Algebraic Aspects of Anyons

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Motivations

- Gain insight into the many-anyon problem
  - Two-anyon problem solved analytically
  - Three- and four-anyon problem partially solved numerically
  - Only a small part of the N-anyon problem is understood
  - The many-body Hilbert space is not tensor product of single-particle Hilbert spaces
- Initiate an algebraic approach (phase-space approach)
  - An anyonic generalization of the harmonic oscillator
  - Very interesting "skew" anyonic weight space
  - Has been done successfully in 1+1 dim

Outline

- Anyons in the LLL and the Calogero model (1+1 dim)
- A new algebraic description of the anyon system (2+1 dim)

} $N = 2$ in detail

- Conclusions and Outlook

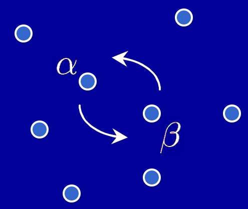
Anyons in the LLL

- Consider N anyons in a B field with Hamiltonian/angular momentum

$$H = \frac{1}{2} \sum_{i=1}^{2} \sum_{\alpha=1}^{N} \left(p_\alpha^i - A_\alpha^i\right)^2 \qquad L = \sum_{i,j} \sum_{\alpha=1}^{N} \varepsilon_{ij} x_\alpha^i p_\alpha^j$$

- Anyonic symmetry condition

$$\sigma_{\alpha\beta} \Psi = e^{i\pi\nu} \Psi$$



- Let $B \to \infty \quad \Rightarrow \quad$ The dynamics in the LLL is effectively one-dimensional
  - Energy is minimized
  - Angular momentum $L \equiv H_{Cal}$

Leinaas, Myrheim
Hansson, Leinaas, Myrheim
Polychronakos

- Calogero model for N particles

$$H_{Cal} = \frac{1}{2} \sum_{\alpha=1}^{N} (p_\alpha^2 + x_\alpha^2) + \sum_{\alpha<\beta}^{N} \frac{g}{(x_\alpha - x_\beta)^2} \qquad g = \nu(\nu - 1)$$

Anyons in the LLL

- Elegant operator solution using the *exchange operator formalism*

Brink, Hansson
Konstein, Vasiliev

Polychronakos

$$K_{\alpha\beta}\mathcal{O}_\alpha = \mathcal{O}_\beta K_{\alpha\beta} \qquad K_{\alpha\beta}K_{\alpha\beta} = 1$$

permutation operator

phase space operator

- Example: Two anyons $(N = 2)$
  - We need two oscillators A_1 and A_2
  - Combine into relative oscillator $a = A_1 - A_2$

$$K_{12}A_1 = A_2 K_{12} \quad \Longrightarrow \quad K_{12}a + aK_{12} = 0$$

"Kleinian"

Two Anyons in the LLL

- Use complex coordinates: $z_\alpha = \frac{1}{\sqrt{2}}(x_\alpha + ip_\alpha)$
- Focus on the relative motion; The c.o.m. $Z = z_1 + z_2$ decouples
- Introduce relative coordinates $z = z_1 - z_2$; denote $K_{12} \equiv K$
- Consider a *deformed Heisenberg algebra*

Wigner

$$\boxed{[a, a^\dagger] = 1 + 2\nu K} \qquad a = A_1 - A_2$$

Vasiliev

$$\boxed{Ka + aK = 0 \qquad Ka^\dagger + a^\dagger K = 0} \qquad \boxed{K^2 = 1}$$

- Complex representation

$$a = \nabla_{Kz} \equiv \partial + \frac{\nu}{z}(1 - K) \qquad a^\dagger = z$$

$$Kz + zK = 0 \qquad K\partial + \partial K = 0$$

Two Anyons in the LLL

- Hamiltonian for the relative motion becomes (after a redef. by z^ν)

$$H_{rel} = \frac{1}{2}\{a, a^\dagger\} = a^\dagger a + \frac{1}{2} + \nu$$

$\mathfrak{sp}(2)$ Cartan generator

$$[H_{rel}, a] = -a \qquad [H_{rel}, a^\dagger] = a^\dagger$$

- Representation theory: define action on g.st. $|0\rangle$

$$a|0\rangle = 0 \qquad K|0\rangle = |0\rangle$$

$$\Rightarrow \qquad H_{rel}|0\rangle = \left(\frac{1}{2} + \nu\right)|0\rangle$$

Shifted g.st. "energy"

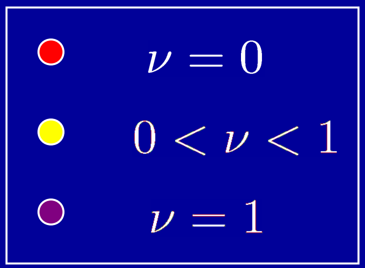
Two Anyons in the LLL

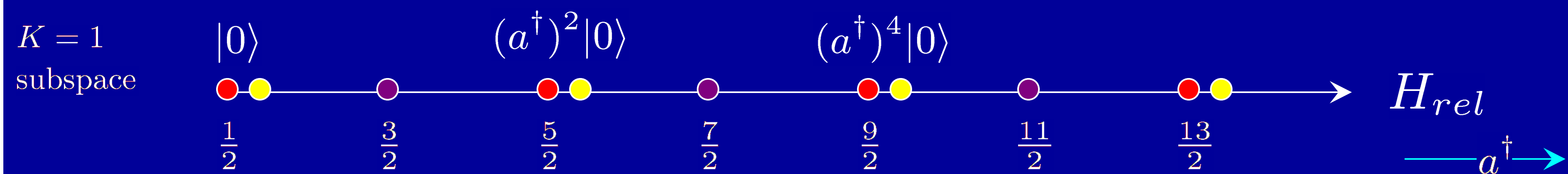
- Oscillator states

$$|n\rangle \equiv (a^\dagger)^n|0\rangle \qquad\qquad K|n\rangle = (-1)^n|n\rangle$$

- Spectrum

$$H_{rel}|n\rangle = \left(n + \frac{1}{2} + \nu\right)|n\rangle$$





- In the coordinate basis, the oscillators act on single-valued wave functions (recall: a factor z^ν has been extracted)

$$\Psi_0 = 1 \qquad\qquad \Psi_n = z^n = (a^\dagger)^n \Psi_0$$

N Anyons in the LLL

- Generalization to N particles (including c.o.m.)

$$[A_\alpha, A^\dagger_\beta] = \delta_{\alpha\beta}(1+\nu\sum_{\gamma=1}^{N} K_{\alpha\gamma}) - \nu K_{\alpha\beta}$$

$$[A_\alpha, A_\beta] = 0 \qquad [A^\dagger_\alpha, A^\dagger_\beta] = 0$$

Brink, Hansson
Vasiliev

$$K_{\alpha\beta} A_\alpha = A_\beta K_{\alpha\beta}$$

- Complex representation

$$A_\alpha = \partial_\alpha + \nu \sum_{\beta\neq\alpha} \frac{1}{z_\alpha - z_\beta}(1 - K_{\alpha\beta}) \qquad\qquad A^\dagger_\alpha = z_\alpha$$

N Anyons in the LLL

- The Hamiltonian takes the form (extract factor $\prod_{\alpha<\beta}^{N}(z_\alpha - z_\beta)^\nu$)

$$H = \frac{1}{2}\sum_{\alpha=1}^{N}\{A_\alpha, A_\alpha^\dagger\} = \sum_{\alpha=1}^{N}\left(z_\alpha\partial_\alpha + \frac{1}{2}\right) + \frac{\nu}{2}N(N-1)$$

$$[H, A_\alpha] = -A_\alpha \qquad [H, A_\alpha^\dagger] = A_\alpha^\dagger$$

- Representation theory: define action on g.st. $|0\rangle$

$$A_\alpha|0\rangle = 0 \qquad K_{\alpha\beta}|0\rangle = |0\rangle$$

- Oscillator states

$$\mathcal{Y}A_{\alpha_1}^\dagger \cdots A_{\alpha_n}^\dagger|0\rangle$$

Reproduce correct spectrum

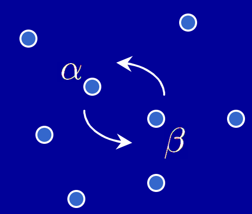
- Question: Can this algebraic construction be generalized to 2+1 dim?

N Anyons in a Harmonic Potential

- For our present purposes ignore the B field (may be included)
- Naive algebraic approach with multi-valued wave functions fails
- Hamiltonian and angular momentum operator

$$H = \frac{1}{2}\sum_{i=1}^{2}\sum_{\alpha=1}^{N} p_\alpha^i p_\alpha^i + x_\alpha^i x_\alpha^i \qquad L = \sum_{i,j}\sum_{\alpha=1}^{N} \varepsilon_{ij} x_\alpha^i p_\alpha^j$$

- Anyonic symmetry condition



$$\sigma_{\alpha\beta}\Psi = e^{i\pi\nu}\Psi$$

- Start with $N = 2$ since this problem has been solved analytically
  - Old approach
  - New approach

Two Anyons in a Harmonic Potential

- Use complex coordinates $z_\alpha = x_\alpha^1 + i x_\alpha^2$
- Focus on the relative motion (the c.o.m. decouple)
- Introduce relative coordinates $z = z_1 - z_2; \quad \bar{z} = \bar{z}_1 - \bar{z}_2$
- Oscillator algebra if we stick to multi-valued wave functions:

$$[a, a^\dagger] = 1 \qquad [b, b^\dagger] = 1$$

$$[a, b^\dagger] = [a, b] = 0$$

$$a = A_1 - A_2$$

$$b = B_1 - B_2$$

- Complex representation

$$a = \sqrt{2}\bar{\partial} + \frac{z}{2\sqrt{2}} \qquad a^\dagger = -\sqrt{2}\partial + \frac{\bar{z}}{2\sqrt{2}}$$

$$b = \sqrt{2}\partial + \frac{\bar{z}}{2\sqrt{2}} \qquad b^\dagger = -\sqrt{2}\bar{\partial} + \frac{z}{2\sqrt{2}}$$

Two Anyons in a Harmonic Potential

- Define $\mathfrak{sp}(4)$ Cartan generators

$$J_1 \equiv \frac{1}{2}\{a, a^\dagger\} \qquad J_2 \equiv \frac{1}{2}\{b, b^\dagger\}$$

- Hamiltonian and angular momentum for the relative motion (put $\hbar = 1$)

$$H_{rel} = J_1 + J_2 \qquad L_{rel} = J_1 - J_2$$

(change coeff. for different theory)

- Representation theory

$$a|0\rangle = 0 \qquad b|0\rangle = 0$$

$$|m, n\rangle \sim \text{ multi-valued wave functions } \Psi_{m,n}$$

Two Anyons in a Harmonic Potential

- Two types of wave functions $\Rightarrow$ skew spectrum

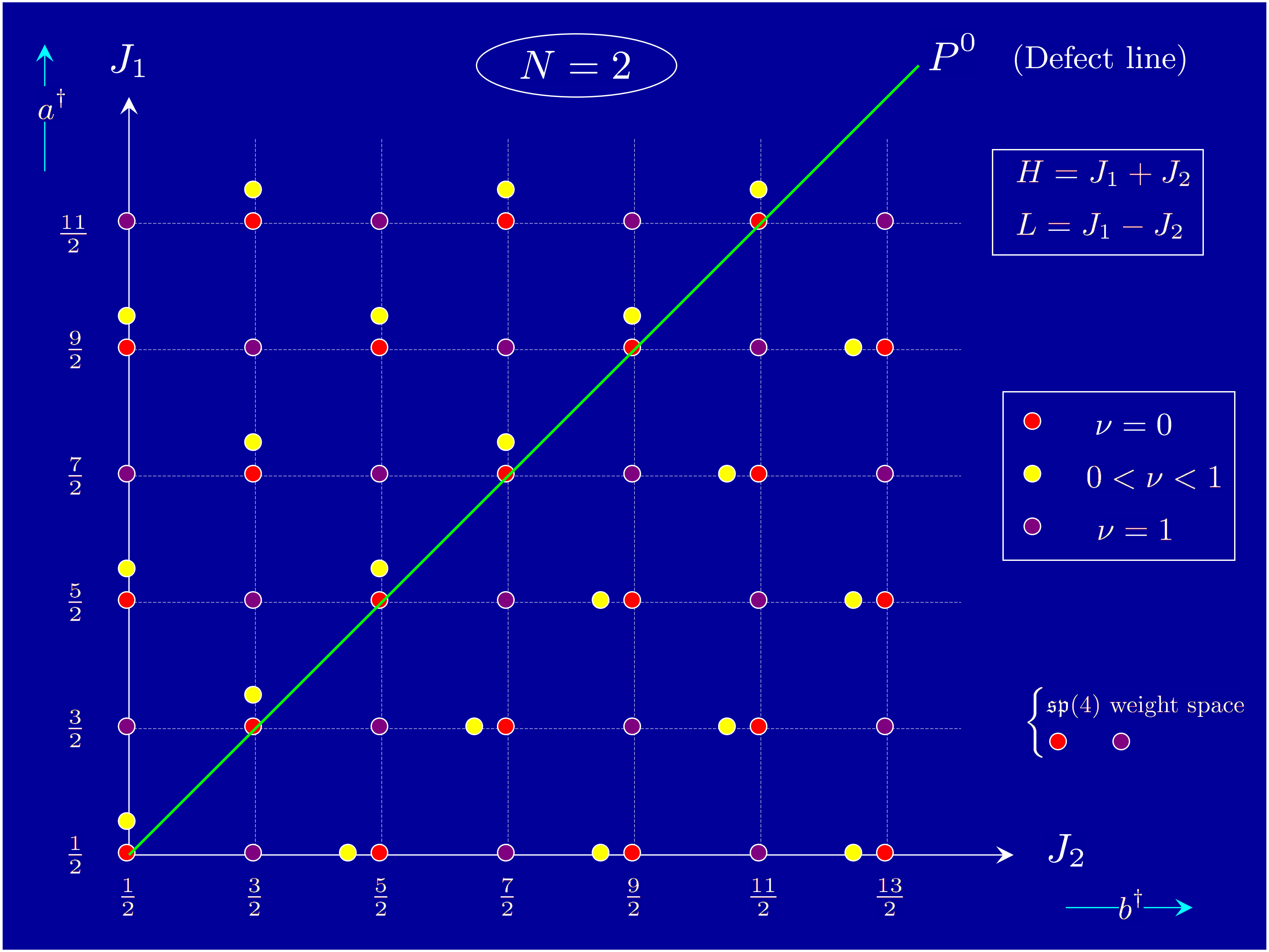
$$\Psi_{m,n} = \begin{cases} z^{\nu} z^{m} \bar{z}^{n} + \cdots & m \geq n \\ \bar{z}^{-\nu} z^{m} \bar{z}^{n} + \cdots & m < n \end{cases}$$

picture

- A general state should take the form $|m, n\rangle = (a^{\dagger})^{m} (b^{\dagger})^{n} |0\rangle$ or equivalently $\Psi_{m,n} = (a^{\dagger})^{m} (b^{\dagger})^{n} \Psi_{0}$

- Problems encountered with naive oscillator construction!
  (i) There is no unique ground state
  (ii) Only certain combinations of the oscillators are allowed
  (iii) Singular wave functions are produced when passing the defect

picture



$N = 2$
P^0
(Defect line)
J_1
$a^\dagger$
J_2
$b^\dagger$
$\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ $\frac{9}{2}$ $\frac{11}{2}$
$\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ $\frac{9}{2}$ $\frac{11}{2}$ $\frac{13}{2}$
$H = J_1 + J_2$
$L = J_1 - J_2$
$\nu = 0$
$0 < \nu < 1$
$\nu = 1$
$\mathfrak{sp}(4)$ weight space

N Anyons

- Problems:
  - Unnatural to have several ground states
  - Singular wave functions are produced
  - Hard to generalize to arbitrary N

- Try different approach: *deform* the algebra
  - Successful in 1+1 dim (or the 1D restriction of the current problem)
  - Harder in 2+1 dim since the spectrum is shifted asymmetrically
  - Construct algebra first for $N = 2$ since there we know the full spectrum
  - Try to generalize to arbitrary N

Deformed Oscillator Algebra: $N = 2$

- Focus on the relative motion
- Introduce relative oscillators $a = A_1 - A_2; \quad b = B_1 - B_2$
- Oscillator algebra

$$[a, a^\dagger] = 1 + 2\nu K^+ \qquad [b, b^\dagger] = 1 + 2\nu K^-$$

$$[a, b] = 2\nu R \qquad [b^\dagger, a^\dagger] = 2\nu R^\dagger$$

$$[a, b^\dagger] = 2\nu S \qquad [b, a^\dagger] = 2\nu S^\dagger$$

- "Diagonal" parts similar to the 1D case
  But: permutation operator $K \equiv K_{12} = K^+ + K^-$
- "Mixed" parts involve new operators R and S

Digression: A projector algebra

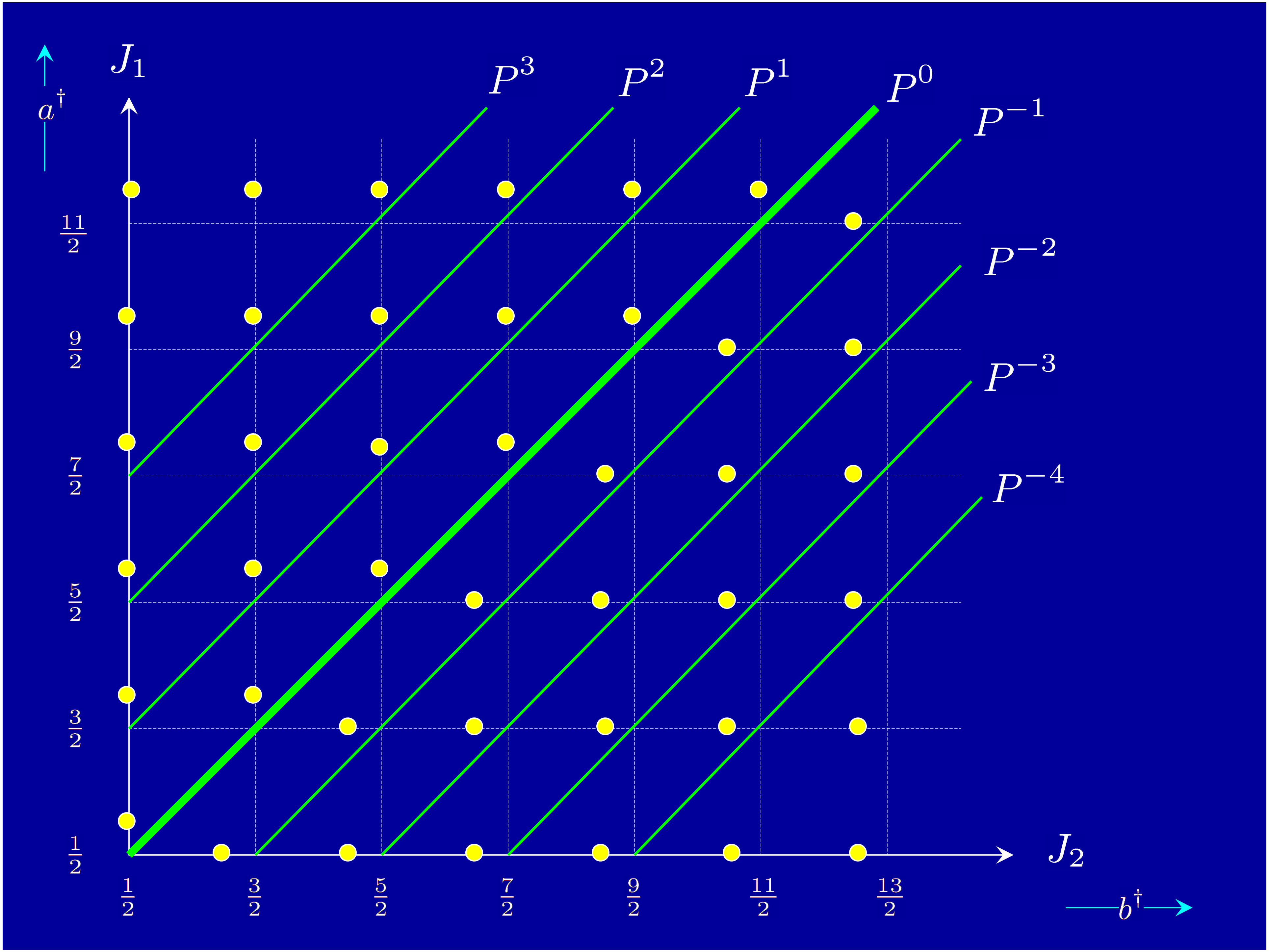
$$\boxed{P^m P^n = \delta_{m,n} P^n} \qquad m, n \in \mathbb{Z}$$

$$a^\dagger P^n = P^{n+1} a^\dagger \qquad b^\dagger P^n = P^{n-1} b^\dagger$$
$$P^n a = a P^{n+1} \qquad P^n b = b P^{n-1}$$

- P^n projects onto states with angular momentum $L = n + \nu$

$$L|\ell\rangle = (\ell + \nu)|\ell\rangle \qquad P^n|\ell\rangle = \delta_{n,\ell}|\ell\rangle$$

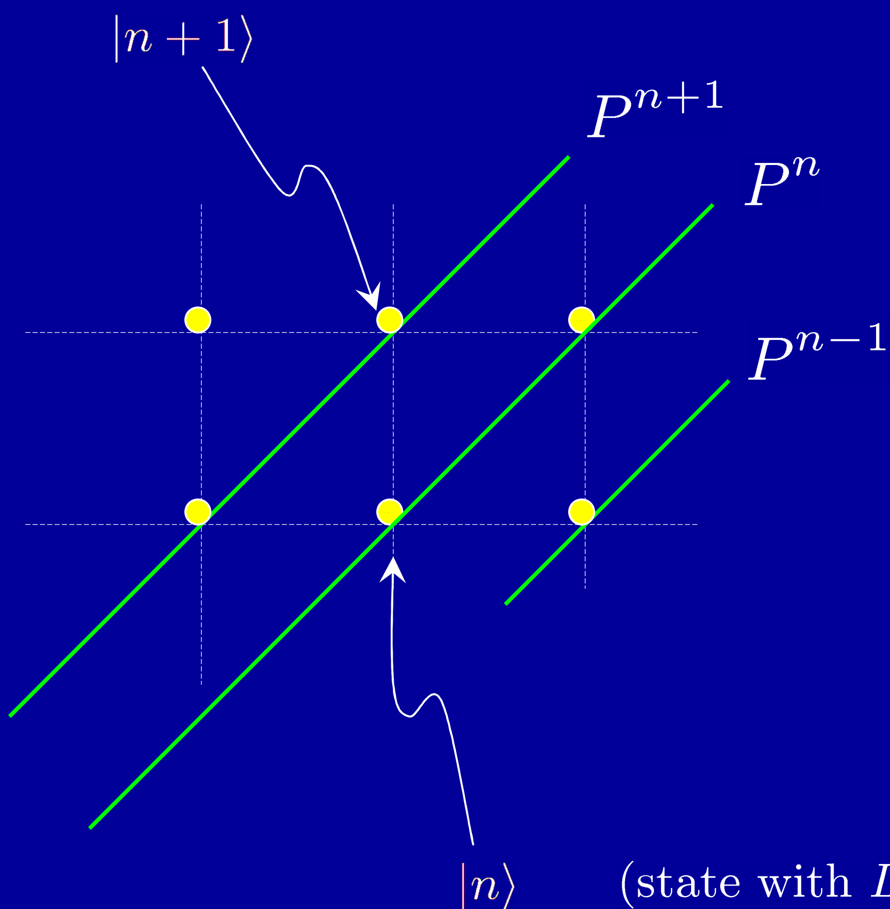
– Proof follows from defining action on the g.st. $P^n|0\rangle = \delta_{n,0}|0\rangle$



J_1
$a^\dagger$
P^3
P^2
P^1
P^0
P^{-1}
P^{-2}
P^{-3}
P^{-4}
$\frac{11}{2}$
$\frac{9}{2}$
$\frac{7}{2}$
$\frac{5}{2}$
$\frac{3}{2}$
$\frac{1}{2}$
J_2
$\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ $\frac{9}{2}$ $\frac{11}{2}$ $\frac{13}{2}$
$b^\dagger$

Digression: A projector algebra

- Illustration of the relation $a^\dagger P^n = P^{n+1} a^\dagger$



(state with $L = n + \nu$)

$$(1) \quad a^\dagger P^n |n\rangle = P^{n+1} \overbrace{a^\dagger |n\rangle}^{=|n+1\rangle} = a^\dagger |n\rangle$$

$$(2) \quad \underbrace{a^\dagger P^n |n+1\rangle}_{=0} = \underbrace{P^{n+1} a^\dagger |n+1\rangle}_{=0}$$

Digression: A projector algebra

- Define projectors – projecting onto states with positive/negative L $\nearrow \sim z^{\nu}$, $\searrow \sim \bar{z}^{-\nu}$

$$\Theta^+ = \sum_{n \geq 0} P^n \qquad \Theta^- = \sum_{n<0} P^n$$

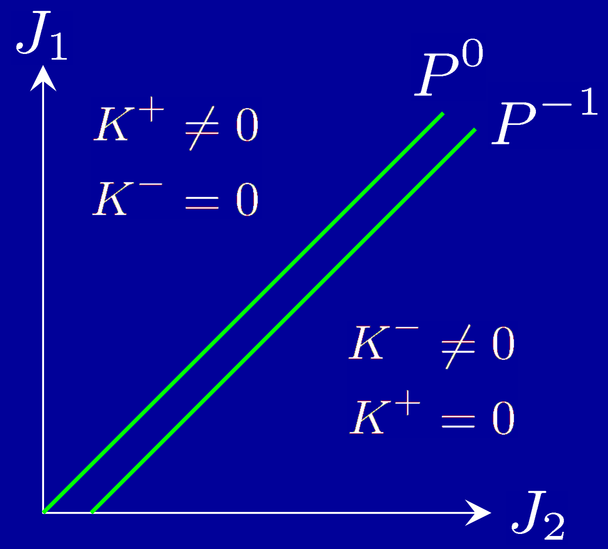
- Completeness relation: $\Theta^+ + \Theta^- = 1$

- Permutation operator $K \equiv K_{12} = K^+ + K^-$

$$\boxed{K^{\pm} \equiv \Theta^{\pm} K}$$

Deformed Oscillator Algebra: $N = 2$

Illustration:



$$[a, a^\dagger] = 1 + 2\nu K^+$$
$$[b, b^\dagger] = 1 + 2\nu K^-$$

- Permutation operator: $Ka + aK = 0$ and $Kb + bK = 0$
  - Project with $\Theta^\pm$
  - Modified permutation relations

$$K^\pm a + aK^\pm = \pm P^{-1} a$$
$$K^\pm b + bK^\pm = \pm P^0 b$$

Deformed Oscillator Algebra: $N = 2$

- Mixed parts in algebra, e.g. $[a, b] = 2\nu R$ and $[a, b^\dagger] = 2\nu S$
  - Define relations between R, S and the oscillators

$$bK^+ - a^\dagger R + S^\dagger a = 0$$
$$aK^- + b^\dagger R + Sb = 0$$

- These relations are consistent with the Jacobi identities and give rise to the commutation relations

$$[J_1, a^\dagger] = a^\dagger + \nu a^\dagger P^{-1} \qquad [J_1, b^\dagger] = -\nu b^\dagger P^0$$
$$[J_2, b^\dagger] = b^\dagger - \nu b^\dagger P^0 \qquad [J_2, a^\dagger] = \nu a^\dagger P^{-1}$$

$$J_1 \equiv \tfrac{1}{2}\{a, a^\dagger\} = a^\dagger a + \frac{1}{2} + \nu K^+$$
$$J_2 \equiv \tfrac{1}{2}\{b, b^\dagger\} = b^\dagger b + \frac{1}{2} + \nu K^-$$

"Deformed" Cartan generators

Deformed Oscillator Algebra: $N = 2$

- Define Hamiltonian and angular momentum operators

$$H_{rel} \equiv J_1 + J_2 \qquad\qquad L_{rel} \equiv J_1 - J_2$$

$$[H_{rel}, a^\dagger] = a^\dagger + 2\nu a^\dagger P^{-1} \qquad\qquad [L_{rel}, a^\dagger] = a^\dagger$$

$$[H_{rel}, b^\dagger] = b^\dagger - 2\nu b^\dagger P^{0} \qquad\qquad [L_{rel}, b^\dagger] = -b^\dagger$$

- Fock vacuum: $a|0\rangle = 0 \qquad b|0\rangle = 0$
  $P^n|0\rangle = \delta_{n,0}|0\rangle \qquad K|0\rangle = |0\rangle$

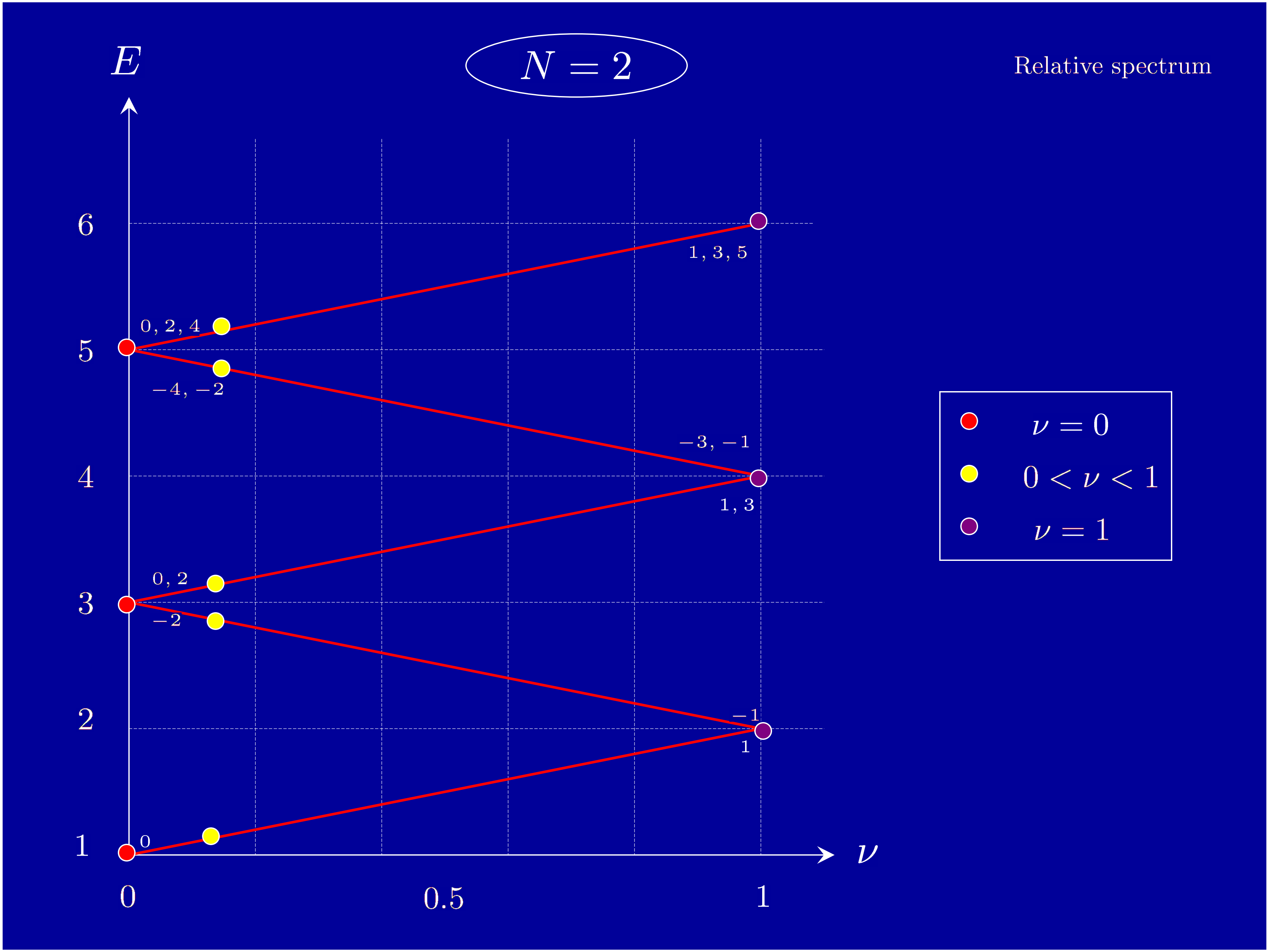
- A general state takes the form $|m,n\rangle = (a^\dagger)^m (b^\dagger)^n |0\rangle$ for which picture

$$H_{rel}|m,n\rangle = (m+n+2+\nu)|m,n\rangle \qquad m \geq n$$ above defect

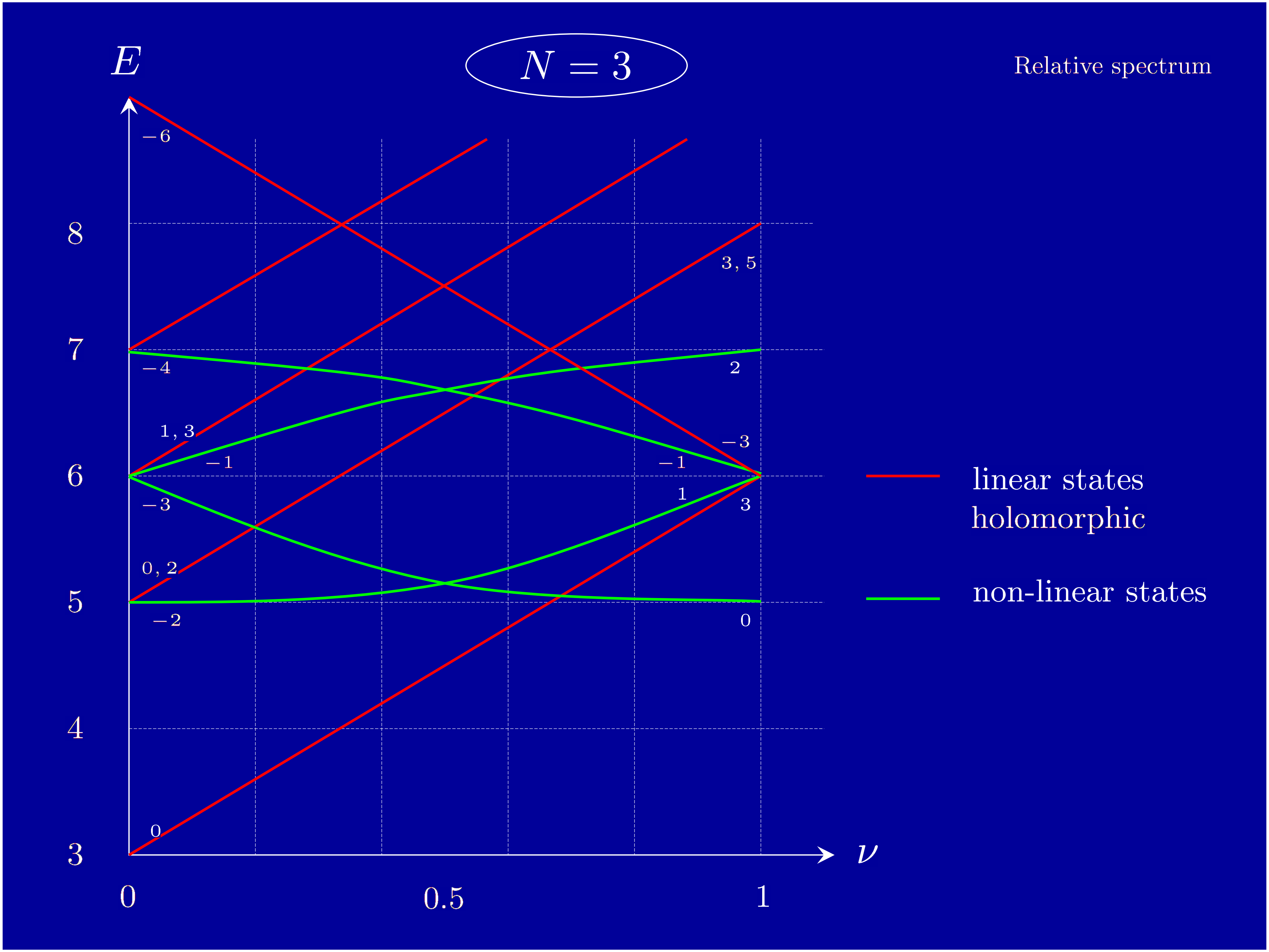
$$H_{rel}|m,n\rangle = (m+n+2-\nu)|m,n\rangle \qquad m < n$$ below defect

$$L_{rel}|m,n\rangle = (m-n+\nu)|m,n\rangle \qquad \forall\ m,n$$

- NB: The weight space is *completely connected* – it is ok to pass the defect



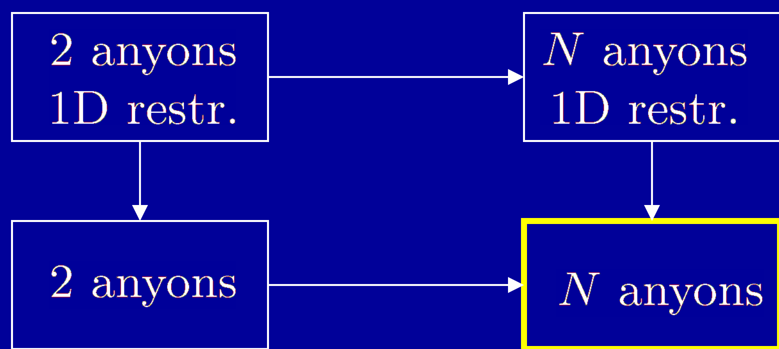
E
$N = 2$
Relative spectrum
6
5
4
3
2
1
$1, 3, 5$
$0, 2, 4$
$-4, -2$
$-3, -1$
$1, 3$
$0, 2$
-2
-1
1
0
0
0.5
1
ν
$\nu = 0$
$0 < \nu < 1$
$\nu = 1$



E
$N = 3$
Relative spectrum
8
7
6
5
4
3
0
0.5
1
ν
-6
$3, 5$
-4
2
$1, 3$
-3
-1
-1
1
-3
3
$0, 2$
-2
0
0
linear states
holomorphic
non-linear states

Deformed Oscillator Algebra: N Anyons

- Strategy:
  - Include c.o.m. motion
  - Generalize result valid for one oscillator
  - Generalize the two-anyon result



- Use notation

$$A_{\alpha i} = (A_\alpha, B_\alpha) \qquad (i = 1, 2)$$
$$A^i_\alpha = (A^\dagger_\alpha, B^\dagger_\alpha) \qquad (\alpha = 1, \ldots, N)$$

Deformed Oscillator Algebra: N Anyons

- The oscillator algebra involves $A_{\alpha i}$, A^i_α, $(R^j_i)_{\alpha\beta}$, $(R^{ij})_{\alpha\beta}$ and $(R_{ij})_{\alpha\beta}$

$$[A_{\alpha i}, A^j_\beta] = \delta_{\alpha\beta}\delta^j_i + \nu\delta_{\alpha\beta}\sum_{\gamma=1}^{N}(R^j_i)_{\alpha\gamma} - \nu(R^j_i)_{\alpha\beta}$$

$$[A_{\alpha i}, A_{\beta j}] = \nu\delta_{\alpha\beta}\sum_{\gamma=1}^{N}(R_{ij})_{\alpha\gamma} - \nu(R_{ij})_{\alpha\beta}$$

$$[A^i_\alpha, A^j_\beta] = \nu\delta_{\alpha\beta}\sum_{\gamma=1}^{N}(R^{ij})_{\alpha\gamma} - \nu(R^{ij})_{\alpha\beta}$$

Deformed Oscillator Algebra: N Anyons

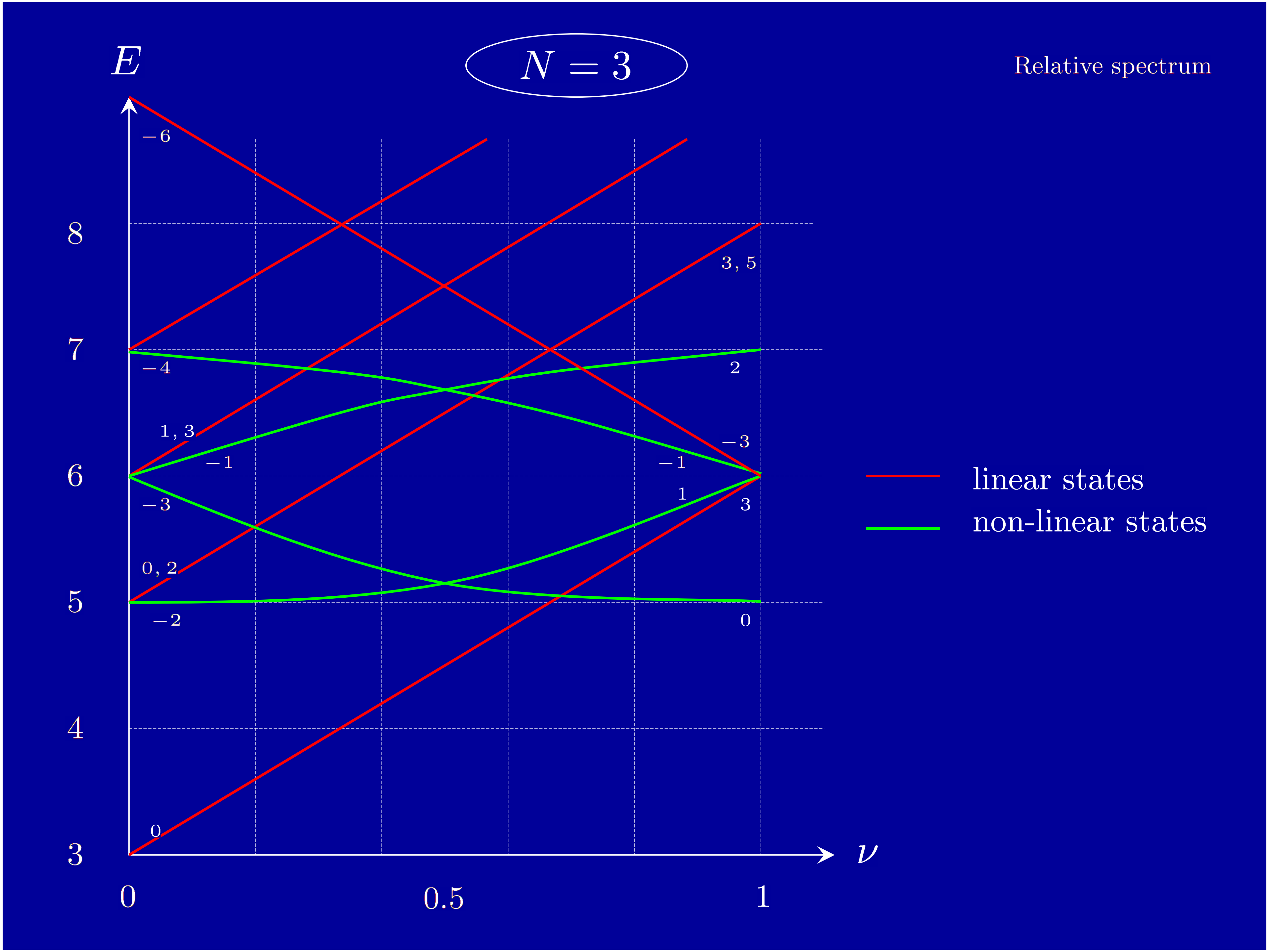
- Permutation relations/projector algebra similar as for $N = 2$
- Hamiltonian and angular momentum operators

$$[H, A^i_\alpha] = A^i_\alpha - \nu \sum_{\beta \neq \alpha} (A^i_\alpha - A^i_\beta) K_{\alpha\beta} P^{2-i}_{\alpha\beta}$$

$$[L, A^i_\alpha] = (-1)^{i+1} A^i_\alpha$$

Representation Theory for 3 Anyons

- Summary $N = 3$:
  - The linear ν dependence is captured by the model
    - ⋆ Spectrum for "linear" states is reproduced: $E_{num} = E_{here}$
    - ⋆ Spectrum for "non-linear" states: $E_{num} = E_{here} + \mathcal{O}(\nu^2)$ Chou, Hua et al
  - The angular momentum spectrum is exactly reproduced (ν)
  - All excitations are connected by the application of oscillators in a (nearly) standard fashion



$N = 3$
Relative spectrum
E
8
7
6
5
4
3
0
0.5
1
ν
-6
-4
$1, 3$
-1
-3
$0, 2$
-2
0
$3, 5$
2
-3
-1
1
3
0
linear states
non-linear states

Conclusions and Outlook

- We have written down an N-anyon algebra which generalizes the 1D one
  - We have explicitly checked the two- and three-anyon cases
  - Straightforward to check representation theory for N anyons
- Things for the future:
  - So far the approach has been purely algebraic
    - ⋆ make contact with the anyonic wave functions in the coord. repr.
  - Better understanding of the projector algebra
  - Include the non-linear dependence on ν in the energy spectrum
    - ⋆ Understand analytical structure of the non-linear wave functions
  - Work algebraically/analytically side-by-side to make progress!