Unified Description of Unitary and Nonunitary FQH States

Jack Polynomials, the Generalized Pauli Principle and Non-Abelian Statistics (or: Everything you always wanted to know about Jack but were afraid to ask)

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Colaboration with: F.D.M. Haldane

Princeton Physics

Nordita, Sweden, Aug 2008

Squeezed Polynomials/Jacks/FQH: Bernevig and Haldane, 2007, PRL

Beyond Jacks: Bernevig and Haldane, 2007,PRB

Nonunitary/Unitary/First Principle Propagators: Bernevig and Haldane, 2008 Submitted

Mathematics of Jacks : Jimbo, Miwa, et al, 2003

Multiple Condition Statistics:Thomale, Bernevig, Greiter, in preparation

Entanglement Entropy Jain VS Jack: Regnault, Bernevig, in preparation

Simplest Nonunitary State: Simon, Rezayi, et al., 2007, PRB

Thin torus/Jain hierarchy: Karlhede, Viefers, Hermanns, Ardonne, Hansson, Bergholtz, Wikberg, Kailasvuori, Haldane, Rezayi, 1990-2008

No Nonunitary FQH: N. Read (3 most recent papers)

Single Component Fractional Quantum Hall States

- Unified description of FQH ground states and excitations in terms of Jack polynomials
- Generalized Pauli principle: exclusion statistics and clustering
- States beyond the Read-Rezayi sequence – at filling k/r
- Quasiparticle (not quasi-hole) excitations
- Non-Abelian Hierarchy States - Revisiting Jain states
- Specific Heat, electron and quasi-hole propagators, a first principle study!
- Connection to Conformal Field Theory.
- Topological entanglement: Jain vs Jack (with Nicolas Regnault)

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Exact Quantization of the Even-Denominator Fractional Quantum Hall State at $\nu = 5/2$ Landau Level Filling Factor

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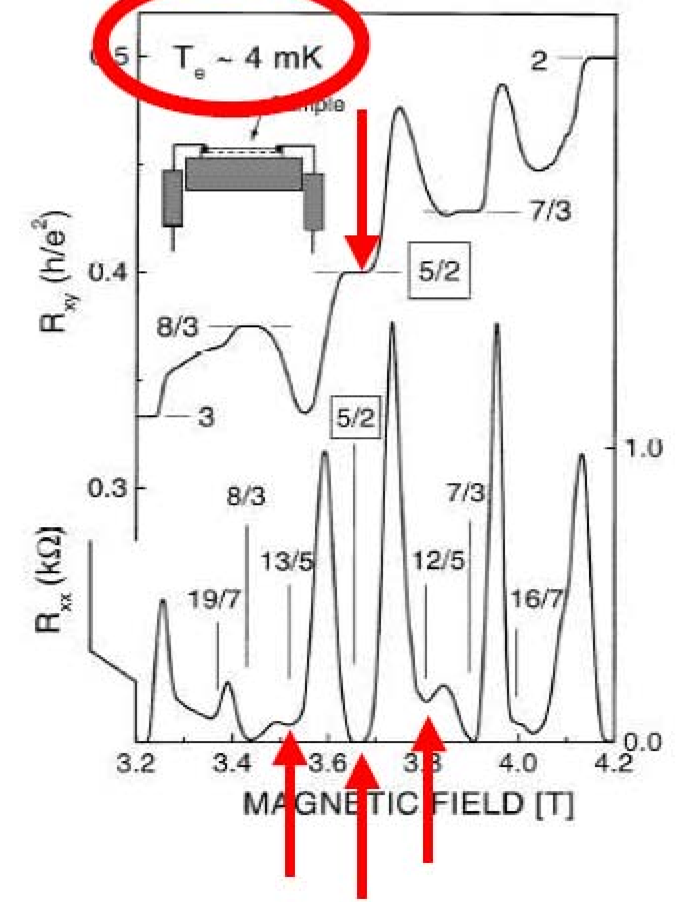
Mobility = 17 million cm^2/V sec

ν = 5/2 is a FQHE state

ν = 13/5 and 12/5

Look Like they might be forming



Model FQH States

Laughlin

$$\psi_L = \prod_{i<j}^{N} (z_i - z_j)^r$$

- Fundamental property: Zeroes of the wavefunction sit on the particles

•Unique quasihole excitations, but not unique quasiparticle excitations (Laughlin VS Jain VS Girvin)

$$\psi_{qp} = \prod_{i=1}^{N} \frac{\partial}{\partial z_i} \prod_{i<j}^{N} (z_i - z_j)^r;$$

$$\psi_{qp} = (z_i - z_j) \prod_{i=1}^{N} \frac{\partial}{\partial z_i} \prod_{i<j}^{N} (z_i - z_j)^{r-1};$$

Moore-Read

$$\psi_{MR} = Pf \left(\frac{1}{z_i - z_j} \right) \prod_{i<j} (z_i - z_j)$$

•Zeroes of the wf NOT on the particles; vanishes for 3 particles together.

•Quasiparticle excitations not known. No connection to the Laughlin state

•Physical properties obtained from CFT, not from wavefunction

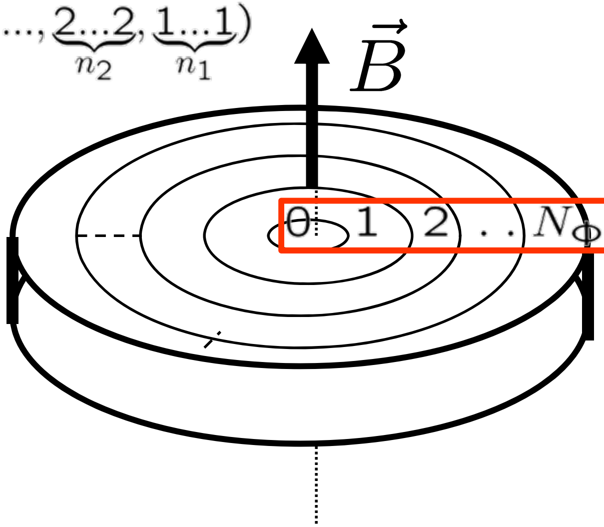
•FQH states beyond Laughlin, Read-Rezayi? Unified picture? Quasiparticles? Relation to the Jain states?

- **UNITARY VS NONUNITARY**

Free Boson Many Body Wavefunctions

- Boson analog of the Slater det. Orbital **occupation** basis $\overset{\boxed{0\quad 1\quad 2\ ...\ N_\Phi}}{[n_0, n_1, n_2 ... n_{N_\Phi}]}$

- Monomial basis; partition: $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N) = (\underbrace{N_\Phi ... N_\Phi}_{n_{N_\Phi}}, ..., \underbrace{2...2}_{n_2}, \underbrace{1...1}_{n_1})$



Orbital occupation $\longrightarrow$ Monomial basis

$\boxed{0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8...}$

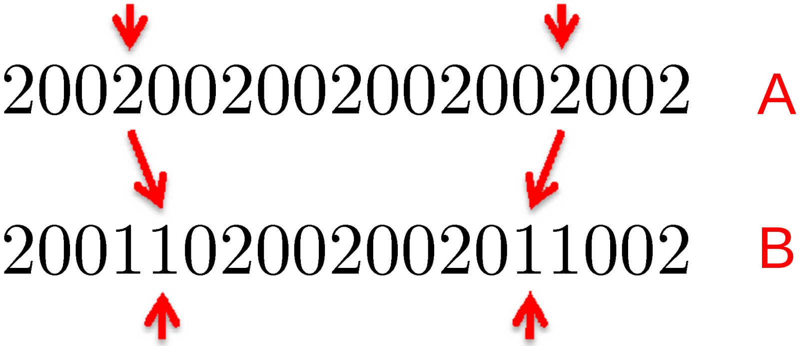
$[101010101...] \longrightarrow (...8, 6, 4, 2, 0)$

- Monomials (Permanents) = Det with all signs positive

$$m_\lambda(z_1, ..., z_N) = Per\left(z_i^{\lambda_j}\right) = Symm(z_1^{\lambda_1} \cdot ... \cdot z_N^{\lambda_N})$$

$$N = 3, \quad \lambda = (4, 2, 0) = [10101] \rightarrow m_{4,2,0} = Symm(z_1^4 z_2^2 z_3^0)$$

- Squeezing Rules in Orbital Space



$200200200200200200 2$ A

2001102002002011002 B

B Squeezed from A (A>B)

Laughlin and Moore-Read FQH States

- Annihilation operators on the Laughlin state $\psi_L = \prod_{i<j}^{N} (z_i - z_j)^r$

$$\boxed{D_i^{L,r} = \frac{\partial}{\partial z_i} - r \sum_{j(\neq i)}{}' \frac{1}{z_i - z_j}; \qquad D_i^{L,r}\psi_L = 0}$$

- Linear combination of the annihilation operators = Laplace Beltrami Operator

$$\sum_{i=1}^{N} z_i D_i^{L,1} z_i D_i^{L,r} = H_{LB}(\alpha_{1,r})$$

$$\alpha_{k,r} = -\tfrac{k+1}{r-1}$$

$$\boxed{\mathcal{H}_{LB}(\alpha) = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i}\right)^2 + \frac{1}{\alpha} \sum_{i<j}^{N} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j}\right)}$$

- Annihilation operators on the Pfaffian state $D_i^{MR\,Pf} \left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j) = 0$

- Laughlin and Moore-Read (also Read-Rezayi): eigenstates of the Laplace-Beltrami

- All single component CFT FQH states are eigenstates of the same operator!

Jack Polynomials (Jacks) $J_\lambda^\alpha(z_1, ..., z_N)$

Henry Jack, 1976

- Eigenstates of the Laplace Beltrami Operator are explicitly known

$$\mathcal{H}_{LB}(\alpha)J_\lambda^\alpha(z_1...z_N) = \epsilon_\lambda J_\lambda^\alpha(z_1...z_N)$$

α = Jack polynomial parameter

λ = monomial root occupation (partition) $=(\lambda_1, \lambda_2, ..., \lambda_N)$

- Decomposition of Jack polynomials in free boson many-body states known

$$J_\lambda^\alpha(z_1, ..., z_N) = m_\lambda(z_1, ..., z_N) + \sum_{\mu<\lambda} v_{\mu\lambda}(\alpha)m_\mu(z_1, ..., z_N)$$

μ squeezed from λ

Coefficients $v_{\mu\lambda}(\alpha)$ are known explicitly

- Jacks at $\alpha > 0$: 1D integrable at RG fixed point. Haldane Shastry, CS eigenst.
- Jacks at $\alpha < 0$ first studied in 2001! (Feigin et al math.QA/0112127)
- **N** particles: **N**-multiplet of operators, starting with the **angular momentum:**

$D_{N-1} = \sum_i z_i \frac{\partial}{\partial z_i}$ = angular momentum $\quad D_{N-2} = \mathcal{H}_{LB}$ = Laplace Beltrami

How the Jacks Look

$$J^{-2}_{10101}$$

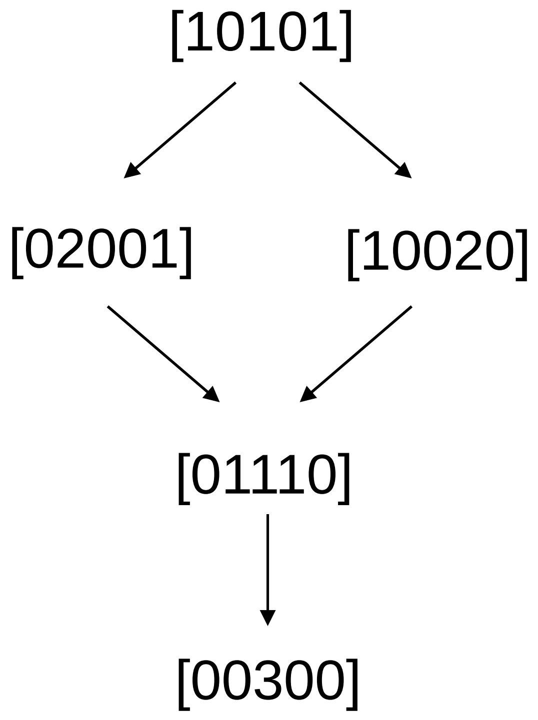
$$m_{4,\,2} - 2\,m_{3,\,3} - 2\,m_{4,\,1,\,1} + 2\,m_{3,\,2,\,1} - 6\,m_{2,\,2,\,2}$$

$$=$$

$$z_3^4 z_2^2 - 2\,z_3^4 z_2 z_1 + z_3^4 z_1^2 - 2\,z_3^3 z_2^3 + 2\,z_3^3 z_2^2 z_1 + 2\,z_3^3 z_2 z_1^2$$
$$- 2\,z_3^3 z_1^3 + z_3^2 z_2^4 + 2\,z_3^2 z_2^3 z_1 - 6\,z_3^2 z_2^2 z_1^2 + 2\,z_3^2 z_2 z_1^3 + z_3^2 z_1^4$$
$$- 2\,z_3 z_2^4 z_1 + 2\,z_3 z_2^3 z_1^2 + 2\,z_3 z_2^2 z_1^3 - 2\,z_3 z_2 z_1^4 + z_2^4 z_1^2 - 2\,z_2^3 z_1^3 + z_2^2 z_1^4$$

$$=$$

$$(z_1 - z_3)^2\,(z_2 - z_3)^2\,(-z_2 + z_1)^2$$



[10101]

[02001]

[10020]

[01110]

[00300]

The Jacks as FQH states

- Laughlin states are single Jack of root orbital occupation:

$$\nu = \frac{1}{2}: \ |[1010101...]\rangle \rightarrow J^{-2}_{1010101...}(z_1, ..., z_N)$$

- Moore-Read state is a single polynomial of root orbital occupation

$$\nu = 1: \ |[2020202...]\rangle \rightarrow J^{-3}_{2020202...}(z_1, ..., z_N)$$

- Read-Rezayi parafermion sequence Quantum Hall states are also Jacks:.

$$\nu = \frac{k}{2}: \ |[k0k0k0k...]\rangle \rightarrow J^{-(k+1)}_{k0k0k0k...}(z_1, ..., z_N)$$

- Density Wave states in the orbital basis**. But NOT Tao-Thouless!**

- Dominance relation (for r=2) numerically observed by Haldane (March Meering 2006 talk) now explained by identification of states with Jacks.

Generalized Pauli Principle: (k,r) statistics

- Model WF: Highest Weight (no quasiholes) and Lowest Weight (no quasiparticles)

- These **uniquely** define ALL good FQH Jacks :

$$\nu = \frac{k}{r}: \ |[k0^{r-1}k0^{r-1}k0^{r-1}...]\rangle \to J_{n^{v}(k,r)}^{-\frac{k+1}{r-1}}(z_1, ..., z_N)$$

$$n^{v}(k,r) = k0^{r-1}k0^{r-1}k0^{r-1}...0^{r-1}k$$

- r=2 is the Read-Rezayi Z_k sequence. Laughlin(k=1), Read-Moore(k=2)

The FQH ground states above are the maximum density states satisfying a **generalized Pauli principle of not more than k particles in r consecutive orbitals!**

- Quasihole excitations also satisfy the Pauli principle and are Jack polynomials

Torus Degeneracy of (k,r) Statistics

Topological Order = ground state degeneracy **on the torus** = how many ways we can put k particles in r boxes

Laughlin GS
(k,r)=(1,2)

$$|[1010101...1010]\rangle$$
$$|[01010101...101]\rangle$$

Pfaffian GS
(k,r)=(2,2)

$$|[2020202...2020]\rangle$$
$$|[11111111...111]\rangle$$
$$|[02020202...202]\rangle$$

Read-Rezayi
(k,r)=(k,2)

$$|[k0k0k0k...k0k0]\rangle$$
$$|[k-11k-11...k-11]\rangle$$
$$\vdots$$
$$|[1k-11k-1...1k-1]\rangle$$
$$|[0k0k0k0k...k0k]\rangle$$

Degeneracy of an SU(2) spin k/2
Spin chain

2/3 GS (k,r)=(2,3)

$$|[2002002...200200]\rangle$$
$$|[0200200...020020]\rangle$$
$$|[0020020...002002]\rangle$$
$$|[1101101...110110]\rangle$$
$$|[0110110...011011]\rangle$$
$$|[1011011...101101]\rangle$$

Simon, Rezayi, Cooper, 2007
Bernevig, Haldane, 2007

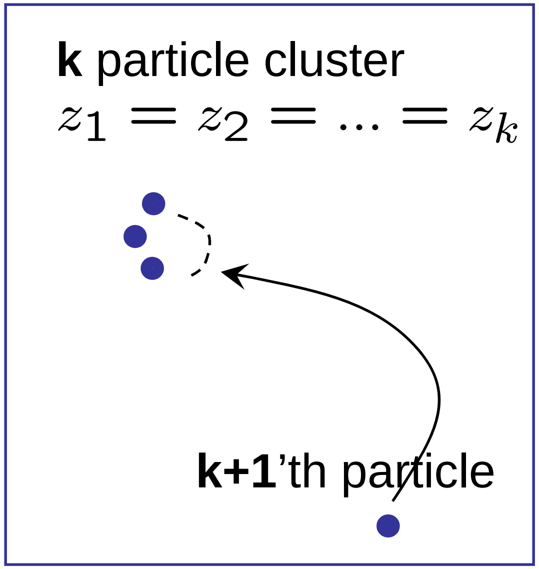
$$N_{GS} = \frac{(k+r-1)!}{k!(r-1)!}$$

Clustering Conditions

- Form a **k** particle cluster

- Bring the **k+1**’th particle close to the **k** particle cluster

- For the **(k,r)** sequence, the GS and quasihole Jack WF vanish as the **r**’th power of the difference



k particle cluster

$z_1 = z_2 = ... = z_k$

k+1’th particle

$$J^{-\frac{k+1}{r-1}}_{n(k,r)}(z_1 = ... = z_k, z_{k+1}, ..., z_N) \sim \prod_{i=k+1}^{N} (z_1 - z_i)^r$$

- Clustering number **k** AND vanishing power **r** are the fundamental properties

- **Feigin et al math.QA/0112127 showed the Jacks span the basis of polynomials that vanish when k+1 particles come together: complete basis**

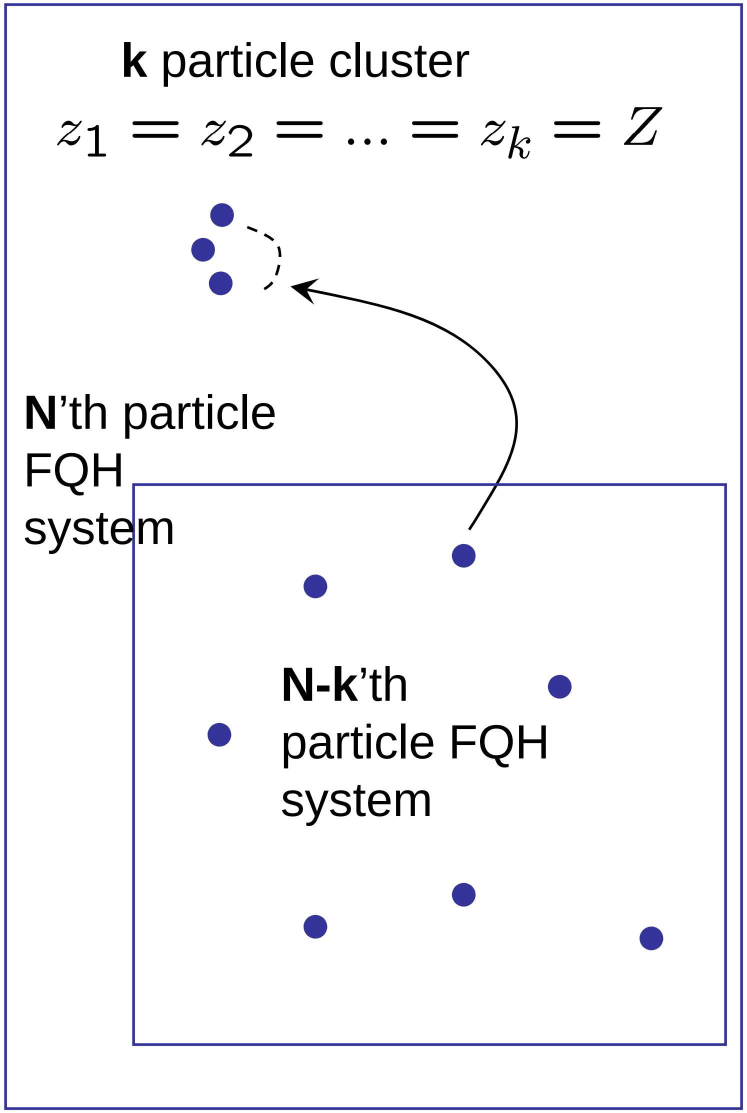
The Jacks and Clustering Conditions

- The (k,r) statistics ground state (maximal density) satisfies a remarkable entanglement property

$$\prod_{i=1}^{N}(Z-z_i)^r J_{n^v(k,r)}^{-\frac{k+1}{r-1}}(z_1...z_N) = J_{n^v(k,r)}^{-\frac{k+1}{r-1}}(z_1...z_N,\underbrace{Z...Z}_{k})$$

- As a corollary, every paired FQH ground-state is a Laughlin state in clustered coordinates

$$J_{n^v(k,r)}^{-\frac{k+1}{r-1}}(\underbrace{z_1,...,z_1}_{},\underbrace{z_2,...,z_2}_{},...) = \prod_{i<j}^{N/k}(z_i-z_j)^{kr}$$



k particle cluster

$$z_1 = z_2 = ... = z_k = Z$$

N'th particle FQH system

N-k'th particle FQH system

Excitations of (k,r) Statistics

- Maintain Pauli principle of (k,r) statistics (not more than k particles in r consecutive orbitals) but add fluxes (zeroes) on the sphere:

Laughlin GS
(k,r)=(1,2)

$|[10101]\rangle$

1-Quasihole
Multiplet
L=N/2

$|[010101]\rangle$
$|[100101]\rangle$
$|[101001]\rangle$
$|[101010]\rangle$

Pfaffian GS
(k,r)=(2,2)

$|[20202]\rangle$

Abelian
Quasiholes

$|[020202]\rangle$
$|[200202]\rangle$
$|[202002]\rangle$
$|[202020]\rangle$

Non-Abelian
Fractionalized
Quasihole

$|[110202]\rangle$
$|[111102]\rangle$
$|[111111]\rangle$
$|[201111]\rangle$
$|[202011]\rangle$

$|[201102]\rangle$

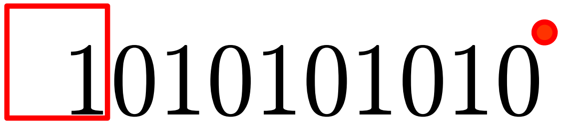
- For r=2 (Read-Rezayi sequence) this gives the counting of states of Conformal Field Theory

Unpinned Quasihole Hilbert Space

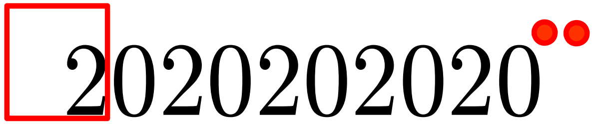
- Number of partitions satisfying (k,r) Pauli Principle

- For k>1, dimension space at 1 flux corresponds to angular momentum addition of more than 1 particle

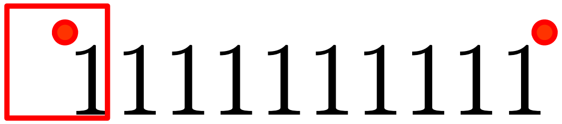
$$D_{1\ qh} = \left(\begin{array}{c} \frac{N}{k} + k \\ k \end{array} \right)$$



k=1 Laughlin Abelian quasihole south pole



k=2; two quasiholes at south pole



k=2; one fractionalized quasihole at north pole, another fractionalized quasihole at south pole

Pinned Quasiholes

- Coherent State superposition of un-pinned quasiholes (Jack polynomials)

$$\prod_i^N (z_i - z_A) \prod_{i<j}^N (z_i - z_j)^r = \sum_{i=1}^N z_A^i J_i(z_1, ..., z_N)$$

- k-1 fractionalized quasiholes at the origin, one at z_A . Example for k=2:

$$|0\rangle \rightarrow |0202...0202\rangle$$

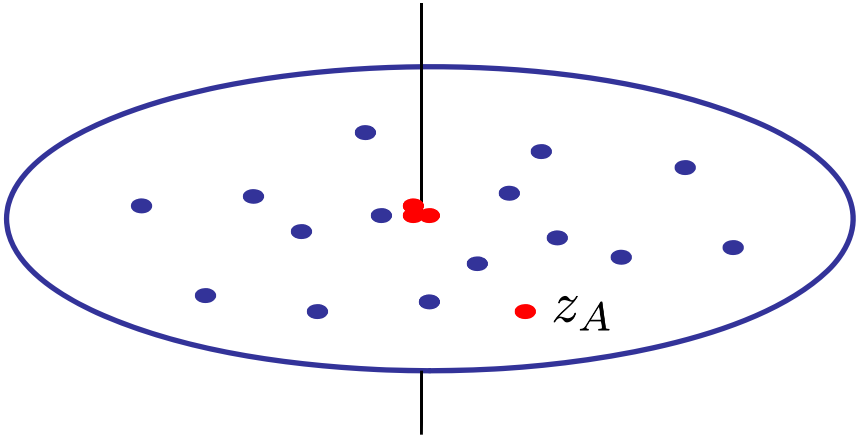
$$|1\rangle \rightarrow |1102...0202\rangle$$

$$|2\rangle \rightarrow |1111...0202\rangle$$

$$\vdots$$

$$|\tfrac{N}{2} - 1\rangle \rightarrow |1111...1102\rangle$$

$$|\tfrac{N}{2}\rangle \rightarrow |1111...1111\rangle$$



$$\boxed{\Psi(z_A, 0^{k-1}; z_1, ..., z_N) = \sum_{i=0}^{\frac{N}{k}} \frac{1}{k^i} z_A^i |i\rangle}$$

- Quantum dimension $d \geq 1$;dimension of pinned quasihole Hilbert space $\sim d^{kn}$

One Quasiparticle States (Abelian)

Quasiparticle States: $L^{+}\psi = \sum_{i=1}^{N} \frac{\partial}{\partial z_i}\psi = 0$

Start with Laughlin state: $|1010101...101\rangle$

Add 3 fluxes: $|0001010101...101\rangle$

Add 2 particles at north pole: $\boxed{|2001010101...101\rangle}$

From Jacks, **Generalized Clustering** properties satisfied by polynomials:

$$P(z_1, z_1, z_1, z_4, z_5, ...) = 0 \text{ as } l = 3$$

$$P(z_1, z_1, z_3, z_3, z_5, z_6 ...) = 0$$

The quasiparticle states **necessarily** break the (k,r) statistics of the parent state

One Quasiparticle States (Abelian)

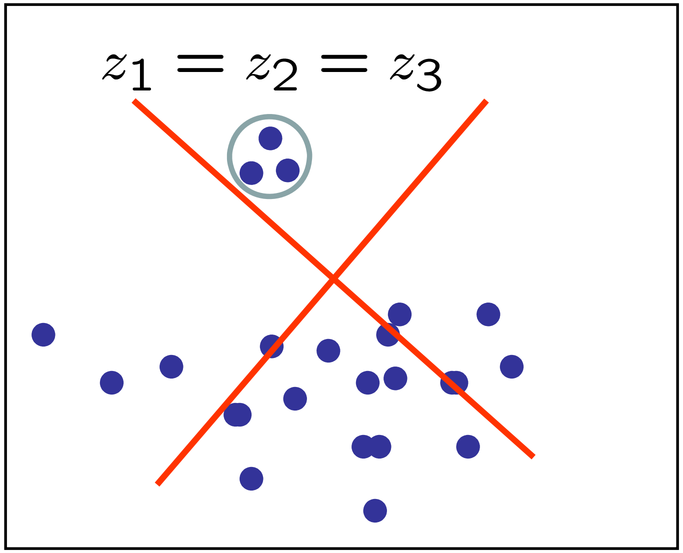
- Laughlin quasiparticle satisfies first clustering but not second

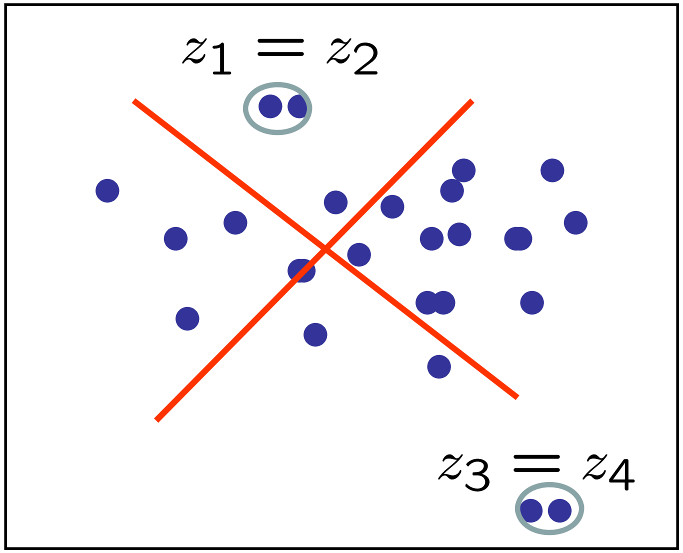
$$\psi_L = \prod_{i=1}^{N} \frac{\partial}{\partial z_i} \prod_{i<j}^{N} (z_i - z_j)^r;$$

- Our quasiparticle has more zeroes, due to generalized clustering

$$P^{1\ qp}_{Jain} = \begin{pmatrix} z_1^\star & z_2^\star & \dots & z_N^\star \\ 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ z_1^{N-2} & z_2^{N-2} & \dots & z_N^{N-2} \end{pmatrix}$$

$$P^{1qp}_{Jain} \prod_{i<j}^{N} (z_i - z_j)$$





Jack Hierarchy States

Bosonic state at $\nu = \frac{2}{3}$ (fermionic $\nu = \frac{2}{5}$) by dumping $\frac{N}{2}$ quasiparticles in Laughlin $\nu = \frac{1}{2}$ (fermionic $\nu = \frac{1}{3}$) state.

	Jack Quasiparticle	Jain Quasiparticle
1	$\vert 20010101010101\rangle$	$\vert 20010101010101\rangle$
2	$\vert 2002001010101\rangle$	$\vert 2010101010101\rangle$
3	$\vert 200200200101\rangle$	$\vert 201011010101\rangle$
4	$\vert 2002002002\rangle$	$\vert 2010110102\rangle$

Hierarchy leads to the Jack polynomial state from before: (k,r)=(2,3) (Simon, Rezayi, 2007)
(Bernevig, Haldane 2007)

$$\psi_{\nu=\frac{2}{5}} = J^{-\frac{3}{2}}_{2002002...2002}(z_1, \ldots, z_N) \cdot \prod_{i<j}(z_i - z_j)$$

New Wavefunctions for k/(k+1) Filling

- Fermionic states at Jain fillings are Vandermonde times r=k+1 Jacks:

- All of them are non-abelian, satisfy (k,k+1) Pauli Generalized Principle

$$\psi_{\nu=\frac{k}{2k+1}} = J^{-\frac{k+1}{k}}_{k0^k k0^k k\ldots k0^k k}(z_1,\ldots,z_N)\cdot\prod_{i<j}(z_i-z_j)$$

- Ground-states of k+1 body pseudopotentials with cluster angular mom < k+1

- Jain 2/3 state is excitation of the (k,r)=(2,3) (Gaffnian) state

- N=4 Particle Jack 2/3 state:

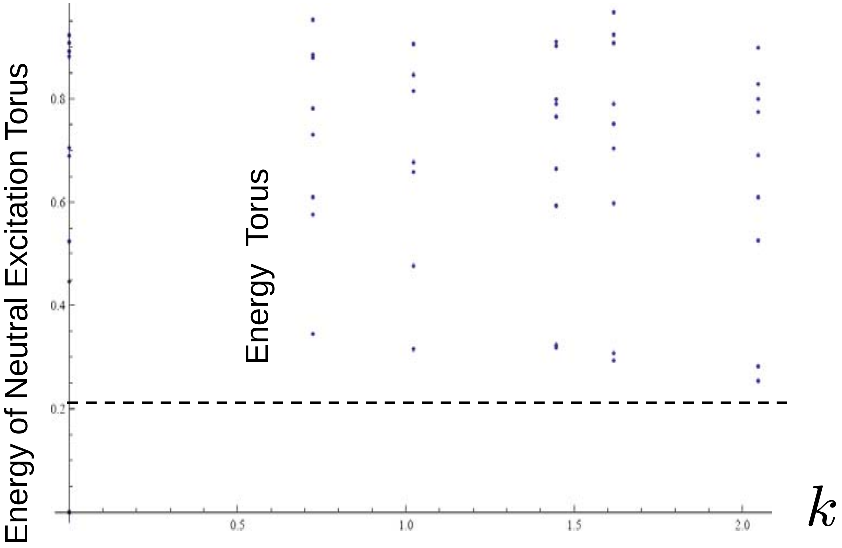
$$|2\;0\;0\;2\;0\;0\;2\;0\;0\;2\;0\;0\ldots.2\;0\;0\;2\;0\;0\;2\rangle$$

$$|2\;0\;1\;0\;1 \qquad 0\;1\;0\;2\rangle$$

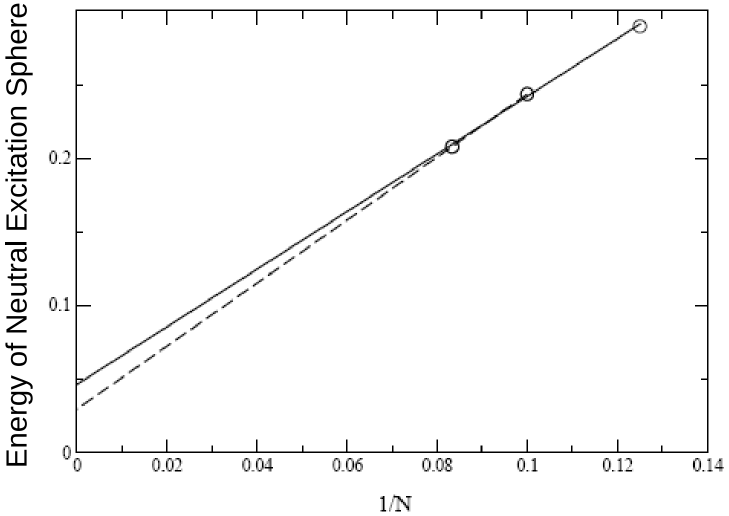
$$(2z_1z_2 - 2z_3z_4 - z_3z_2 - z_1z_3 - z_1z_4 - z_4z_2)\times$$
$$(2z_1z_3 + 2z_4z_2 - z_3z_2 - z_3z_4 - z_1z_4 - z_1z_2)\times$$
$$(2z_1z_4 + 2z_3z_2 - z_1z_3 - z_3z_4 - z_1z_2 - z_4z_2)$$

Numerics on the 2/3 state

- Overlaps >0.96 on sphere (Rezayi; Regnault) for N=12, 14



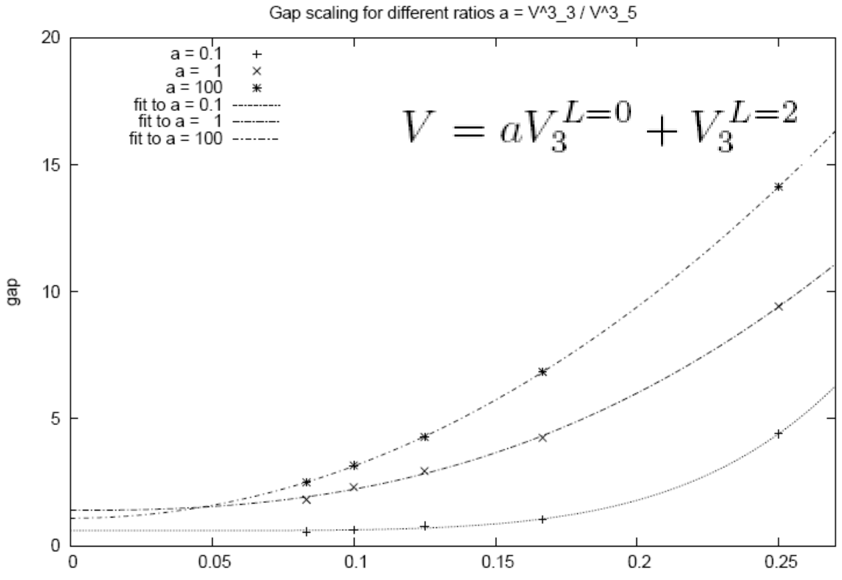
Simon, Rezayi, Cooper, Berdnikov, PRB 2007



Coulomb

N	6	8	10	12	14
$\mathcal{O}_{\nu=2/5}$	0.991 (3)	0.982 (4)	0.977 (5)	0.972 (5)	0.968 (6)
N	6	9	12	15	
$\mathcal{O}_{\nu=3/7}$	0.993 (3)	0.979 (4)	0.963 (5)	0.954 (7)	

N. Regnault[1], M. O. Goerbig[2] and Th. Jolicoeur[3]

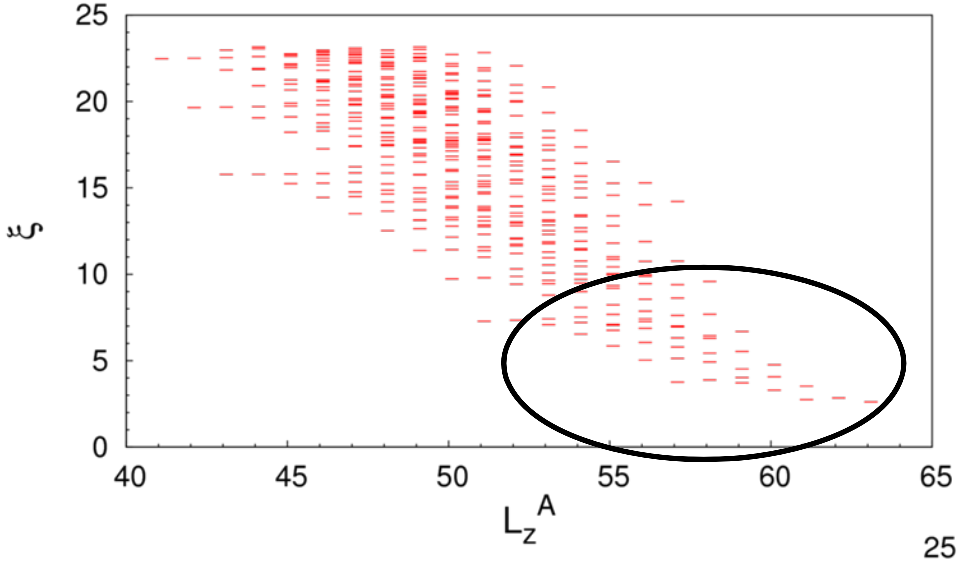


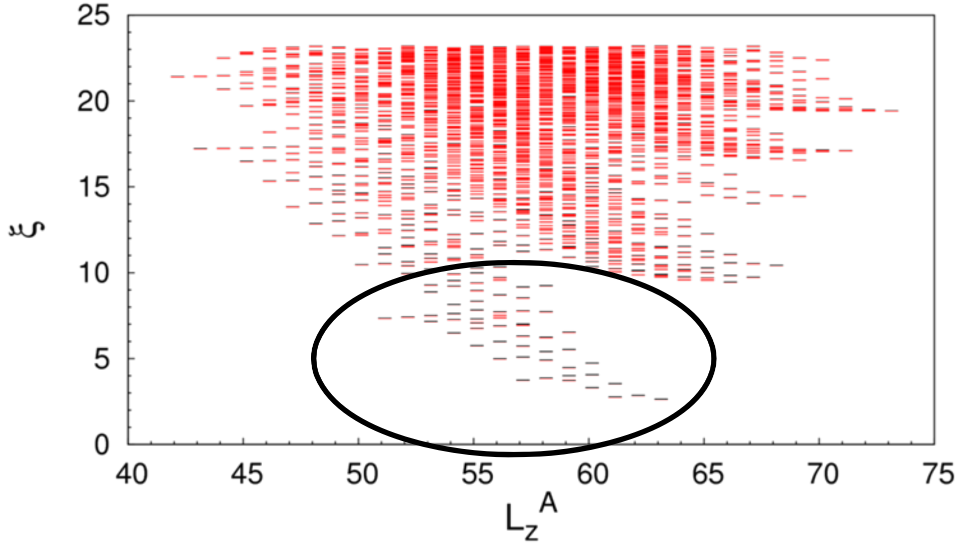
(With **Ronny Thomale** and M. Greiter)

Topological Entanglement

Hui and Haldane, 2008:

$$|\psi\rangle = \sum_{\alpha} e^{-\xi_\alpha/2} |\psi_{N_\alpha}\rangle \otimes |\psi_{S_\alpha}\rangle$$

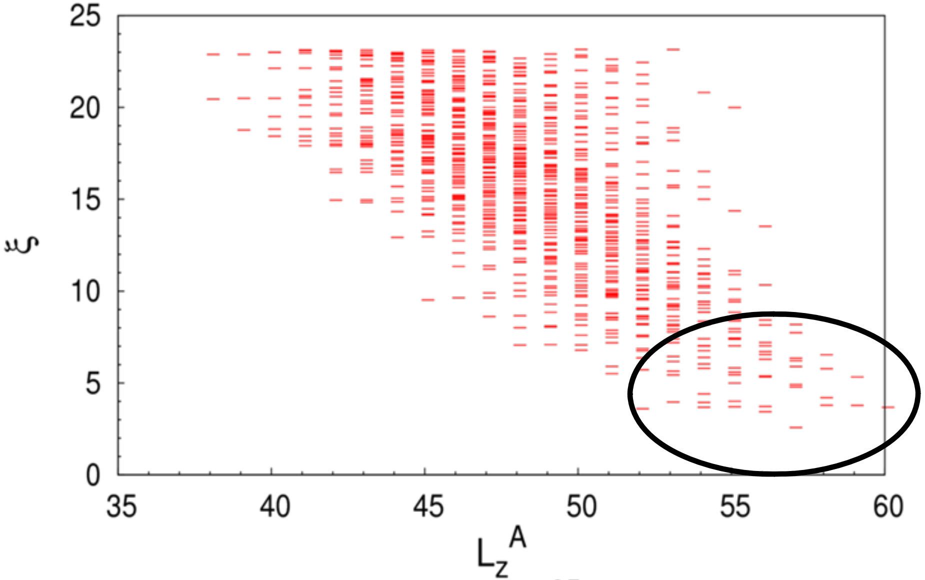


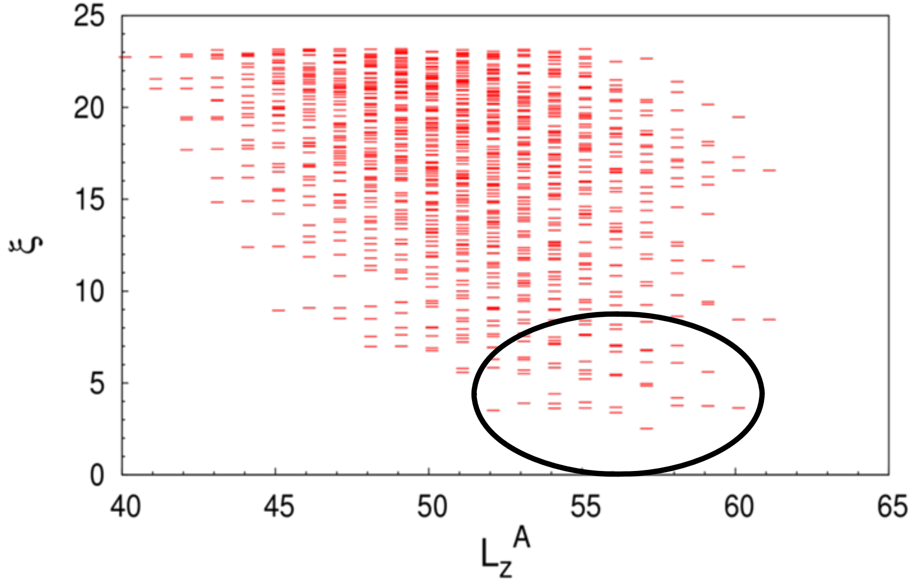


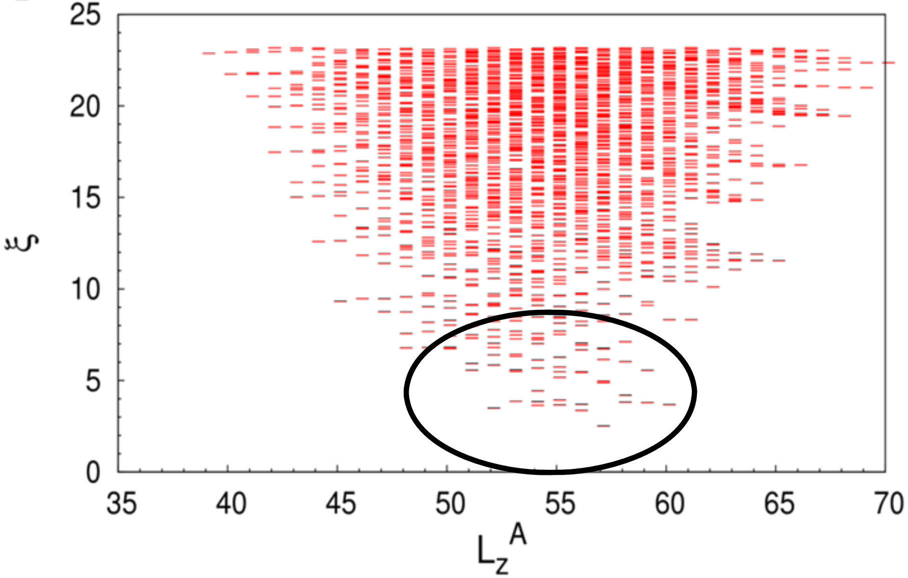
Topological Entanglement for 2/5

Collaboration with N. Regnault

$$|\psi\rangle = \sum_{\alpha} e^{-\xi_\alpha/2} |\psi_{N_\alpha}\rangle \otimes |\psi_{S_\alpha}\rangle$$







The Quantum Numbers of Topological Order in FQH States

Filling $\nu = \frac{k}{r}$

Degeneracy on the torus $D_{T^2} = \frac{(k+r-1)!}{k!(r-1)!}$

Quantum dimensions of primary fields: d

Hall Thermal coefficient $\equiv$ central charge

From (k,r) Generalized Pauli Principle, using

- counting of partitions

- existence of one-to one map to wavefunctions (Jack Polyn)

Particle propagator exponent g_e

Quasi-hole propagator exponent g_{qh}

Involves taking norms, hence scalar products

Conformal Field Theory Connection

- FQH states can empirically be written as CFT correlators:

$$\prod_{i<j}(z_i - z_j)^r = \langle \psi_e(z_1)...\psi_e(z_N)\rangle; \quad \psi_e(z_N) = e^{i\sqrt{r}\phi}$$

- The clustering conditions as well as the quantum dimensions point to (k,r) Jacks as correlation functions of $W_k(k+1, k+r)$ algebras (conjectured: Feigin, et al,2003; “proved” Bernevig, Haldane 2008).

- The W CFTs are unitary for A: k=1 and any r B: r=2 and any k

- All other W’s are non-unitary: really bad stuff happens (see papers by Read)

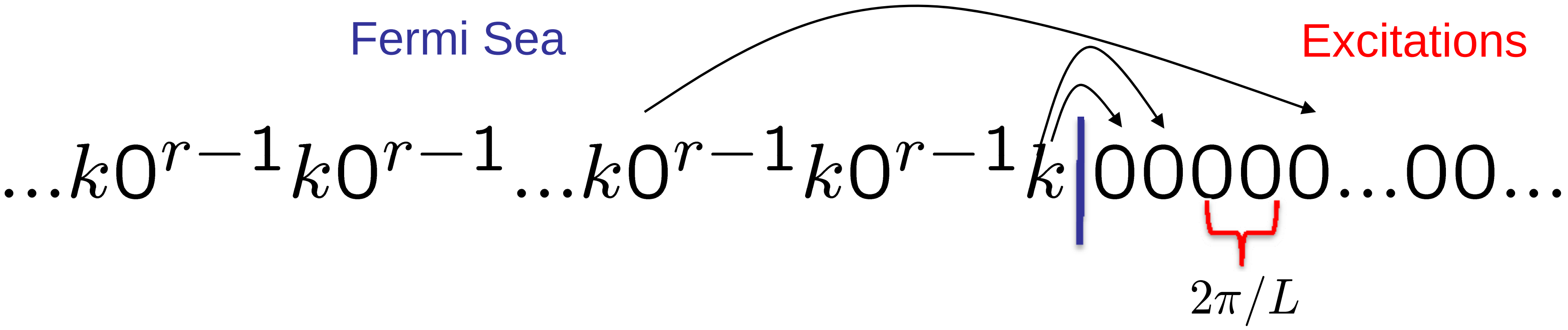
- Negative specific heat

- Negative scaling dimension: field correlators blow up at large distances

- Plasma in non-screening phase (conjecture)

Edge Thermal Hall Coefficient

- Compute entropy of our non-abelian k/r states: High Temperature expansion



$$\ldots k0^{r-1}k0^{r-1}\ldots k0^{r-1}k0^{r-1}k\,|\,0\underbrace{00}_{2\pi/L}00\ldots 00\ldots$$

$$Z = Tr\,(\exp(-\beta H)) = \textstyle\sum_{D=1}^{\infty} N_D q^D \qquad q = e^{-\frac{2\pi\beta v_F}{L}}$$

$$F = -Tln(Z) \qquad C = -T\frac{\partial^2 F}{\partial T^2} = \frac{\pi L T}{3 v_F} c$$

- c = central charge in CFT
- We computed N_D using the theory of partitions (ex Andrews book)

Edge Specific Heat and Quantum Dimensions

- The (k,r) Pauli Principle also gives the quantum dimensions d_i (Chebyshev Polyn)!

$$c = 1 + \sum_i L\left(\frac{1}{d_i^2}\right) \qquad L(z) = \sum_{i=1}^{\infty} \frac{z^n}{n^2} + \frac{1}{2} ln(z) ln(1-z)$$

- Using dilogarithm identities: (Kirilov, 1992, Nahm et al, 1992)

effective central charge of $W_k(k+1, k+r)$

$$c = 1 + \frac{r(k-1)}{r+k}$$

U(1) charge part

Non-abelian part >0

- Hence positive specific heat from the Pauli Principle of the FQH states, even though non-unitary CFT

- For 2/3 (or 2/5) Non-Abelian state:

$$c = 1 + \frac{3}{5} \qquad c_{Jain} = 1 + 1 \qquad d = 2\cos\left(\frac{\pi}{5}\right)$$

Golden Number,
Fibonacci Anyons

Central Charge

- Is a coefficient embedded deep in the polynomial **ground-state** wave-function

$$\psi(z_1)\psi(z_2) = \frac{1}{(z_1-z_2)^{2h}}\left(1+\frac{2h}{c}(z_1-z_2)^2 T(z_2) + O((z_1-z_2)^3)\right)$$

- Amazing fact: c is identical to the (physical) one obtained on the edge from counting **excitations**, but only for **unitary** theories. For non-unitary, they are different.

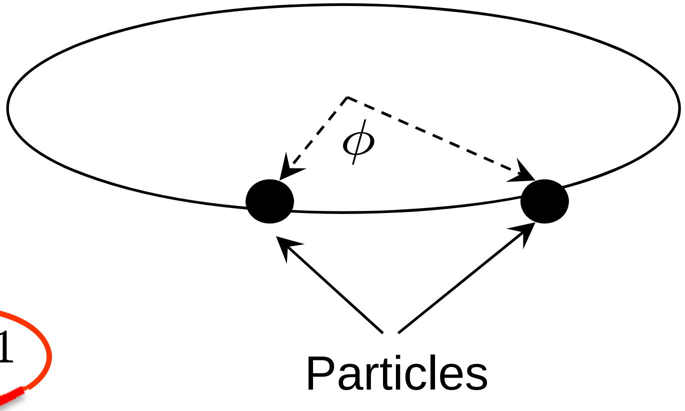
- For the Jacks, I obtain: $c=(k-1)\left(1-\frac{k(r-1)^2}{k+r}\right)$

- Same as c for W models. Means we now have both c and c_eff identical to W models, which means h_{min} also matches. c=c_{eff} for r=2

Particle Propagator

- For Laughlin States:

$$G(\phi) \sim \frac{1}{\left(\sin\left(\frac{\phi}{2}\right)\right)^{ge}}$$



$$n_M = \int_0^{2\pi} e^{-iM\phi} G(\phi) \sim (N_\phi - M)^{g_e - 1}$$

Luttinger Liquid Behavior

$$n_M \sim \frac{(N_\phi - M + g_e - 1)!}{(g_e - 1)!(N_\phi - M)!}$$

$$\boxed{g_e = r}$$

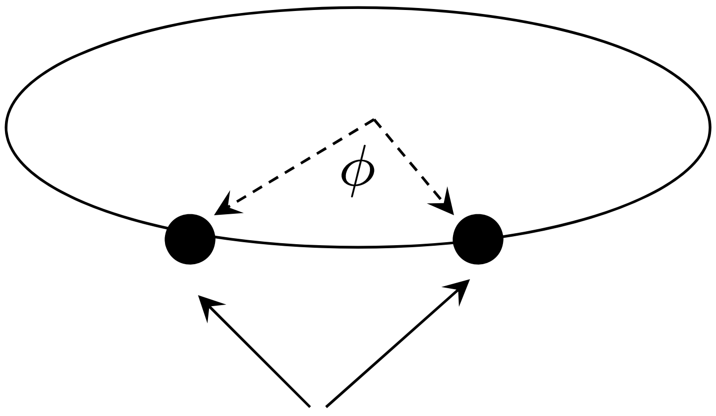
g_e	n_{N_ϕ}	$n_{N_\phi-1}$	$n_{N_\phi-2}$	$n_{N_\phi-3}$	$n_{N_\phi-4}$	$n_{N_\phi-5}$
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126

k	r	n_{N_ϕ}	$n_{N_\phi-1}$	$n_{N_\phi-2}$
1	2	1	2	2.88
2	2	1	2.21	3.4
4	3	1	3.07	6.24
3	4	1	3.93	8.43

Quasihole Propagators

- For Laughlin States:

$$G(\phi) = \frac{1}{(\sin(\phi/2))^{g_{qh}}}$$



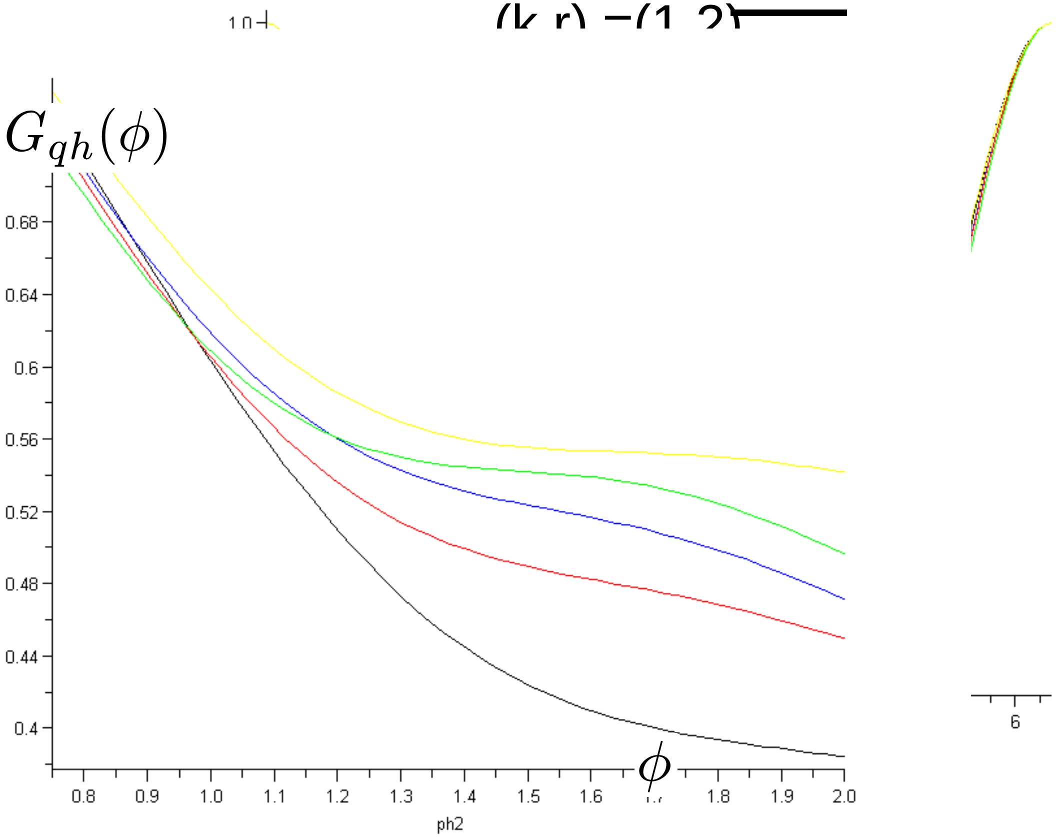
Quasiholes

- Remarkable fact: for FQH states described by unitary CFTs, the CFT and quantum mechanical scalar product give the same $g_{qh} > 0$. Very mysterious!!!

- If calculated using non-unitary CFT $g_{qh} < 0$; Using many-body WF $g_{qh} > 0$

- New Proposal: Non-Unitary CFT’s have “effective” Quantum Mechanical scaling dimensions $g_{qh} > 0$, just like the well-known “effective central charge”

Quasihole Propagators

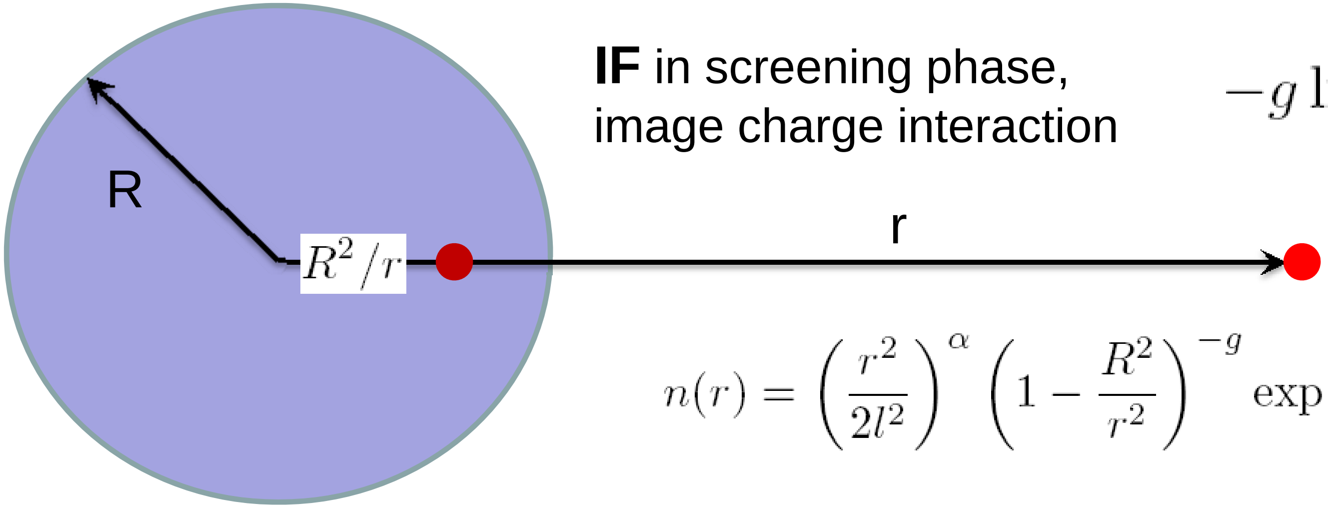


Comparing states at the same $\bar{N} = N/k$

The $\nu = \frac{2}{3}$ state quasihole exponent is bounded by Read-Rezayi states:

$$\frac{3}{8} > g_{qh} > \frac{3}{10}$$

Quasihole Propagators and Plasma Screening



IF in screening phase,
image charge interaction

$$-g \ln\left(r - R^2/r\right)$$

$$n(r) = \left(\frac{r^2}{2l^2}\right)^{\alpha} \left(1 - \frac{R^2}{r^2}\right)^{-g} \exp\left(-\frac{qr^2}{2l^2}\right)$$

For (k,r) sequence quasiholes, can prove exactly $\alpha = \frac{N}{k},\ g = g_{qh},\ q = 1/r$

g cannot be obtained exactly (yet) but: We know the expression of n(r) in terms of Jacks: $|j\rangle$

$$n_{qh}(r) \exp(qr^2/2l^2) = \sum_{j=0}^{N/k} \left(\frac{r}{k}\right)^{2j} \langle j|j\rangle$$

k=2, r=2 Moore-Read

k=3, r=2 Read-Rezayi

k=2, r=3 Gaffnian

$$\langle \frac{N}{k} - i | \frac{N}{k} - i \rangle = c \binom{q+i-1}{i} \left(\frac{R^{2i}}{k^{2i}}\right)$$

N=[illegible]: g=0.[illegible]; N=[illegible]: g=[illegible]; [illegible] Conjecture N=large: g=0.33333

CFT prediction [illegible] calculation

Non-Abelian Qp Qh State – Read Moore

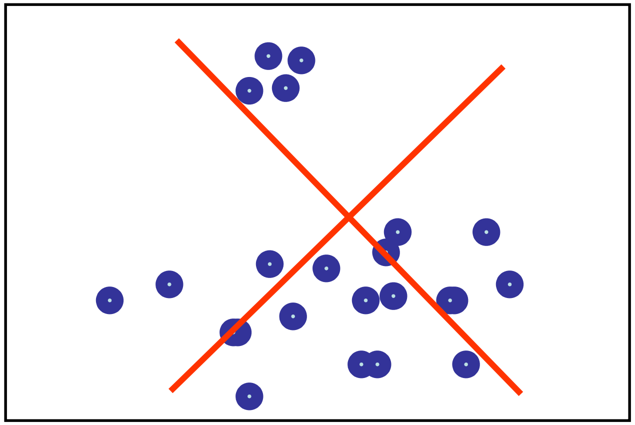
Start with Read-Moore state: $|2020202...202\rangle$

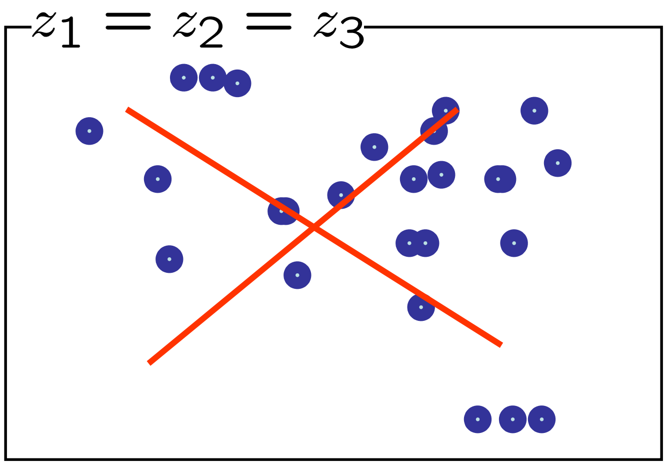
Add 1 particle from 2nd to 0th orbital: $|3010202...202\rangle$

Make a non-abelian string: $|3011111...111\rangle$

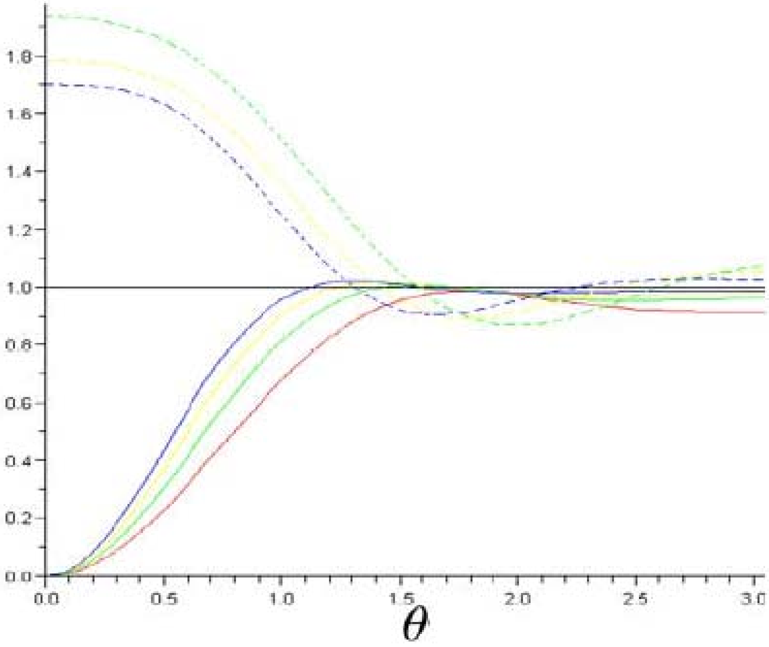
From Jacks, **Generalized Clustering** properties satisfied by polynomials:

$$P(z_1, z_1, z_1, z_1, z_5, z_6...) = 0 \text{ as } l = 5$$

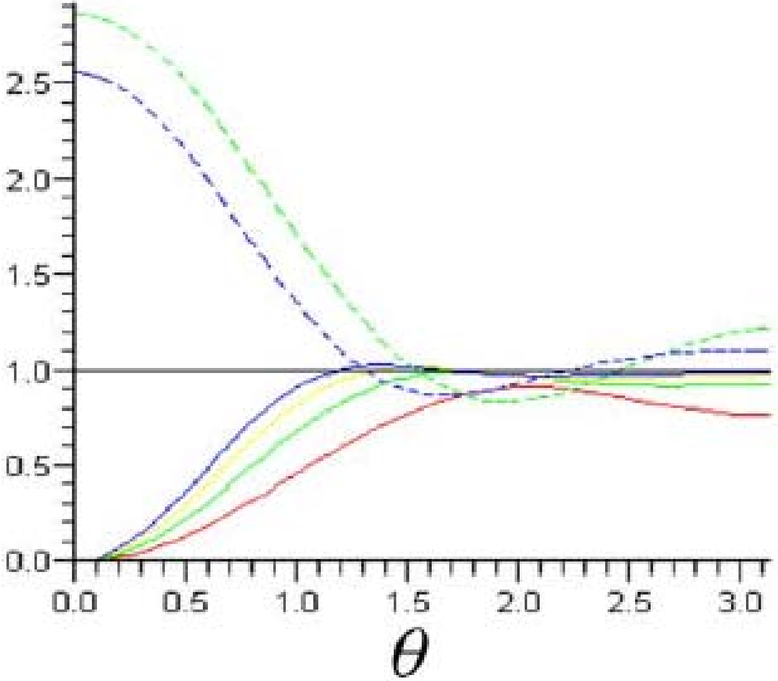




Exact Density Profiles



Laughlin qh and qp density profiles



Read-Moore qh and qp density profiles

Conclusions

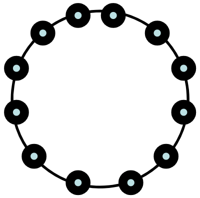
- Unified description of FQH states; explicit decomposition in monomials
- Generalized Pauli principle; clustering conditions
- Series beyond Read-Rezayi
- Quasiparticles
- New Hierarchy scheme leads to nonabelian LLL states, 2/5,3/7,…
- Specific heat, propagators
- Can non-unitary CFT’s describe FQH states

Integrable 1D models and Spin Chains

- Haldane Shastry (1D lattice)

$$z_j = \exp\left(\frac{2\pi i}{N} j\right)$$

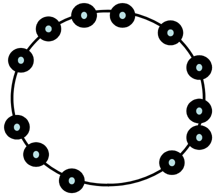
$$\mathcal{H} = \left(\frac{2\pi}{N}\right)^2 \sum_{i<j} \frac{\vec{S}_i \vec{S}_j}{|z_i - z_j|^2}$$

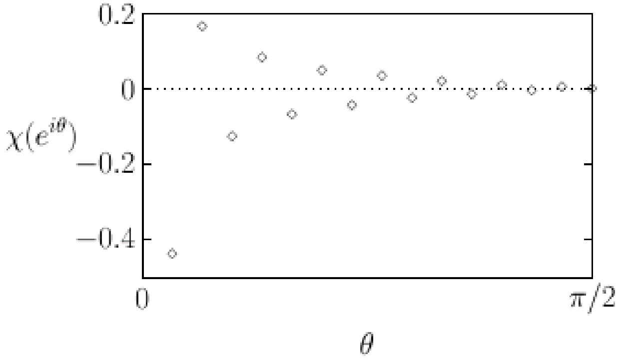


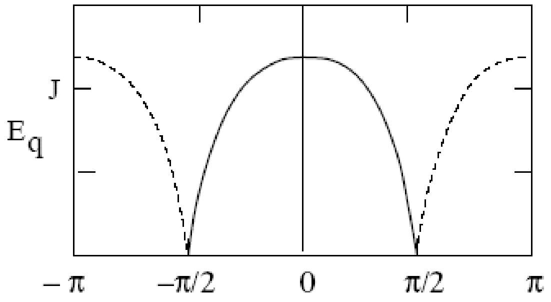
- Calogero Sutherland (1D continuum)

$$z_j = \exp(i\theta_j)$$

$$\mathcal{H} = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i}\right)^2 + \frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right) \sum_{i<j}^{N} \frac{1}{|z_i - z_j|^2}$$







Beyond Parafermions

- Model Fractional Quantum Hall states satisfy 2 conditions:

Highest Weight
(absence of quasiholes)

$$L^{+}\psi = \sum_{i=1}^{N} \frac{\partial}{\partial z_i}\psi = 0$$

Lowest Weight
(absence of quasiparticles)

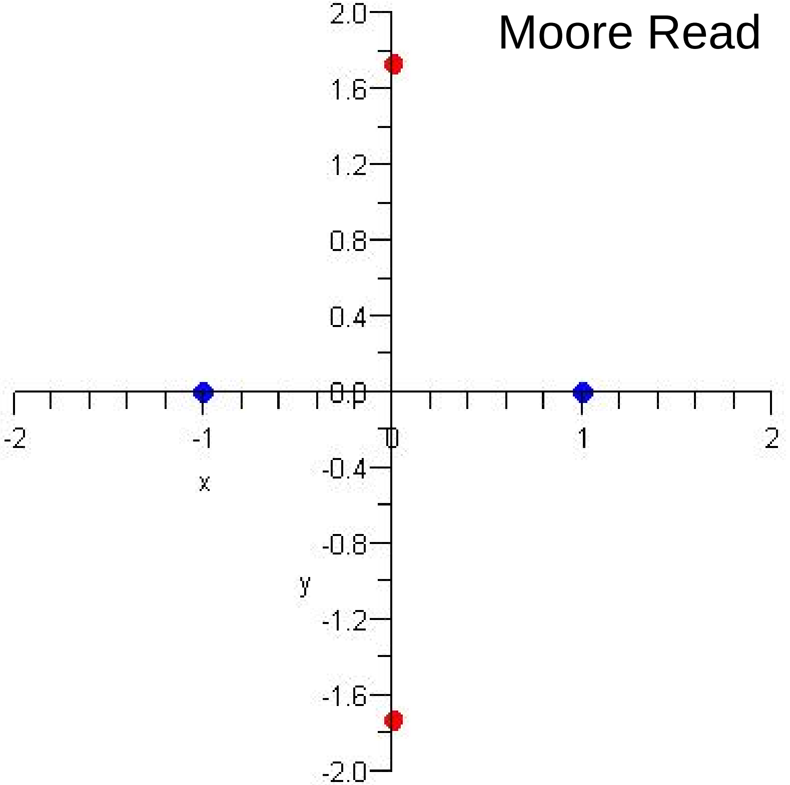
$$L^{-}\psi = \sum_{i=1}^{N} (N_{\Phi} z_i - z_i^2 \frac{\partial}{\partial z_i})\psi = 0$$

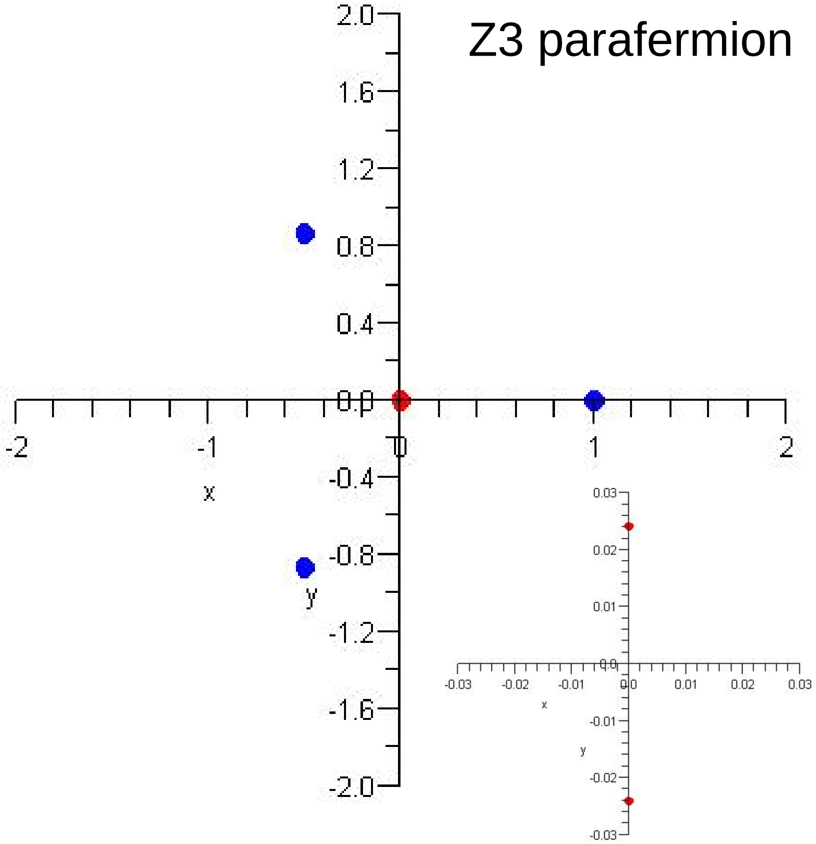
- Highest and Lowest Weight uniquely define ALL good FQH Jacks :

$$\nu = \tfrac{k}{r}\text{:}\ \ |[k0^{r-1}k0^{r-1}k0^{r-1}...]\rangle \rightarrow J^{-\frac{k+1}{r-1}}_{n^{v}(k,r)}(z_1, ..., z_N)$$

$$n^{v}(k,r) = k0^{r-1}k0^{r-1}k0^{r-1}...0^{r-1}k$$

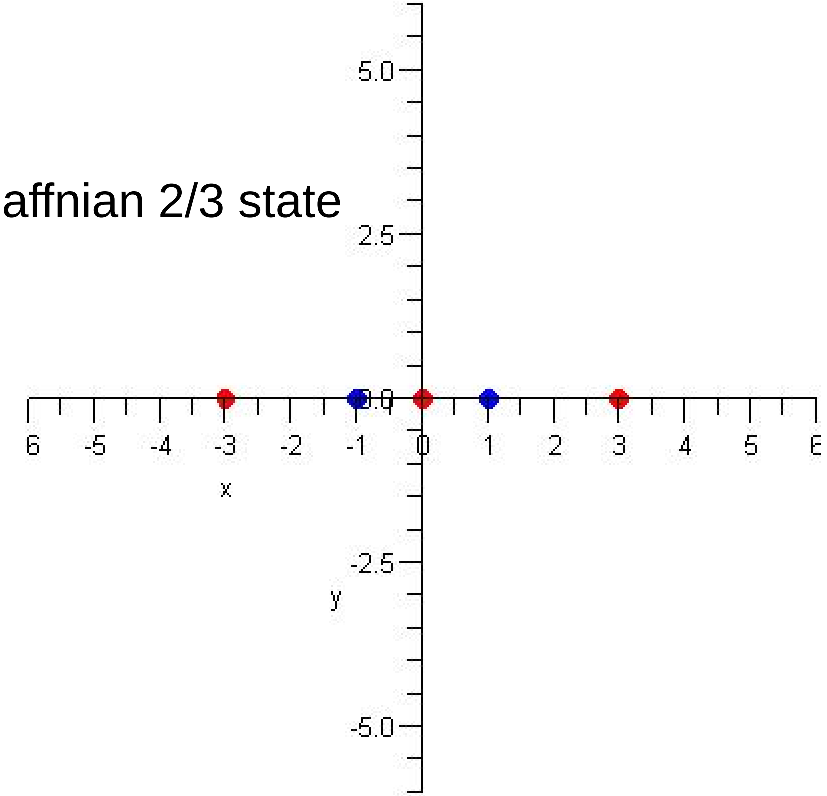
Zeroes of the FQH States





Zeroes of the FQH States

Gaffnian 2/3 state



Zero energy states of k-body potentials:

$$V_0^{k+1}, V_2^{k+1}, \ldots V_{r-1}^{k+1}$$

Abelian Quasiparticles– Read Moore

Start with Read-Moore state: $|2020202...202\rangle$

Add 3 fluxes: $|0002020202...202\rangle$

Add 4 particles at north pole: $|4002020202...202\rangle$

From Jacks, **Generalized Clustering** properties satisfied by polynomials:

$$P(z_1, z_1, z_1, z_1, z_1, z_2, z_3 ...) = 0 \text{ as } l = 3$$

$$P(z_1, z_1, z_1, z_2, z_2, z_2, z_3, z_4 ...) = 0$$

$$\begin{pmatrix} z_1^\star & z_2^\star & \cdots & z_N^\star \\ 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^{N-2} & z_2^{N-2} & \cdots & z_N^{N-2} \end{pmatrix} J^{-3}_{[20202...0202]}$$

Quasiparticle States Read-Rezayi

$$l = \frac{N}{k}; \quad |k+10k-11k-11...1k-11k-1\rangle;$$
$$l = \frac{N}{k}-1; \quad |k+10k-11k-11...1k-10k\rangle;$$
$$l = \frac{N}{k}-2; \quad |k+10k-11k-11...1k-10k0k\rangle;$$
$$l = 2; \quad |k+10k-10k0k...k0k0k\rangle;$$

$P(z_1, z_1, z_1...z_1, z_{k+3}, z_{k+3}...) = 0$ as $l = 3$

$P(z_1, z_1, ..., z_1, z_{k+2}, z_{k+2}, ..., z_{k+2}, z_{2k+3}, z_{2k+4}...) =$
0