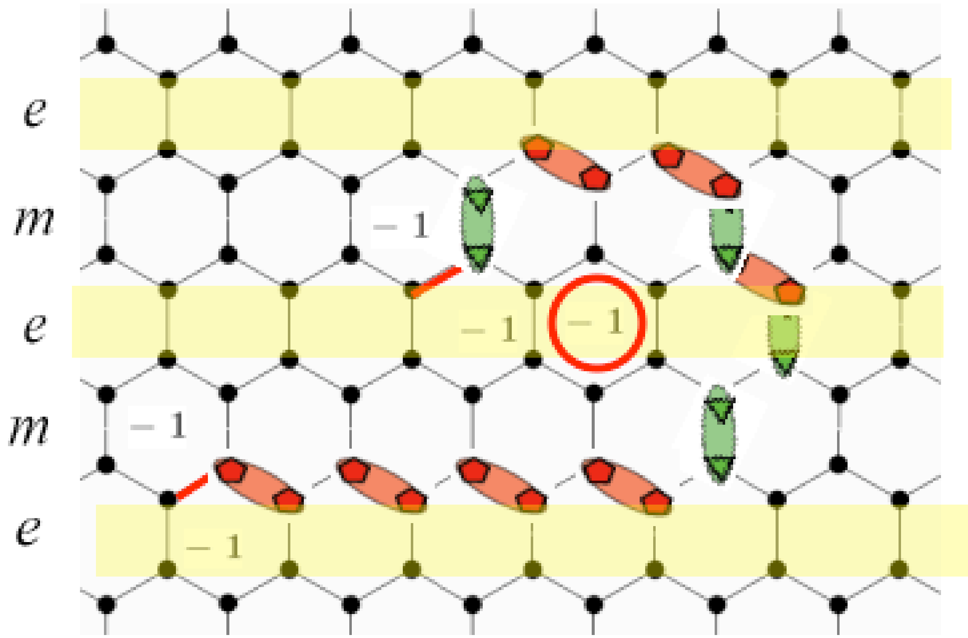


We can move the pair of vortices around the other vortex without changing the energy of the system (only two links are violated at a given time).

Mutual statistics of fermions and vortices



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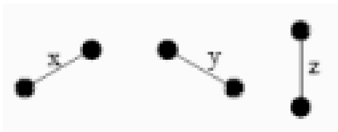
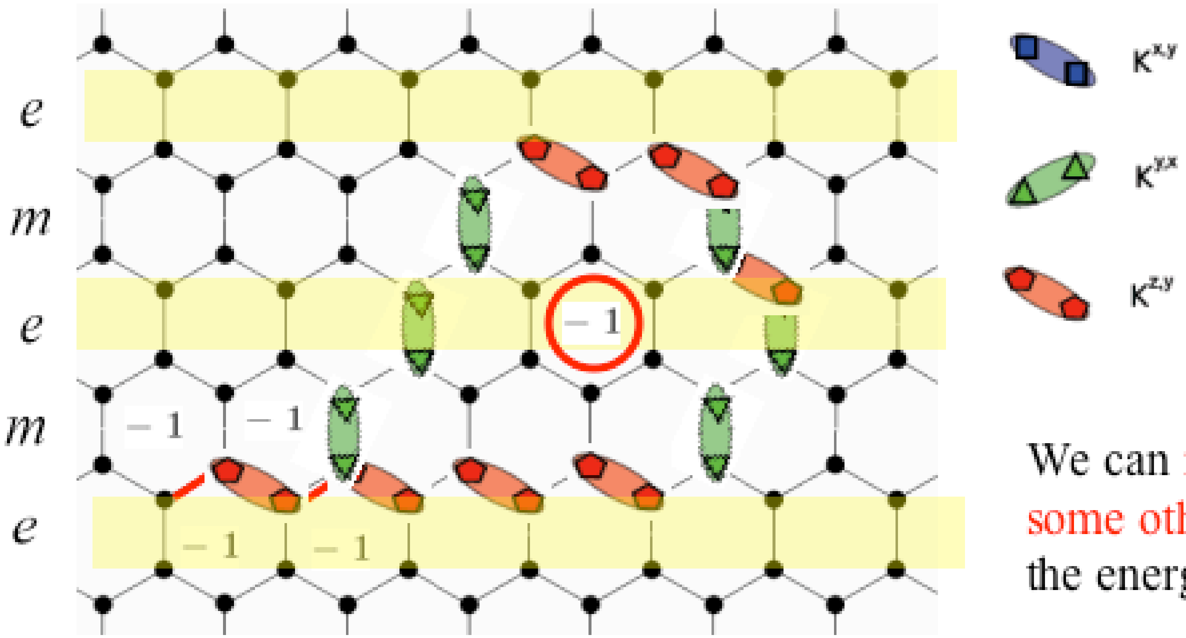


We can rotate the pair of vortices around some other vortices without changing the energy of the system.

Mutual statistics of fermions and vortices



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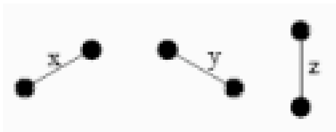
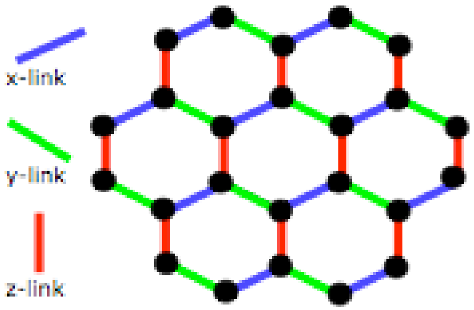


We can rotate the pair of vortices around some other vortices without changing the energy of the system.

The net operation corresponds to the multiplication of the plaquette operators, i.e. vortices, inside the loop.

This is the cleanest way to observe the statistics of vortices of the honeycomb model, circumventing approximations associated with direct mapping of the toric code operations.

Kitaev honeycomb lattice model



With magnetic field:

$$H = H_0 + \Sigma_i \Sigma_{\alpha=x,y,z} B_\alpha \sigma_{\alpha,i}$$

Phase diagram

- phase A - abelian topological phase $Z_2 \times Z_2$
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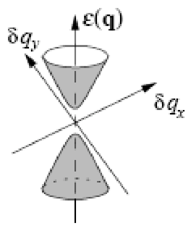
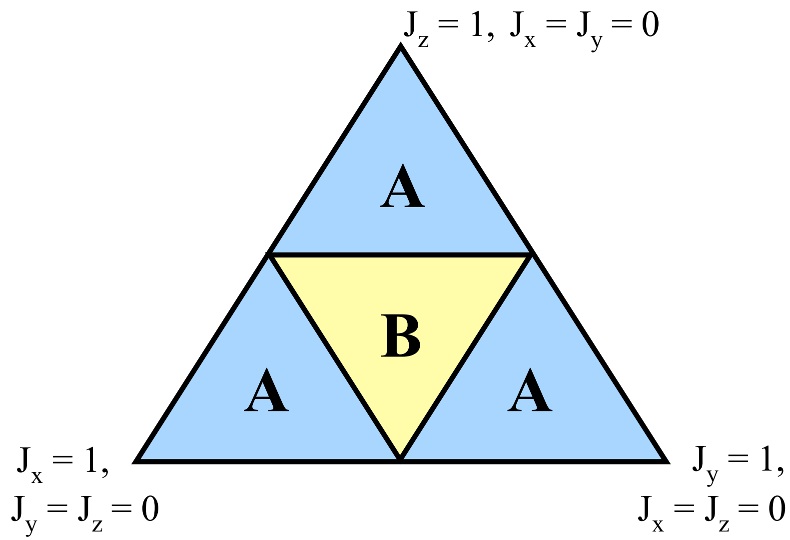
- FQH state at ν=5/2, p-wave sc graphene



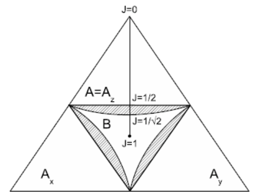


A.Y.Kitaev, *Anyons in an exactly sovable model and beyond*, Ann. Phys. 321, 2 (2006).c

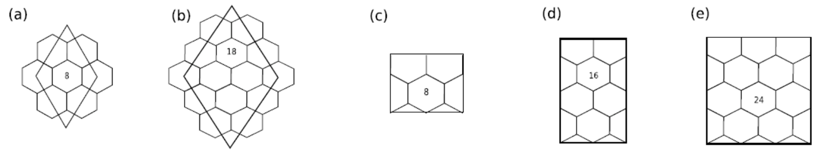
Effect of magnetic field

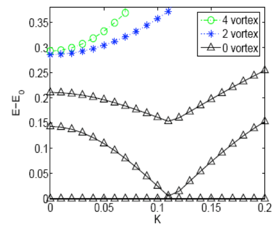
Perturbative effect of magnetic field in all vortex sectors

$$H = -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y - J_z \sum_{z-links} \sigma_j^z \sigma_k^z + K \sum_{ijk} \sigma_i^x \sigma_j^y \sigma_k^z$$

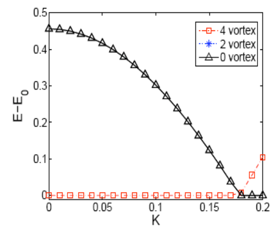


Exact numerical diagonalisation of finite size toroidal systems:





Confirmation of opening and closing of gaps within particular vortex sectors.



Exact numerical confirmation of analytical results.

V. Lahtinen, G. Kells, A. Carollo, T. Stitt, J. Vala, J. K. Pachos
Spectrum of the non-abelian phase in Kitaev's honeycomb lattice model
arXiv:0712.1164 (to appear in Annals of Physics)

Technical developments

Code development for generating parallel multi-spin-lattice operators

- Excellent scalability on shared and distributed memory machines

Eigensolvers and analysis

- Linear scaling exact diagonalization techniques
  ARPACK, PETSc and SLEPc libraries

  Capabilities: > 36 spin lattice systems (real, using symmetries)
  ~ 32 spin (complex, e.g. Kitaev model with B field)

- Approximate techniques:
  ~ 2-D open b.c 100-spin systems with local Hamiltonian
  Projected Entangled Pair States (PEPS) approach

Quantum propagation

- Chebyshev polynomial expansion of quantum evolution operator

T. Stitt, G. Kells and J. Vala ‘***LAW: A Tool for Improved Productivity with High-Performance Linear Algebra Codes. Design and Applications***’, arXiv:0710.4896 *(To be submitted to Computer Physics Communications)*

Schroedinger/Lanczos
ICHEC BlueGene/P/L

4096/2048 cores
33 TB memory
3D toroidal network

Walton
IBM cluster 1350

958 processing cores
14 TB memory

Technical developments

	Method is exact	1D systems	2D systems	Fermionic systems (sign)	Parallel	Momentum sectors	Sz sectors	> 2 Site interactions	Stores full Hamiltonian	Established technique	Periodic bondary conditions	Typical system size in spins
In-House Exact Diagonalisation Code	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	32
ALPS Exact Diagonalisation	✓	✓	✓	✓	✗	✓	✗	✗	✗	✓	✓	20
ALPS DMRG	✗	✓	✗	✓	✗	✗	✗	✗	✗	✓	✓	80
PEPS	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗	-
ALPS Quantum Monte Carlo	✗	✓	✓	✗	✗	✗	✗	✗	✗	✓	✓	-

Under development

- extension of the exact diagonalization code to include Bloch basis representation
- integration of our exact diagonalization code with ALPS
- PEPS representation for topologically ordered systems with open and closed boundary

Work in progress

Kitaev honeycomb lattice model

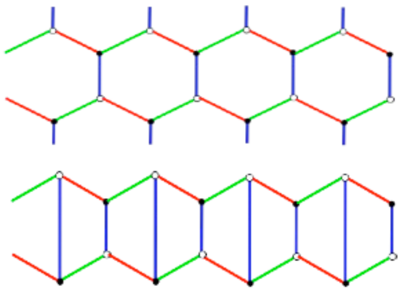
- thin torus limit of the model
- quasiparticle properties and dynamics in phase B in perturbative magnetic field and beyond
- spectral properties of phase B in magnetic field
- quantum phase transitions between topological phases
- modified and alternative models (with Joost Slingerland and Hector Bombin)

Fendley quantum loop gas models - modified inner product allows for nonabelian phases in contrast to Freedman models

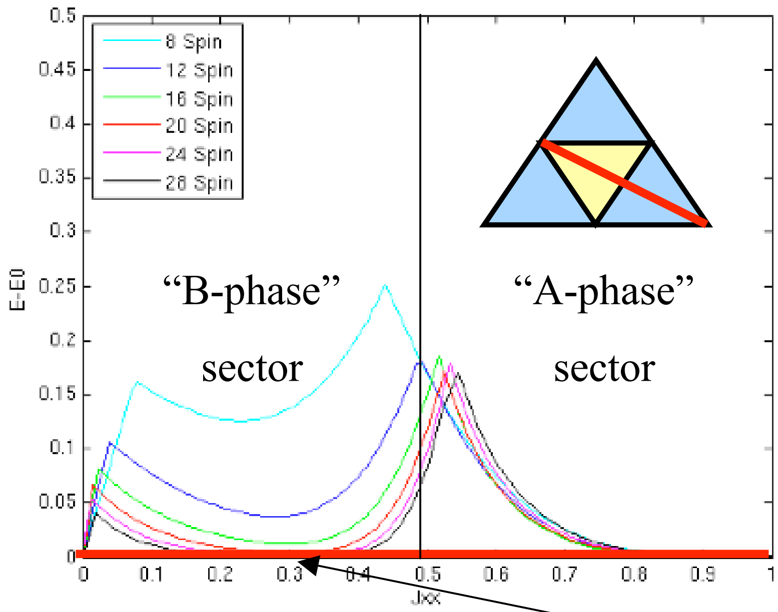
- low energy spectral properties of the Fendley Hamiltonian at k=2 (Toric code)
- extension to k=3 theory –Fibonacci anyons

Thin torus limit of the Kitaev model - preliminary results

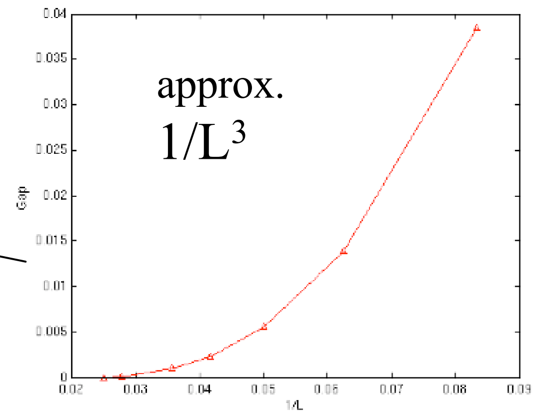
One-dimensional limit that is more accessible by both analytical (CFT) and numerical (DMRG) techniques.



Low energy spectrum in the vortex free sector appear to mimic the spectral properties of the uncompactified system:

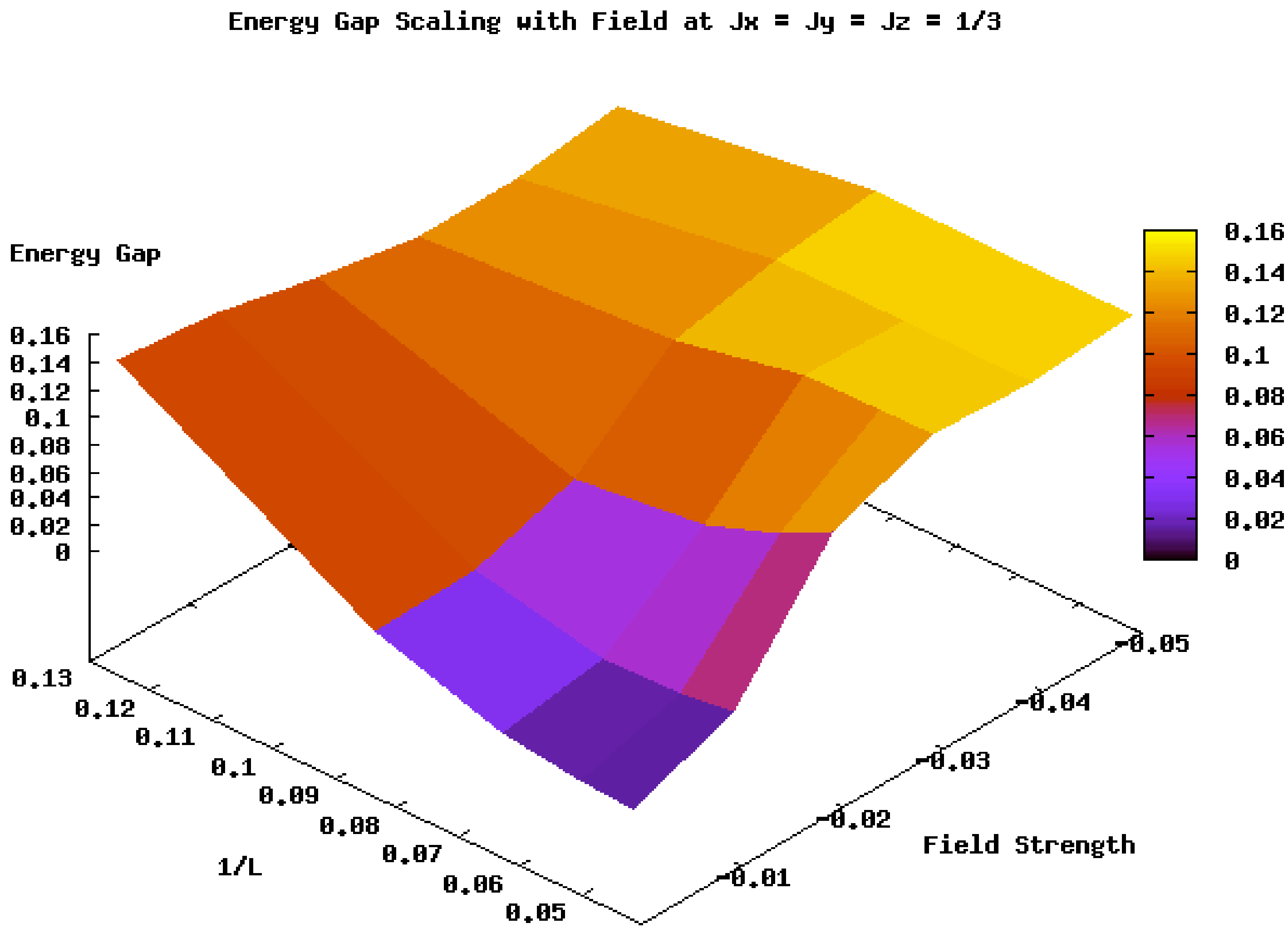


Scaling of the gap with 1/L was studied using both exact diagonalization (up to N=28) and DMRG (up to N=80):



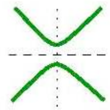
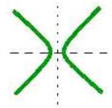
Thin torus with magnetic field - preliminary results

Magnetic field opens a gap in the “B-phase” sector of the model



Fendley quantum loop gas models

Two state quantum system at each vertex of the completely packed quantum loop model

$|1\rangle =$  $\quad |\hat{1}\rangle =$ 

Paul Fendley
arXiv/0804.0625

Introducing topological inner product and new orthogonal basis

$$\begin{pmatrix} \langle 1|1\rangle & \langle 1|\hat{1}\rangle \\ \langle \hat{1}|1\rangle & \langle \hat{1}|\hat{1}\rangle \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

$$|0\rangle = \frac{1}{\sqrt{d^2-1}}\left(d|\hat{1}\rangle + |1\rangle\right)$$
$$|\hat{0}\rangle = \frac{1}{\sqrt{d^2-1}}\left(d|1\rangle + |\hat{1}\rangle\right)$$

$$\lambda = \frac{\langle \hat{1}|1\rangle}{\sqrt{\langle 1|1\rangle\langle \hat{1}|\hat{1}\rangle}} = \frac{\langle \chi|\eta\rangle}{\sqrt{\langle \chi|\chi\rangle\langle \eta|\eta\rangle}} = \pm\frac{1}{d}$$

leads to cracking $2^{1/2}$ barrier encountered in quantum loop gas models by Freedman.

Starting point: toric code

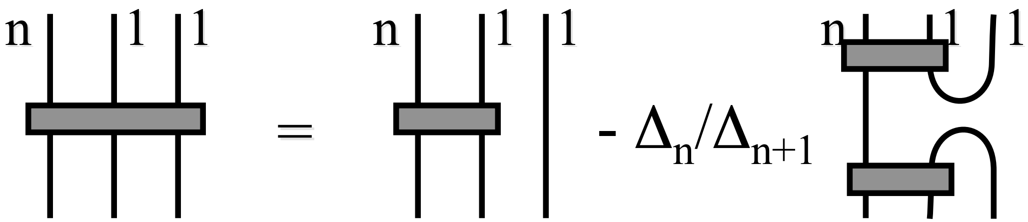
$$H = H_{toric} + uH_u \qquad H_{toric} = \sum_V W_V + \overline{W}_F\ .$$

$$W_V = \frac{1}{2}(1 - \sigma^z_{V1}\sigma^z_{V2}\sigma^z_{V3}\sigma^z_{V4}),$$
$$\overline{W}_F = \frac{1}{2}(1 - \sigma^x_{F1}\sigma^x_{F2}\sigma^x_{F3}\sigma^x_{F4})$$

vertex and face operators

$$H_u = \sum_V W_V \sum_{a=1}^{4} \sigma^z_{Va} + \sum_F \overline{W}_F \sum_{a=1}^{4} \sigma^x_{Fa}\ .$$

adding Jones-Wenzl projectors provides $SO(3)_k$ theory at arbitrary level of theory k.



$d = 2\cos(\pi/(k+2))$

$\Delta_{-1} = 0$
$\Delta_0 = 1$
$\Delta_{n+1} = d\,\Delta_n - \Delta_{n-1}$

Fendley quantum loop gas models - preliminary results

Starting point: toric code

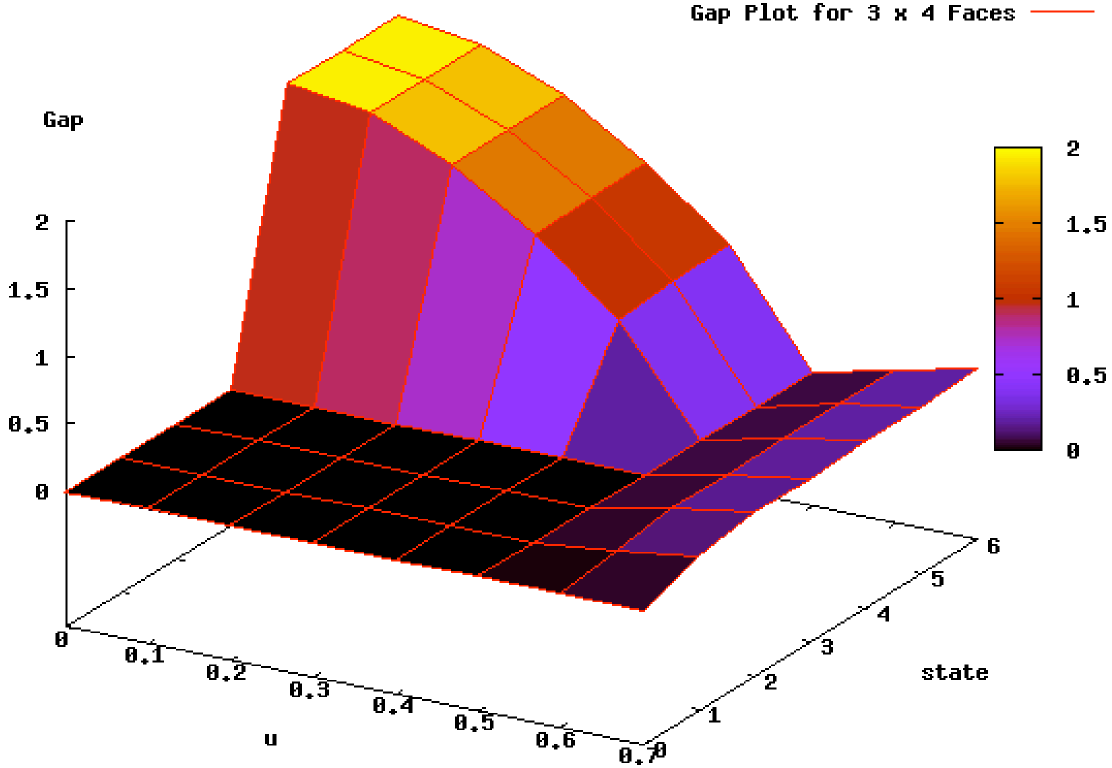
$$H = H_{toric} + uH_u \qquad H_{toric} = \sum_V W_V + \overline{W}_F \,.$$

$$\begin{aligned} W_V &= \frac{1}{2}(1-\sigma^z_{V1}\sigma^z_{V2}\sigma^z_{V3}\sigma^z_{V4}), \\ \overline{W}_F &= \frac{1}{2}(1-\sigma^x_{F1}\sigma^x_{F2}\sigma^x_{F3}\sigma^x_{F4}) \end{aligned}$$

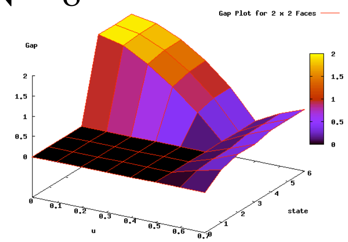
vertex and face operators

$$H_u = \sum_V W_V \sum_{a=1}^{4} \sigma^z_{Va} + \sum_F \overline{W}_F \sum_{a=1}^{4} \sigma^x_{Fa} \,.$$

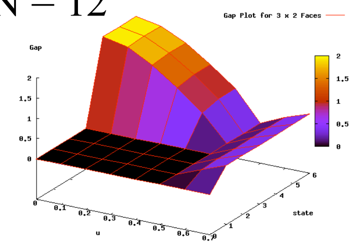
N = 24



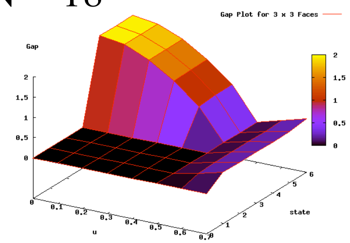
N = 8



N = 12



N = 18



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