Entanglement Entropy at 2D quantum critical points, topological fluids and quantum Hall fluids

Invited Talk at the Nordita Workshop, Stockholm 2008

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Collaborators and References

- Stefanos Papanikolaou, Kumar Raman, Benjamin Hsu, Shiying Dong, Robert G. Leigh and Sean Nowling (UIUC),
- Michael Mulligan and Eun-Ah Kim (Stanford), Joel E. Moore (University of California Berkeley), Paul Fendley (Virginia)
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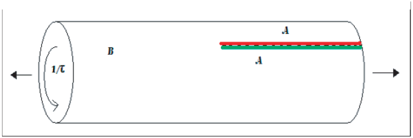

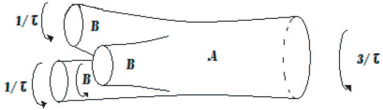

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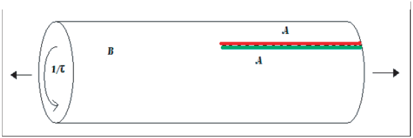

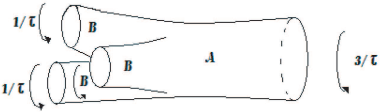

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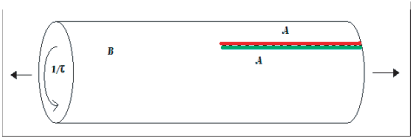

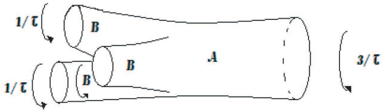

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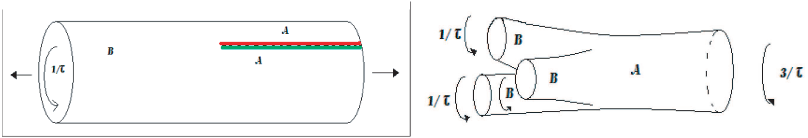

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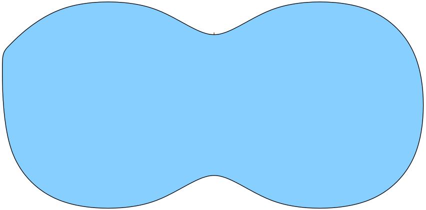

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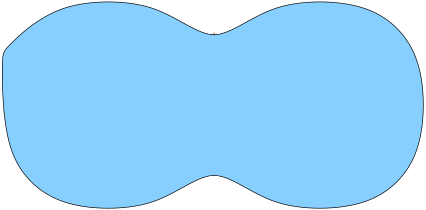

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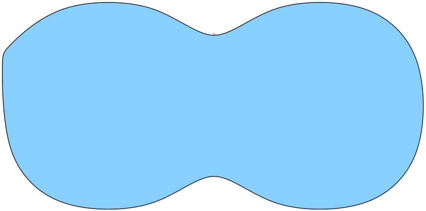

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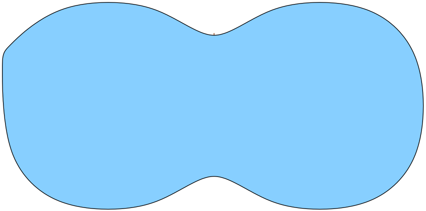

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- A and B are physically separate and have no common intersection, $\chi_A + \chi_B - \chi_{A \cup B} \neq 0$.
  The system physically splits in two disjoint parts $\Rightarrow$ $\log L$ term in the entanglement entropy
- A and B share a common boundary $\Rightarrow$ $\log L$ term whose coefficient is determined by the angles at the intersections
- If the boundary of A is not smooth, the coefficient depends on the angles α_i for both regions

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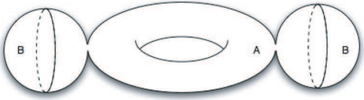

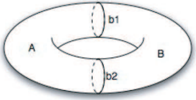

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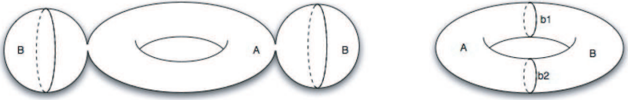

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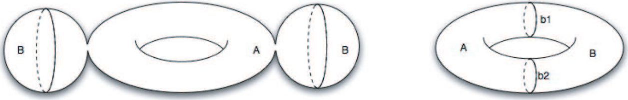

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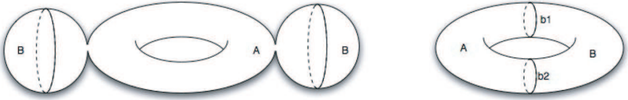

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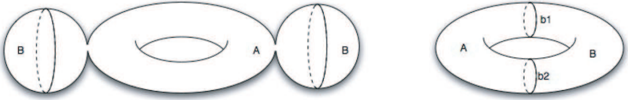

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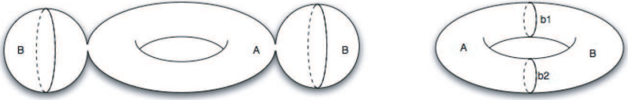

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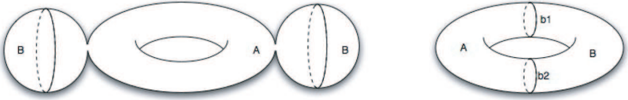

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