Condensation Transitions between Topological Phases

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Based on work with Sander Bais, arXiv:0808.0627

Earlier work which we extend:

- Bais, Schroers, JKS. PRL 89:18601, 2002
- Bais, Schroers, JKS. JHEP 05:068, 2003
- Bais, Mathy, cond-mat/0602101, cond-mat/0602109, cond-mat/0602115

Topological Symmetry Breaking and Bose Condensation

Want to study transitions between topological phases (threats/features for TQC)

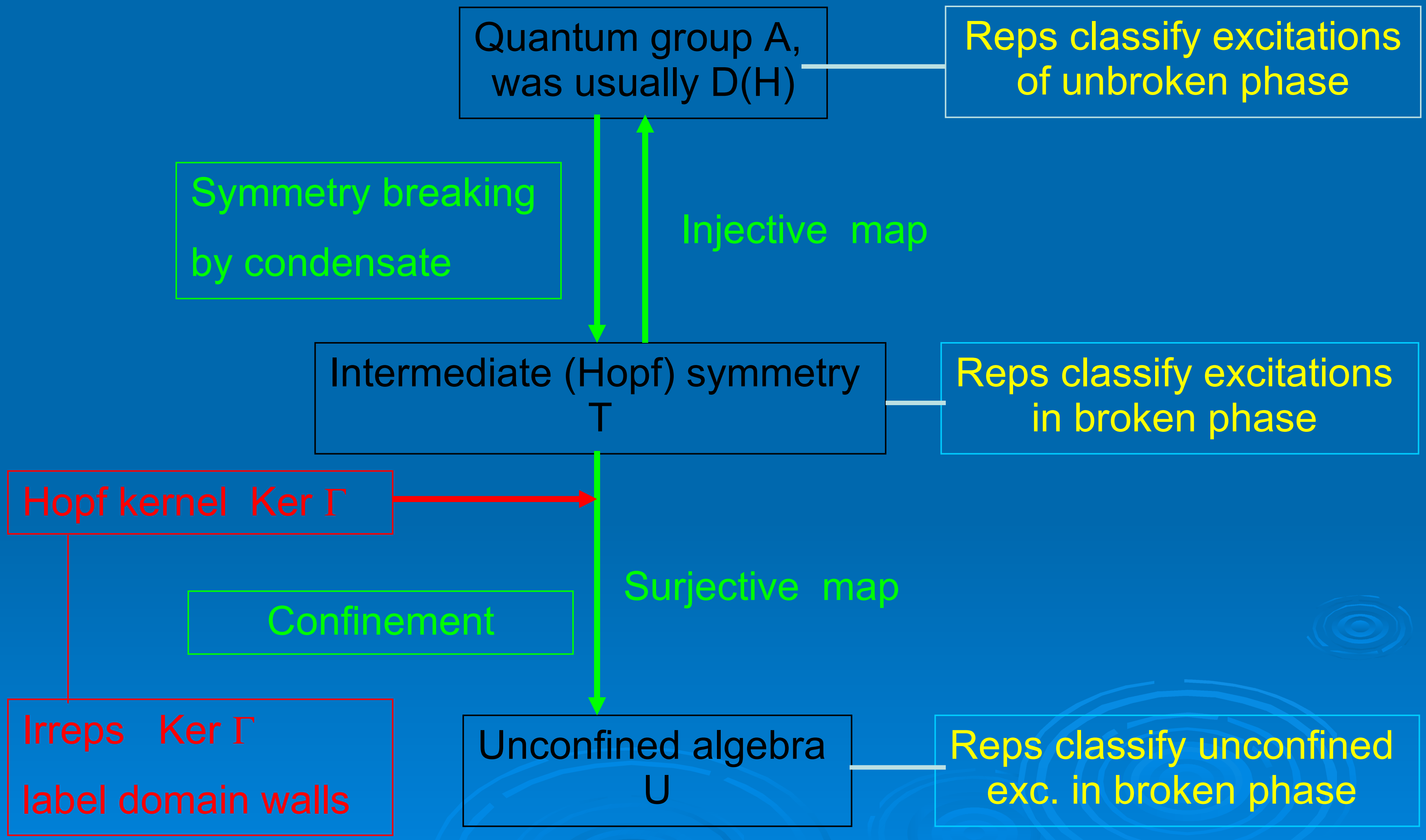
→ Need a guiding principle!

- Can describe topological order by extended (nonlocal) “symmetry” concepts: TQFTs, Tensor Categories, Hopf Algebras, Quantum Groups

Particle types	⟷	Irreducible representations
Fusion	⟷	Tensor Product
Braiding	⟷	R-matrix
Twist	⟷	Ribbon Element

- IDEA:
  Relate topological phases by “Symmetry Breaking”
- Mechanism? Bose Condensation!
  Break the Quantum Group to the “Stabiliser” of the condensate’s order parameter

Quantum Groups for Symmetry Breaking and Confinement



Quantum group symmetry breaking: What we will use here

(Not Quantum Groups or full TQFTs! Though almost...)

Fusion rules

$$a \times b = \sum_c N_c^{ab} c$$

Twist factors

$$= e^{2\pi i h_a}$$

Monodromy,
determined by twist

$$= e^{2\pi i (h_c - h_a - h_b)}$$

Note: From these, can get S and M matrices, central charge c (mod 8), quantum dimensions and total qdim D, and scaling dimensions (mod1)

On Bosons

Q: What is a boson in 2+1 TQFT?

A: FAPP, a particle with

- trivial twist factor/ integer conformal weight
- trivial self monodromy in at least one fusion channel,
  i.e. at least one of the fusion products also has trivial twist/integer weight

(really should require that states with totally trivial self-monodromy exist
for any number of 'bosonic' quasiparticles, but often this can be proved from the above)

Relevance: Bosons naturally occur in many topological models:

- In most proposed exotic Hall states, including the filling 5/2 state (but <u>not</u> 12/5...)
- In all "doubled" theories of the form Theory x Theory *
  (includes most "picture theories", loop/dimer/string net models)
- In all discrete gauge theories (toric code and generalisations)

"Symmetry breaking" from the dual side

Inspired by usual group or algebra symmetry breaking,
introduce branchings for topological sectors (cf. restriction of representations):

$$a \to \sum_i n_{a,i} a_i \qquad a_i \text{ are T-sectors}, \quad n_{a,i} \text{ are multiplicities}$$

Note: condensate must branch to vacuum (cf. trivial representation)

Requirements

1. The new labels themselves have good fusion rules
   (need associativity, vacuum and charge conjugation)
2. Branching and fusion commute,

$$a \otimes_A b \to \left(\sum_i n_{a,i} a_i\right) \otimes_T \left(\sum_i n_{b,i} b_i\right)$$

This implies preservation of quantum dimensions (useful in calculations)

Note: no requirements on braiding at this point (allow for confined excitations).

Example: Breaking $SU(2)_4$

$SU(2)_4$ unbroken		
0	$d_0 = 1$	$h_0 = 0$
1	$d_1 = \sqrt{3}$	$h_1 = \frac{1}{8}$
2	$d_2 = 2$	$h_2 = \frac{1}{3}$
3	$d_3 = \sqrt{3}$	$h_3 = \frac{5}{8}$
4	$d_4 = 1$	$h_4 = 1$

$1 \times 1 = 0 + 2$			
$1 \times 2 = 1 + 3$	$2 \times 2 = 0 + 2 + 4$		
$1 \times 3 = 2 + 4$	$2 \times 3 = 1 + 3$	$3 \times 3 = 0 + 2$	
$1 \times 4 = 3$	$2 \times 4 = 2$	$3 \times 4 = 1$	$4 \times 4 = 0$

Quantum dimensions, scaling dimensions (hence spins) and fusion for the irreps 0,1,2,3,4 of $SU(2)_4$

Notes:

- Has non-integer quantum dimensions (did not know how to do that before).
- Shows interesting relation to CFT construction (conformal embedding)
- Relevant to k=4 Read-Rezayi state
- Simple (but can do much more complicated by the same techniques)

Condensate, splitting and identification

Assume a bosonic condensate forms in the 4 rep of $SU(2)_4$:

$$\begin{aligned}1\times 4 &= 3\\ 4\times 1 &= 3\end{aligned} \quad \text{so} \quad 1 \Leftrightarrow 3$$

$$2\times 2 = 0+2+4 \;\rightarrow\; 0+2+0 \;\Rightarrow\; 2 \;\rightarrow\; 2_1+2_2$$

Note: 2 can indeed split into two parts, since $d_2=2$

Now:

$$2\times 2 \;\rightarrow\; 2_1\times 2_1 + 2_1\times 2_2 + 2_2\times 2_1 + 2_2\times 2_2 = 0+0+2_1+2_2$$

Must have either $2_1\times 2_1 = 2_2\times 2_2 = 0$

or $2_1\times 2_2 = 2_2\times 2_1 = 0$

The first choice gives violation of associativity.

Would have to have either $\begin{aligned}2_1\times 2_2 &= 2_1\\ 2_2\times 2_1 &= 2_2\end{aligned}$ or $\begin{aligned}2_1\times 2_2 &= 2_2\\ 2_2\times 2_1 &= 2_1\end{aligned}$

With first of these get $(2_1\times 2_2)\times 2_1 = 0$ and $2_1\times(2_2\times 2_1) = 2_1$

Contradicts associativity! Similarly for the second option.

Conclusion: $2_1\times 2_2 = 2_2\times 2_1 = 0$

$SU(2)_4$ breaking summary

$SU(2)_4$ broken	
$0 \rightarrow 0$	$d_0 = 1$
$1 \rightarrow 1$	$d_1 = \sqrt{3}$
$2 \rightarrow 2_1 + 2_2$	$d_{2_1} = d_{2_2} = 1$
$3 \rightarrow 1$	
$4 \rightarrow 0$	

$$1 \times 1 = 0 + 2_1 + 2_2$$

$$1 \times 2_1 = 1 \qquad 2_1 \times 2_1 = 2_2$$

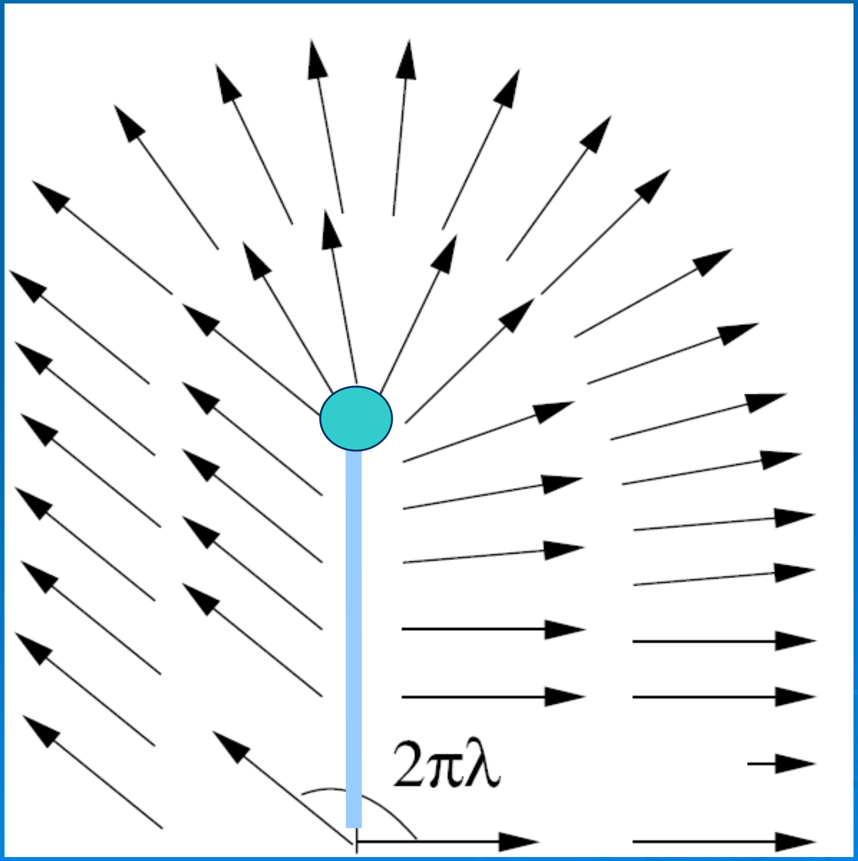
$$1 \times 2_2 = 1 \qquad 2_1 \times 2_2 = 0 \qquad 2_2 \times 2_2 = 2_1$$

Sectors and fusion for the broken SU(2)_4 theory, after condensation of 4

Confinement

Excitations of the broken theory may be confined (or boundary excitations);
If they have nontrivial monodromy with the condensed excitation,
they will pull a string where the condensate is locally destroyed

Confined Excitation



Confinement and Spin/Monodromy

For unconfined particles, the condensate should not interfere with the monodromy or spin factors. Hence the following requirement on sectors of the broken theory.

a_i is not confined iff all fields that branch to a_i have equal twist factors (i.e. conformal dimensions that differ by integers).

The non-confined particles also have well defined monodromies with each other, given by their twist factors (which are unambiguously defined from the branching).

Note 1:
the sectors which branch to unconfined particles have trivial braiding with the condensed sector(s), (i.e. no strings)

Note 2:
the confined 'particles' do not have well defined spin factors
(and they should not, because they pull strings).

The vacuum of the broken theory (T) should not be confined
(this is a requirement on T)

Confinement for $SU(2)_4$

From branching rules and twist factors (conformal weights) one finds that the 1 (or 3) is confined,

The unconfined algebra becomes $SU(3)_1$.
This is uniquely determined from the fusion and spins of the remaining sectors.

Compare broken $SU(2)_4$ with $SU(3)_1$ below.

Note $\theta = e^{2\pi i h}$

$SU(2)_4$ after confinement	
0	$\theta_0 = 1$
1	confined
2_1	$\theta_{2_1} = e^{2\pi i/3}$
2_2	$\theta_{2_2} = e^{2\pi i/3}$

$$2_1 \times 2_1 = 2_2$$

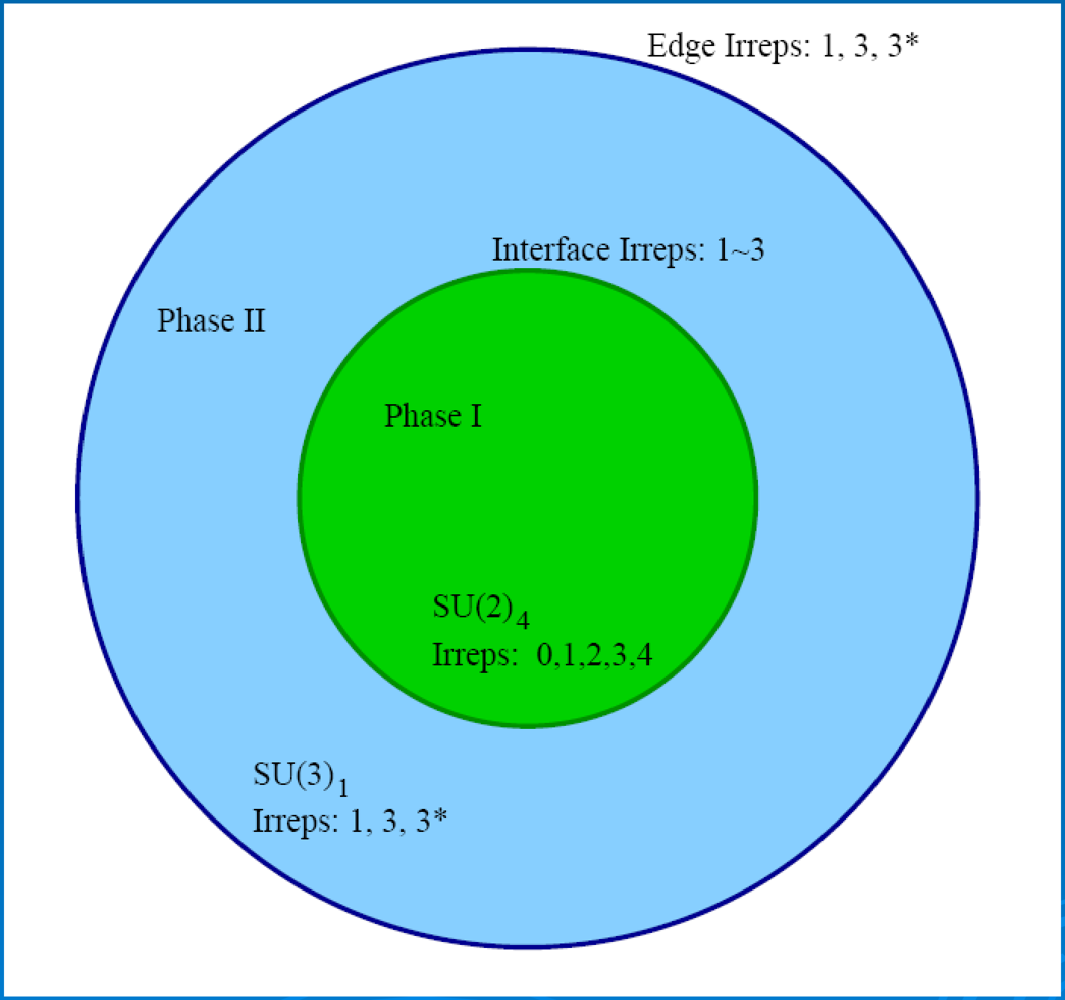
$$2_1 \times 2_2 = 0 \qquad 2_2 \times 2_2 = 2_1$$

$SU(3)_1$	
1	$d_1 = 1 \quad h_0 = 0$
3	$d_3 = 1 \quad h_3 = \frac{1}{3}$
$\bar{3}$	$d_{\bar{3}} = 1 \quad h_3 = \frac{1}{3}$

$$3 \times 3 = \bar{3}$$

$$3 \times \bar{3} = 1 \quad \bar{3} \times \bar{3} = 3$$

Overall Result for breaking $SU(2)_4$



Note: the broken theory (T-level) describes the interface between Phases I and II

Relation to CFT constructions

General Idea: the Quantum Group normalizes the Chiral Algebra so Quantum Group breaking is Chiral Algebra extension

Conformal embeddings:

For conformal embedding of H_k into $G_{1,}$ can obtain G_1 from H_k by condensation of all available bosons.
Can also find conformal branching rules and modular invariants this way.

Cosets:

Spectrum and spin factors for the coset G_k/H_l are those of $G_k x H_{-l}$ after condensation of all available bosons.
For the simplest cosets, we recover a known construction (the Identification group method – Schellekens, Yankielowicz a.o.)
But our method also works with identification fixed points and even for Maverick cosets – possibly the simplest way yet of treating cosets.

Orbifolds

Probably more complicated.
In some cases can (partially) undo the orbifold construction

Conformal Embedding related to $SU(2)_4$

In general, for conformal embedding of H_k into G_1

Central charges satisfy c(G) = c(H) ==> c(G/H) = 0

Coset algebra is trivial

==> get finite branching of inf. Dim. KM representations

Our Example: $SU(2)_4$ ==> $SU(3)_1$ (both c=2), Compare:

$SU(3)_1$ conf. emb. Branching

$$
\begin{aligned}
1 &\rightarrow 0 + 4 \\
3 &\rightarrow 2 \\
\bar{3} &\rightarrow 2
\end{aligned}
$$

$SU(2)_4$ QG breaking branching

$$
\begin{aligned}
0 &\rightarrow 0 \\
1 &\rightarrow 1 \\
2 &\rightarrow 2_1 + 2_2 \\
3 &\rightarrow 1 \\
4 &\rightarrow 0
\end{aligned}
$$

Can get these from each other!

A coset example: Producing the Ising model

Ising = $(SU(2)_1 \times SU(2)_1) / SU(2)_2$ coset

Form the product $SU(2)_1 \times SU(2)_1 \times (SU(2)_2)^*$

$SU(2)_1$	
0	$d_0 = 1 \quad h_0 = 0$
1	$d_1 = 1 \quad h_1 = \frac{1}{4}$

$$1 \times 1 = 0$$

$SU(2)_2$	
0	$d_0 = 1 \quad h_0 = 0$
1	$d_1 = \sqrt{2} \quad h_1 = \frac{3}{16}$
2	$d_2 = 1 \quad h_2 = \frac{1}{2}$

$$1 \times 1 = 0 + 2$$

$$1 \times 2 = 1 \qquad 2 \times 2 = 0$$

Note: have 1 boson: (1,1,2) has d=1 and h=1/4+1/4-1/2=0

Condense this, do breaking, result: (next slide)

Ising model

$(00;0) \sim 1$	$d_1 = 1$	$h_1 = 0$
$(00;2) \sim \varepsilon$	$d_\varepsilon = 1$	$h_\varepsilon = \frac{1}{2}$
$(01;1) \sim \sigma$	$d_\sigma = \sqrt{2}$	$h_\sigma = \frac{1}{16}$

$$\varepsilon \times \varepsilon = 1$$

$$\varepsilon \times \sigma = \sigma \quad \sigma \times \sigma = 1 + \varepsilon$$

General features of QG breaking: observations on c and D

Reminder:
c= topological central charge
D= total quantum dimension
A=Anyon model before condensation
T=Fusion model after breaking (with confined particles)
U=Anyon model after breaking and confinement

For modular theories A, we observe that

$$C_A = C_U$$

$$D_A/D_T = D_T/D_U$$

Summary and Outlook

Results

- Extended Topological symmetry breaking to TQFTs with non-integer quantum dimensions
- Found connection to conformal embeddings and cosets

Questions/Future Work

- Found Fusion and twist factors.
  How to determine the rest of the TQFT (half-braidings, F-symbols…) ?
  Note: often fixed by consistency (always?)
- Work suggests conformal embeddings of coset chiral algebras.
  Interesting CFT problem…
- Physical applications….