Auxiliary material for the Letter: Collective States of Interacting Anyons, Edge States, and the Nucleation of Topological Liquids

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In this auxiliary note we describe in full detail the Hilbert space and the explicit form of the Hamiltonian of the anyonic spin-1 chains discussed in the Letter.

The constituent particles of the anyonic spin chains discussed in the main article are non-Abelian anyons. These can be described by so-called $su(2)_k$ theories, which themselves are 'quantum deformations' of $SU(2)$ [1]. The anyons or particles present in these theories are labeled by a generalized 'angular momenta', $j = 0, 1/2, 1, \cdots, k/2$, with the largest allowed angular momentum characterizing the particular $su(2)_k$ theory. In the limit $k \to \infty$ we recover SU(2) and its infinite number of representations. The analog of combining two ordinary spins, and reducing the tensor product, corresponds to the 'fusion' of two anyons which obey the fusion rules

$$j_1 \times j_2 = |j_1 - j_2| + (|j_1 - j_2| + 1) + \ldots + \min(j_1 + j_2, k - j_1 - j_2)\,, \tag{1}$$

where the fusion outcomes on the right hand side are consistent with respect to the 'cutoff' k present in the anyonic theories. For simplicity, we can actually restrict ourselves to the 'integer spin' subset of the full theory when k is odd, and hence only consider integer spins $j = 0, 1, 2, \ldots, (k-1)/2$. In particular, we note that this subset is a closed set under the fusion rules Eq. (1).

Since the construction of anyonic spin-1/2 chains has been discussed in great detail in an introductory paper [2], we will focus on anyonic spin-1 chains in the following.

THE HILBERT SPACE

The ayonic chains we consider are formed by a set of spin-1 anyons (with generalized angular momentum $j = 1$). The Hilbert space of these chains can be described in terms of a 'fusion chain' as depicted in Fig. 1. Here the 'incoming' particles (depicted

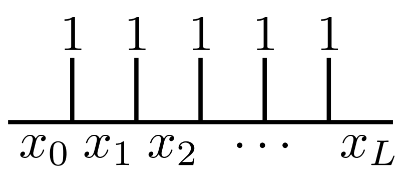


FIG. 1: The fusion chain of the anyonic spin-1 chains. The Hilbert space is spanned by all admissible labelings $|x_0, x_1, \ldots, x_{L-1}\rangle$ of this fusion chain.

at the top) are the spin-1 anyons constituting the chain. The elements of the Hilbert space describing this chain are then given by all admissible labelings $x_0, x_1, \ldots, x_{L-1}, x_L$ of the fusion chain with a length L. A labeling of the fusion chain is called admissible when at each trivalent vertex the fusion rules of Eq. (1) are satisfied, e.g. the label x_i can be generated in the fusion of x_{i-1} and 1. Throughout this note we will consider periodic boundary conditions, $x_L = x_0$.

To give an explicit example of such a Hilbert space, we will consider the case of $k = 5$, where using the integer subset the x_i can take the values $0, 1, 2$. The fusion rules will then read

$$\begin{array}{lll} 0 \times 0 = 0 & 0 \times 1 = 1 & 0 \times 2 = 2 \\ & 1 \times 1 = 0 + 1 + 2 & 1 \times 2 = 1 + 2 \\ & & 2 \times 2 = 0 + 1\,. \end{array}$$

Close inspection of these fusion rules will reveal certain local constraints for the admissible labelings $x_0, x_1, \ldots, x_{L-1}, x_L = x_0$. In this case of $k = 5$ we note that a 0 in an admissible labeling always has neighbors with label 1 on either side. The presence of such a local constraint is also the underlying reason that there exists no tensor product decomposition of this anyonic Hilbert space. To provide an explicit example, the Hilbert space of a periodic spin-1 chain of length $L = 4$ contains 26 states, which can be listed as

$\|0,1,0,1\rangle$	$\|0,1,1,1\rangle$	$\|0,1,2,1\rangle$	$\|1,0,1,0\rangle$	$\|1,0,1,1\rangle$	$\|1,0,1,2\rangle$
$\|1,1,0,1\rangle$	$\|1,1,1,0\rangle$	$\|1,1,1,1\rangle$	$\|1,1,1,2\rangle$	$\|1,1,2,1\rangle$	$\|1,1,2,2\rangle$
$\|1,2,1,0\rangle$	$\|1,2,1,1\rangle$	$\|1,2,1,2\rangle$	$\|1,2,2,1\rangle$	$\|1,2,2,2\rangle$	$\|2,1,0,1\rangle$
$\|2,1,1,1\rangle$	$\|2,1,1,2\rangle$	$\|2,1,2,1\rangle$	$\|2,1,2,2\rangle$	$\|2,2,1,1\rangle$	$\|2,2,1,2\rangle$
$\|2,2,2,1\rangle$	$\|2,2,2,2\rangle$				

THE HAMILTONIAN

As we discuss in the main text, the Heisenberg Hamiltonian of the anyonic systems closely resembles that of ordinary spin systems. Specifically, the Hamiltonian energetically splits the fusion outcomes in Eq. (1). However, in the fusion chain basis described above, the fusion of two neighboring anyons in the chain (depicted at the top in Fig. 1) is not explicit. In order to make the fusion of two such neighboring anyons explicit we need to perform a basis transformation, as depicted in Fig. 2. Similar to

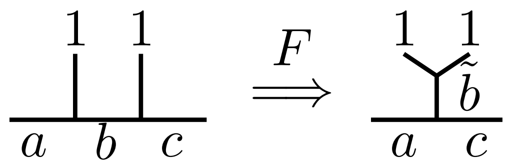


FIG. 2: The basis transformation needed in the construction of the Hamiltonian. The F-symbol carries the labels $\left(F_c^{a11}\right)_{\tilde{b}}^{b}$

a basis transformation for ordinary spins we can describe such a transformation by 'deformations' of the ordinary $6j$-symbols for $SU(2)$, which are commonly called F-matrices. For a more detailed introduction and the general computation of these F-matrices, we again refer to Ref. 2. Having performed such a basis transformation one can assign an energy according to the fusion outcome $\tilde{b}$, e.g. by applying the corresponding projector, and subsequently transform back to the original basis. For instance, the matrix elements of a Hamiltonian Π_i^1 assigning an energy $+1$ to the fusion channel $\tilde{b} = 1$ at site i (and an energy 0 to all the other fusion channels $\tilde{b}$) read

$$\langle a', b', c'| \, \Pi_i^1 \, |a, b, c\rangle = (F_c^{a,1,1})_{b'}^{1} (F_c^{a,1,1})_{1}^{b} \delta_{a,a'} \delta_{c,c'} \,, \tag{2}$$

where we use the notation $a = x_{i-1}$, $b = x_i$, $c = x_{i+1}$ to describe a consecutive triple of labelings $x_{i-1}x_ix_{i+1}$ in the fusion chain. Similarly, the matrix elements of a Hamiltonian Π_i^2 assigning an energy $+1$ to the fusion channel $\tilde{b} = 2$ at site i (and an energy 0 to all the other fusion channels $\tilde{b}$) read

$$\langle a', b', c'| \, \Pi_i^2 \, |a, b, c\rangle = (F_c^{a,1,1})_{b'}^{2} (F_c^{a,1,1})_{2}^{b} \delta_{a,a'} \delta_{c,c'} \,. \tag{3}$$

Here we made use of the fact that $F = F^{-1} = F^t$ in these cases.

To give an explicit expression of the two Hamiltonians Π_i^1 and Π_i^2 given above we again consider the case of $k = 5$. The possible labelings of the triples $|a, b, c\rangle$ then become

$$\begin{aligned} &\{|0,1,0\rangle\,, |0,1,1\rangle\,, |0,1,2\rangle\,, |1,1,0\rangle\,, |1,0,1\rangle\,, |1,1,1\rangle\,, |1,2,1\rangle\,, \\ &|1,1,2\rangle\,, |1,2,2\rangle\,, |2,1,0\rangle\,, |2,1,1\rangle\,, |2,2,1\rangle\,, |2,1,2\rangle\,, |2,2,2\rangle\}\,. \end{aligned} \tag{4}$$

Performing the basis transformation depicted in Fig. 2, the labelings of the triples $\left|a, \tilde{b}, c\right\rangle$ are then given by

$$\begin{aligned} &\{|0,0,0\rangle\,, |0,1,1\rangle\,, |0,2,2\rangle\,, |1,1,0\rangle\,, |1,0,1\rangle\,, |1,1,1\rangle\,, |1,2,1\rangle\,, \\ &|1,1,2\rangle\,, |1,2,2\rangle\,, |2,2,0\rangle\,, |2,1,1\rangle\,, |2,2,1\rangle\,, |2,0,2\rangle\,, |2,1,2\rangle\}\,, \end{aligned} \tag{5}$$

where the value of $\tilde{b}$ is again the fusion channel of the two neighboring spin-1 anyons in the chain. Concentrating on the Hamiltonians first, we will provide the explicit form for the F-matrices for $k = 5$ in the following section. Using those, we can now write down the Hamiltonians Π_i^1 and Π_i^2 in the basis of Eq. (4). The Hamiltonians Π_i^1 projecting onto the spin-1 channel becomes

$$\Pi_i^1 = \begin{pmatrix}
0 \\
& 1 \\
&& 0 \\
&&& 1 \\
&&&& 1/d_1 & -1/d_1^{5/2} & -d_2^{3/2}/d_1^2 \\
&&&& -1/d_1^{5/2} & 1/d_1^4 & d_2^{3/2}/d_1^{7/2} \\
&&&& -d_2^{3/2}/d_1^2 & d_2^{3/2}/d_1^{7/2} & d_2^3/d_1^3 \\
&&&&&&& d_2^2/d_1^2 & -d_2^{3/2}/d_1^2 \\
&&&&&&& -d_2^{3/2}/d_1^2 & d_2/d_1^2 \\
&&&&&&&&& 0 \\
&&&&&&&&&& d_2^2/d_1^2 & -d_2^{3/2}/d_1^2 \\
&&&&&&&&&& -d_2^{3/2}/d_1^2 & d_2/d_1^2 \\
&&&&&&&&&&&& 1/d_1 & 1/\sqrt{d_1 d_2} \\
&&&&&&&&&&&& 1/\sqrt{d_1 d_2} & 1/d_2
\end{pmatrix} \tag{6}$$

in terms of the quantum dimensions $d_1 = 1 + 2\cos(2\pi/7)$ and $d_2 = 2\cos(\pi/7)$.
The Hamiltonians Π_i^2 projecting onto the spin-2 fusion channel then reads

$$\Pi_i^2 = \begin{pmatrix}
0 \\
& 0 \\
&& 1 \\
&&& 0 \\
&&&& d_2/d_1^2 & d_2^2/d_1^{5/2} & d_2^{3/2}/d_1^3 \\
&&&& d_2^2/d_1^{5/2} & d_2^3/d_1^3 & d_2^{5/2}/d_1^{7/2} \\
&&&& d_2^{3/2}/d_1^3 & d_2^{5/2}/d_1^{7/2} & d_2^2/d_1^4 \\
&&&&&&& d_2/d_1^2 & d_2^{3/2}/d_1^2 \\
&&&&&&& d_2^{3/2}/d_1^2 & d_2^2/d_1^2 \\
&&&&&&&&& 1 \\
&&&&&&&&&& d_2/d_1^2 & d_2^{3/2}/d_1^2 \\
&&&&&&&&&& d_2^{3/2}/d_1^2 & d_2^2/d_1^2 \\
&&&&&&&&&&&& 0 \\
&&&&&&&&&&&&& 0
\end{pmatrix} . \tag{7}$$

Connection to the SU(2) bilinear-biquadratic spin-1 Hamiltonian

Finally, the most general Hamiltonian for the anyonic 'spin-1' chain can be written in terms of these projectors as

$$\mathcal{H}_{\nu=1} = \sum_i \cos\theta\, \Pi_i^2 - \sin\theta\, \Pi_i^1 \tag{8}$$

where an angle θ parameterizes the relative coupling strengths of the two projectors introduced above.

This Hamiltonian is closely related to the bilinear-biquadratic Hamiltonian commonly discussed in the context of ordinary SU(2) spin-1 chains

$$H_{\mathrm{bil-biq}} = \sum_i \cos(\theta_{\mathrm{b-b}})(\mathbf{S}_i \cdot \mathbf{S}_{i+1}) + \sin(\theta_{\mathrm{b-b}})(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 \,. \tag{9}$$

In particular, we can rewrite this SU(2) spin Hamiltonian in terms of projectors Π_i^1 and Π_i^2, where the projector Π_i^1 assigns an energy +1 if two spin-1's at sites i and $i+1$ combine (or 'fuse') into a total spin 1 and energy 0 otherwise. Similarly, the projector Π_i^2 assigns an energy +1 if two spin-1's at sites i and $i+1$ combine (or 'fuse') into a total spin 2 and energy 0 otherwise. Before we rewrite Hamiltonian (9) in terms of these projectors, we remind ourselves of the following relations

$$(\mathbf{S}_i \cdot \mathbf{S}_{i+1}) = \Pi_i^1 + 3\Pi_i^2 - 2 \qquad\qquad (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 = -3\Pi_i^1 - 3\Pi_i^2 + 4 \,,$$

and

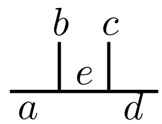
$$(\mathbf{S}_i + \mathbf{S}_{i+1})^2 = 2\Pi_i^1 + 6\Pi_i^2 \qquad\qquad (\mathbf{S}_i + \mathbf{S}_{i+1})^4 = 4\Pi_i^1 + 36\Pi_i^2 \; .$$

It then follows that we can write the relation between the two angles θ and $\theta_{\mathrm{b-b}}$ as

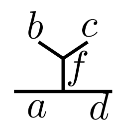
$$\tan\theta = \frac{\tan\theta_{\mathrm{b-b}} - 1/3}{1 - \tan\theta_{\mathrm{b-b}}} \qquad\qquad \tan\theta_{\mathrm{b-b}} = \frac{\tan\theta + 1/3}{1 + \tan\theta} \; . \tag{10}$$

F-MATRICES

Finally, we provide the explicit form of the F-matrices needed to perform a basis transformation of the general form



$$= \sum_f \left(F_d^{a,b,c}\right)_f^e$$



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Before detailing the explicit, albeit rather complex, expression for the F-matrices for arbitrary $su(2)_k$ theories [4], which could be used to construct Hamiltonians Eqs. (2) and (3) for arbitrary values of k, we again concentrate on the case $k = 5$. In doing so we will only state the 'outer labels' a, b, c, and d of the F-matrices. The 'inner labels' e and f follow from the fusion rules, and will always be ordered from low to high values. The number of allowed values for e (and consequently also f), then denotes the 'dimensionality' of the F-matrix. In the case of $k = 5$, we have one-, two- and three-dimensional F-matrices as listed below.

There are many symmetry relations, but for simplicity, we will state all matrices here. First, the one dimensional matrices are all equal to $+1$ and explicitly given by

$$\begin{aligned}
&F_0^{000} = F_1^{001} = F_2^{002} = F_1^{010} = F_0^{011} = F_1^{011} = F_2^{011} = F_1^{012} = F_2^{012} = \\
&F_2^{020} = F_1^{021} = F_2^{021} = F_0^{022} = F_1^{022} = F_1^{100} = F_0^{101} = F_1^{101} = F_2^{101} = \\
&F_1^{102} = F_2^{102} = F_0^{110} = F_1^{110} = F_2^{110} = F_0^{111} = F_0^{112} = F_1^{120} = F_2^{120} = \\
&F_0^{121} = F_0^{122} = F_2^{122} = F_2^{200} = F_1^{201} = F_2^{201} = F_0^{202} = F_1^{202} = F_1^{210} = \\
&F_2^{210} = F_0^{211} = F_0^{212} = F_2^{212} = F_0^{220} = F_1^{220} = F_0^{221} = F_2^{221} = F_1^{222} = 1
\end{aligned} \tag{11}$$

The two-dimensional matrices are

$$F_2^{111} = F_1^{112} = F_1^{121} = F_1^{211} = \begin{pmatrix} -d_2/d_1 & \sqrt{d_2}/d_1 \\ \sqrt{d_2}/d_1 & d_2/d_1 \end{pmatrix} \tag{12}$$

$$F_2^{112} = F_1^{122} = F_1^{221} = F_2^{211} = \begin{pmatrix} 1/\sqrt{d_2} & -1/\sqrt{d_1} \\ -1/\sqrt{d_1} & -1/\sqrt{d_2} \end{pmatrix} \tag{13}$$

$$F_2^{121} = F_1^{212} = \begin{pmatrix} 1/d_1 & -\sqrt{d_2/d_1} \\ -\sqrt{d_2/d_1} & -1/d_1 \end{pmatrix} \tag{14}$$

$$F_2^{222} = \begin{pmatrix} 1/d_2 & -\sqrt{d_1}/d_2 \\ -\sqrt{d_1}/d_2 & -1/d_2 \end{pmatrix} \; . \tag{15}$$

Finally, we have one three-dimensional F-matrix

$$F_1^{111} = \begin{pmatrix} 1/d_1 & -1/\sqrt{d_1} & \sqrt{d_2}/d_1 \\ -1/\sqrt{d_1} & 1/d_1^2 & (d_2/d_1)^{3/2} \\ \sqrt{d_2}/d_1 & (d_2/d_1)^{3/2} & d_2/d_1^2 \end{pmatrix} \; . \tag{16}$$

F-matrices for arbitrary $su(2)_k$ anyon theories

To be able to write an explicit expression for the F-matrices for arbitrary $su(2)_k$ theories, we will first introduce some notation. With $q = \exp\frac{2\pi i}{k+2}$ we define so-called q-numbers as $\lfloor n \rfloor = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$, and thus one has $\lfloor n \rfloor = \sum_{i=1}^{n} q^{\frac{n+1}{2} - i}$.

We also define the q-factorials as $\lfloor n\rfloor! = \lfloor n\rfloor\lfloor n-1\rfloor\cdots\lfloor 1\rfloor$, for integer $n>0$, and $\lfloor 0\rfloor! = 1$. The labels of the anyons $a, b, \ldots$ take the values $0, 1/2, 1, \ldots k/2$. The quantum dimensions of the particles are $d_j = \lfloor 2j+1\rfloor = \sin\left(\frac{(2j+1)\pi}{k+2}\right)/\sin\left(\frac{\pi}{k+2}\right) = d_{k/2-j}$.
In addition we define, for $a \leq b+c$, $b \leq a+c$, $c \leq a+b$ and $a+b+c = 0 \bmod 1$,

$$\Delta(a,b,c) = \sqrt{\frac{\lfloor a+b-c\rfloor!\lfloor a-b+c\rfloor!\lfloor -a+b+c\rfloor!}{\lfloor a+b+c+1\rfloor!}} \tag{17}$$

With these definitions, one way of writing the F-matrices is as follows [3]

$$\begin{aligned}\left(F_d^{abc}\right)_f^e = &(-1)^{a+b+c+d}\Delta(a,b,e)\Delta(c,d,e)\Delta(b,c,f)\Delta(a,d,f)\sqrt{\lfloor 2e+1\rfloor}\sqrt{\lfloor 2f+1\rfloor}\\ &\sum_n{}' \frac{(-1)^n\lfloor n+1\rfloor!}{\lfloor a+b+c+d-n\rfloor!\lfloor a+c+e+f-n\rfloor!\lfloor b+d+e+f-n\rfloor!}\\ &\times\frac{1}{\lfloor n-a-b-e\rfloor!\lfloor n-c-d-e\rfloor!\lfloor n-b-c-f\rfloor!\lfloor n-a-d-f\rfloor!}\ ,\end{aligned} \tag{18}$$

where the sum over n is over (non-negative) integers, such that

$$\max(a+b+e, c+d+e, b+c+f, a+d+f) \leq n \leq \min(a+b+c+d, a+c+e+f, b+d+e+f)\ ,$$

which guarantees that the arguments of the q-factorials are non-negative integers.

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[1] More precisely $U_q[SU(2)]$ with q a root of unity. See, e.g., C. Kassel, *Quantum Groups*, Springer-Verlag, New York (1995).

[2] S. Trebst, M. Troyer, Z. Wang, A.W.W. Ludwig, Prog. Theor. Phys. Suppl. **176**, 384 (2008); arXiv:0902.3275.

[3] The gauge we use is one in which those coefficients with the highest weight are positive in the limit $k\to\infty$. In particular, all the elements of the projectors onto the highest spin representation are non-negative. In addition, the one-dimensional F-symbols with only integer spin labels are all $+1$. This choice of gauge differs from the one used by Kirillov and Reshetikhin [4] by a factor of $(-1)^{(a+b+c+d)}$.

[4] A.N. Kirillov, N.Y. Reshetikhin, *Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links*, in V.G. Kac, ed., *Infinite dimensional Lie algebras and groups, Proceedings of the conference held at CIRM, Luminy, Marseille*, p. 285, World Scientific, Singapore (1988).