Collective States of Interacting Anyons, Edge States, and the Nucleation of Topological Liquids

Charlotte Gils,[1] Eddy Ardonne,[2] Simon Trebst,[3] Andreas W. W. Ludwig,[4] Matthias Troyer,[1] and Zhenghan Wang[3]

[1]*Theoretische Physik, Eidgenössische Technische Hochschule Zürich, 8093 Zürich, Switzerland*

[2]*Nordita, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden*

[3]*Microsoft Research, Station Q, University of California, Santa Barbara, California 93106, USA*

[4]*Physics Department, University of California, Santa Barbara, California 93106, USA*

(Received 20 October 2008; revised manuscript received 19 June 2009; published 10 August 2009)

Quantum mechanical systems, whose degrees of freedom are so-called $\mathrm{su}(2)_k$ anyons, form a bridge between ordinary SU(2) quantum magnets (of arbitrary spin-S) and systems of interacting non-Abelian anyons. Anyonic spin-$1/2$ chains exhibit a topological protection mechanism that stabilizes their gapless ground states and which vanishes only in the limit ($k \to \infty$) of the ordinary spin-$1/2$ Heisenberg chain. For anyonic spin-1 chains the phase diagram closely mirrors the one of the biquadratic SU(2) spin-1 chain. Our results describe, at the same time, nucleation of different 2D topological quantum fluids within a "parent" non-Abelian quantum Hall state, arising from a macroscopic occupation with localized, interacting anyons. The edge states between the "nucleated" and the parent liquids are neutral, and correspond precisely to the gapless modes of the anyonic chains.

DOI: 10.1103/PhysRevLett.103.070401 PACS numbers: 05.30.Pr, 03.65.Vf

SU(2) quantum spin degrees of freedom are ubiquitous in condensed matter physics describing the elementary quantum mechanical properties of many magnetic materials. For ordinary SU(2) spins there is—mathematically spoken—an infinite number of representations, or in other words arbitrarily large spins. Here we consider a "quantum deformation" [1] of SU(2), where we limit the number of representations to $k+1$ "angular momenta" which take values $j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$. The degrees of freedom of these so-called $\mathrm{su}(2)_k$ theories are examples of non-Abelian anyons with the Ising ($k=2$) and Fibonacci ($k=3$) anyons being studied in a variety of contexts such as unconventional $p_x + ip_y$ superconductors [2], fractional quantum Hall states [3], and proposals for topological quantum computation [4]. We will therefore refer to these generalized angular momenta also as anyon types. The analog of combining two ordinary spins, and reducing the tensor product, corresponds to the "fusion" of two anyons which obey the "fusion rules" $j_1 \times j_2 = |j_1 - j_2| + (|j_1 - j_2| + 1) + \ldots + \min(j_1 + j_2, k - j_1 - j_2)$. For example, fusing two anyons with generalized angular momenta $j_{1,2} = \frac{1}{2}$, these rules imply $\frac{1}{2} \times \frac{1}{2} = 0 + 1$ for $k \geq 2$, which in the limit $k \to \infty$ describes the coupling of two ordinary spin-$1/2$'s into a singlet or triplet.

Here we consider the collective ground state formed by a set of interacting anyons in the presence of an interaction that energetically splits the possible fusion outcomes—similar to the Heisenberg Hamiltonian for ordinary spins. In particular, we address the formation of this collective state in the context of the original topological liquid of which the anyons are excitations, such as a non-Abelian quantum Hall liquid. Summarizing our results we find that the occupation of a "parent" topological liquid with a finite density of interacting anyons leads to the nucleation of a *distinct*, gapped topological liquid within the original parent liquid. Specifically, we make an explicit connection between the *gapless* collective states of chains of interacting anyons and the *edge state* between these two liquids, which in turn allows us to fully characterize the nucleated liquid. This general correspondence also points to the possible occurrence of various unconventional quantum Hall states that arise from interactions between anyons with generalized angular momentum $j > 1/2$.

A first step towards understanding the collective ground states of interacting anyons has been taken by studying chains of interacting Fibonacci anyons [5–7]: Uniform chains with pairwise interactions that favor either the "singlet" or "triplet" channel are gapless and can be mapped exactly onto the tricritical Ising and critical 3-state Potts models, respectively [5]. In conventional systems these gapless theories would be completely unstable to the formation of a gap, but a more subtle mechanism is at play in the Fibonacci chain where an additional topological symmetry stabilizes the gaplessness of the system against local perturbations [5]. However, if we consider the more general anyonic systems described by $\mathrm{su}(2)_k$ theories and take the "undeformed" limit $k \to \infty$, we recover a system of ordinary SU(2) spins for which there is no such notion of a topological symmetry. This observation naturally raises the question of whether the observed topological protection is unique to the Fibonacci theory $\mathrm{su}(2)_3$. In this Letter, we show that the existence of a topological symmetry is a common feature in all $\mathrm{su}(2)_k$ anyonic theories and that it protects the gaplessness of generalized spin-$1/2$ chains in these theories for *all finite levels k*. In fact, it is the ordinary SU(2) spin-$1/2$ chain that stands out in this series as it looses this special symmetry and protection mechanism. We also discuss $\mathrm{su}(2)_k$ generalizations of the spin-1 chain, a similar topological symmetry protection there, and close connections between the phase diagram of the anyonic spin-1 chains and the biquadratic SU(2) spin-1 chain.

Anyonic spin-1/2 chains.—We first turn to $su(2)_k$ quantum deformations of ordinary quantum spin-1/2 chains which have been introduced in Ref. [5]. These "golden chains" consist of a linear arrangement of non-Abelian anyons with "angular momentum" $\nu = 1/2$ in the $su(2)_k$ theory. Pairwise interactions between adjacent anyons favor fusion into the singlet $j = 0$ for "antiferromagnetic" (AFM) exchange and into the triplet $j = 1$ for "ferromagnetic" (FM) exchange. These $su(2)_k$ spin-1/2 chains turn out to be gapless for all levels k [5]. The critical theory is a conformal field theory (CFT) closely related to the original $su(2)_k$ theory, namely, a particular coset theory [8]. We first concentrate on AFM couplings for which this CFT description is given by the $(k-1)$th unitary minimal model [9]. For $k = 2$ this is the Ising model, for $k = 3$ the tricritical Ising model, and in the limit $k \to \infty$ becomes the $c = 1$ theory describing the ordinary Heisenberg chain.

For all finite levels k, there exists an additional topological symmetry that defines $k + 1$ symmetry sectors. This symmetry corresponds to the operation of commuting a spin through all the spins in the chain. While this symmetry operation also exists in the limit $k \to \infty$ of the ordinary Heisenberg chain, we find that it plays a fundamental role only in the case of k being finite. To be explicit, we give the matrix elements of the topological symmetry operator Y in terms of the F matrices (which we define in the auxiliary material [10]):

$$\langle x'_1, \ldots, x'_L | Y | x_1, \ldots, x_L \rangle = \prod_{i=1}^{L} (F^{x'_{i+1}}_{\nu x_i \nu})^{x'_i}_{x_{i+1}}. \quad (1)$$

This topological symmetry becomes important for the anyonic spin chains as it protects their gapless states against instabilities arising from local perturbations. Since the operator Y commutes with the Hamiltonian, we can assign each perturbation to one of its symmetry sectors. Small, translationally invariant perturbations preserving this symmetry can only drive the critical system into a gapped phase if they correspond to an operator, relevant in the renormalization group sense, which is in the topologically trivial sector. Relevant operators in other sectors break the topological symmetry and are thus prohibited from opening a gap. The number of translationally invariant relevant operators grows as $k - 1$, while the number of topological sectors grows as $k + 1$. The question thus is whether these two diametrical effects result in a cancellation, as it is the case for $k = 3$, and lead to a topological protection for all $su(2)_k$ chains with $k > 3$.

We explain that this topological protection indeed exists for all finite levels k by observing a powerful connection between the coset theories describing the gapless state and the assignments of topological symmetry sectors to all operators in these theories: The primary operators in these coset theories carry (a pair of) $su(2)_k$ labels like those of the original anyonic degrees of freedom. This observation allows us to obtain topological sectors for all operators in the gapless theory and identify those operators which can drive the system into a gapped phase. In the limit $k \to \infty$ we recover the behavior of the ordinary spin-1/2 chain. We have checked, for $k = 2$, 3, 4, 5, that the so-obtained topological assignments agree with results from exact diagonalization of chains with up to $L = 24$ anyons (for $k = 5$) using Eq. (1).

In describing the details of the above topological symmetry assignments we will concentrate for brevity on the case that k is odd. In this case we can restrict ourselves to the "integer spin" representations $0, 1, \ldots, (k-1)/2$ of $su(2)_k$. The generalized spin-1/2 chains are then based on anyons carrying angular momentum $j = (k-1)/2$ [11]. In the gapless theories of the AFM chains, the (primary) operators $\phi^{j_1}_{j_2}$ carry two labels j_1, j_2. In the coset construction for these theories, i.e., $su(2)_{k-1} \times su(2)_1/su(2)_k$, the label j_1 corresponds to the representations of $su(2)_k$, the label j_2 to those of $su(2)_{k-1}$, and the label i_2 to those of $su(2)_1$, where $i_2 = j_1 - j_2 \bmod 1$. The label j_1 turns out to determine the topological sector of the operators. In particular, the operators in the topologically trivial sector turn out to be $\phi^0_{j_2}$. We consider the "character decomposition" $\chi^{(1)}_{i_2}\chi^{(k-1)}_{j_2} = \sum_{j_1} B^{j_1}_{j_2}\chi^{(k)}_{j_1}$, where $\chi^{(k)}_j$ denotes the ("affine") character of $su(2)_k$, and the $B^{j_1}_{j_2}$'s the (Virasoro) characters [12] of a unitary minimal model [8]. The Z_2 symmetry of these coset models also allows us to identify the sublattice symmetry of the primary operators. The states at $K = 0$ correspond to operators with integer j_2, while the states at $K = \pi$ correspond to operators with half-integer j_2. The scaling dimensions which result from this decomposition are those of the "Kac-table," $x(k, j_1, j_2) = 2\{1 + j_2(j_2+1)/(k+1) - j_1(j_1+1)/(k+2)\}$. It follows that the ground state at $K = 0$ is in the trivial topological sector. The lowest lying operator at $K = 0$ in this sector is ϕ^0_1, with scaling dimension $2(k+3)/(k+1)$. Thus, for any finite k, there is no relevant operator (with scaling dimension <2) which can drive the system into a gapped phase. In the limit $k \to \infty$ of the ordinary spin-1/2 chain, this operator becomes exactly marginal. At momentum $K = \pi$, the most relevant operator in the trivial sector is $\phi^0_{1/2}$, with scaling dimension $\frac{1}{2}(k+4)/(k+1)$, bounded by one for $k \geq 2$: staggered perturbations can always drive the system into a gapped "dimerized" phase. The results above hold for k even as well.

The critical behavior for FM interactions is described by the Z_k-parafermion coset $su(2)_k/u(1)$, whose operators also carry two labels, ψ^j_m. Again $j = 0, 1, \ldots, (k-1)/2$ (for k odd) originates from $su(2)_k$ and determines the topological sector as before, and $m = 0, 1, \ldots, k-1$ determines the momentum, $K = 2\pi m/k$. The operators in the trivial sector are the Z_k-parafermion operators, ψ^0_m, of scaling dimensions $2m(k-m)/k$. The only state whose corresponding scaling dimension is less than two at momentum $K = 0$ in the trivial sector is the ground state ψ^0_0, implying that for all k this critical phase is stable against small, translationally invariant perturbations.

Anyonic spin-1 chains.—We now turn to chains formed by a set of anyons with generalized angular momentum $\nu = 1$ in the $\mathrm{su}(2)_k$ theory for $k \geq 4$. There are three fusion channels of two $\nu = 1$ anyons, namely $1 \times 1 = 0 + 1 + 2$, so the most general Hamiltonian can be written in terms of an angle θ as $\mathcal{H}_{\nu=1} = \sum_i \cos\theta \Pi_i^2 - \sin\theta \Pi_i^1$, where the projectors Π_i^1 and Π_i^2 assign an energy $+1$ for anyons at sites i and $i + 1$ fusing into 1 or 2, respectively (see also [10]). We recover the ordinary biquadratic SU(2) spin-1 chain in the limit $k \to \infty$ which has a rich phase diagram [13–16] as shown in Fig. 1(a).

We have calculated the phase diagrams of anyonic spin-1 chains for the $\mathrm{su}(2)_5$ and $\mathrm{su}(2)_7$ theories using exact diagonalization, which allows us to generalize this phase diagram to all levels $k \geq 5$. As illustrated in Fig. 1(b), there are four phases that closely mirror their $k \to \infty$ counterparts: The Haldane phase survives as a gapped phase and includes a generalized *AKLT* point where the exact ground states are known [17]. It is surrounded by two gapless phases which have the same sublattice symmetries as their SU(2) counterparts, e.g., a Z_3 sublattice symmetry for the phase in the upper wedge which becomes the nematic phase, and a Z_2 sublattice symmetry for the phase in the lower wedge which turns into the dimerized phase. For arbitrary $k \geq 5$ the gapless phase in the upper wedge is the coset model $\mathrm{su}(2)_{k-4} \times \mathrm{su}(2)_4/\mathrm{su}(2)_k$ verified numerically by matching the low-energy spectra for $k = 5$ and 7. Similarly, our numerical results suggest that the gapless phase in the lower wedge is for arbitrary $k \geq 5$ the kth member of the family of the so-called off-diagonal modular invariants [18] of the unitary minimal models $\mathcal{M}_k$. The critical end points of this phase with the Haldane phase can be mapped to an integrable model [19] and correspond to the $N = 1$ supersymmetric minimal models with central charge $c = 3/2 - 12/[k(k + 2)]$. In the $k \to \infty$ limit this critical point turns into the $\mathrm{su}(2)_2$ Wess-Zumino-Witten point of the ordinary biquadratic spin-1 chain. All critical phases are protected by the topological symmetry (1).

Quantum Hall liquids.—Our program of exploring collective states of anyonic spin chains is, at the same time, a tool to systematically study topological phases which can occur inside non-Abelian quantum Hall liquids (at the same filling) due to population of such liquids by a macroscopic number of localized, interacting non-Abelian anyons. This also provides us with the properties of the (so far unexplored) edge states appearing at the interfaces between these two liquids. Let us focus for brevity on *bosonic* quantum Hall fluids [20]. Corresponding statements for *fermionic* states with the same non-Abelian statistics [3] involve only differences in (trivial) Abelian factors.

First reconsider the case of a linear arrangement of $j = 1/2$ anyons in a surrounding $\mathrm{su}(2)_k$ topological 2D fluid with AFM interactions, described by the minimal model $\mathcal{M}_k$. We can think of this critical state as two noninteracting counterpropagating (neutral) edge states of central charge $c_k = 1 - 6/(k + 1)(k + 2)$, which are basically located "on top of each other," residing in the surrounding $\mathrm{su}(2)_k$ topological bulk fluid. Let us now imagine separating these two edge states slightly in space, so that a narrow strip opens between them, see Fig. 2(b). We find that it is possible to place *another* topological quantum Hall fluid X into this narrow strip such that the above pair of counterpropagating edges of central charge c_k are precisely the edge states between the surrounding topological $\mathrm{su}(2)_k$ fluid and the new, intervening fluid X in the strip. The intervening liquid X is a topological fluid characterized by $\mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1$. This can be seen [21] from the coset representation [8] of the minimal model $\mathcal{M}_k = \mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1/\mathrm{su}(2)_k$. Recall that the central charges of $X = \mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1$ and the surrounding fluid $\mathrm{su}(2)_k$ differ by c_k, thus resulting in an edge state between the two liquids of central charge c_k [22]. To summarize, the (AFM) interactions between an array of $j = 1/2$ anyons in an $\mathrm{su}(2)_k$ liquid nucleate a new intervening liquid characterized by the topological properties of $\mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1$ [23].

Even though our anyons were initially confined to one dimension, these results will also apply to a macroscopic number of interacting non-Abelian anyons occupying two-dimensional (2D) regions of the surrounding liquid, thereby nucleating larger, 2D regions of the intervening liquid. The simplest example of this phenomenon was observed [24] for the Moore-Read quantum Hall liquid, which, when occupied with a macroscopic number of interacting quasiholes (even at random positions), turns into the Abelian "strong-pairing" state. In general, for FM anyon interactions the liquid X is Abelian and the neutral edge CFT is the Z_k parafermion theory.

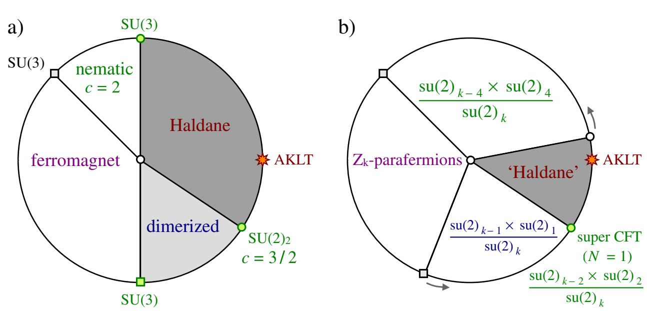

FIG. 1 (color online). Phase diagrams of the "biquadratic" spin-1 chain for (a) SU(2) and (b) $\mathrm{su}(2)_k$ with $k \geq 5$. Projections onto the triplet ($j = 1$) and quintuplet ($j = 2$) states are parametrized by an angle θ as $J_1 = -\sin(\theta)$ and $J_2 = \cos(\theta)$.

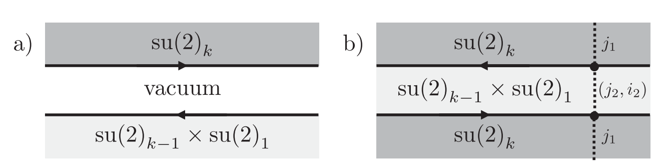

FIG. 2. Topological liquids and their edge states.

Even more interesting examples arise when we express our results for the anyonic spin-1 chains in terms of intervening liquids. A family of novel liquids with topological character [25] $\mathrm{su}(2)_{k-4} \times \mathrm{su}(2)_4$ nucleated within a $\mathrm{su}(2)_k$ surrounding liquid corresponds to the family of critical phases in the upper wedge of Fig. 1(b) with corresponding edge states described by the coset $\mathrm{su}(2)_{k-4} \times \mathrm{su}(2)_4/\mathrm{su}(2)_k$. Interestingly, the critical phases in the lower wedge of Fig. 1(b) describe the appearance of edge states between the $\mathrm{su}(2)_k$ bulk fluid and a novel topological quantum liquid of type $\mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1$, characterized by the *off-diagonal* modular invariant [18] of the minimal model $\mathcal{M}_k$. In all cases, the new resulting liquid has *less* anyon types than its $\mathrm{su}(2)_k$ parent liquid, thus reducing the non-Abelian statistics and the CFT of the edge state is given by the coset construction [8]. Our results fully agree with the case studied by a complementary approach in [26], and can also be applied to the hierarchy states discussed in [27].

In the context of topological liquids the topological symmetry (1) acquires a very physical meaning: local perturbations of the chain Hamiltonians correspond to tunneling events across the intervening liquid. It is precisely the perturbations in the trivial sector which correspond to tunneling processes that are not accompanied by simultaneous ejection into the surrounding liquid of anyon particles with nontrivial topological quantum numbers [such processes correspond to $j_1 = 0$ in Fig. 2(b)].

We thank D. A. Huse for collaboration on a closely related project and M. Freedman, A. Kitaev, M. P. A. Fisher, A. Läuchli, and K. Schoutens for discussions. Our numerical work employed the ALPS libraries [28]. A. W. W. L. was supported, in part, by NSF DMR-0706140.

Note added.—After the submission of this manuscript, a closely related paper on edges between different non-Abelian quantum Hall states appeared [29].

[1] More precisely $U_q[\mathrm{SU}(2)]$ with q a root of unity. See, e.g., C. Kassel, *Quantum Groups* (Springer-Verlag, New York, 1995).
[2] N. Read and D. Green, Phys. Rev. B **61**, 10 267 (2000).
[3] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[4] C. Nayak *et al.*, Rev. Mod. Phys. **80**, 1083 (2008).
[5] A. Feiguin *et al.*, Phys. Rev. Lett. **98**, 160409 (2007).
[6] S. Trebst *et al.*, Phys. Rev. Lett. **101**, 050401 (2008).
[7] S. Trebst *et al.*, Prog. Theor. Phys. Suppl. **176**, 384 (2008).
[8] P. Goddard *et al.*, Commun. Math. Phys. **103**, 105 (1986).
[9] D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984).
[10] See EPAPS document No. E-PRLTAO-103-075934 for detailed information on the Hamiltonians which we discuss in the main text. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.
[11] The fusion rules imply that $1/2 \times k/2 = (k-1)/2$ and $(k-1)/2 \times (k-1)/2 = 0 + 1$. The spin-1/2 chains we consider for odd k are based on the $j = (k-1)/2$ representation.
[12] Satisfying standard field identifications, e.g., D. Gepner, Phys. Lett. B **222**, 207 (1989).
[13] F. D. M. Haldane, Phys. Lett. A **93**, 464 (1983); Phys. Rev. Lett. **50**, 1153 (1983).
[14] M. N. Barber and M. T. Batchelor, Phys. Rev. B **40**, 4621 (1989); A. Klümper, Europhys. Lett. **9**, 815 (1989).
[15] G. Fáth and J. Sólyom, Phys. Rev. B **44**, 11 836 (1991).
[16] A. Läuchli, G. Schmid, and S. Trebst, Phys. Rev. B **74**, 144426 (2006).
[17] C. Gils *et al.* (to be published).
[18] A. Cappelli, C. Itzykson, and J.-B. Zuber, Nucl. Phys. **B280**, 445 (1987).
[19] E. Date *et al.*, Nucl. Phys. **B290**, 231 (1987).
[20] See, e.g., N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, Phys. Rev. Lett. **87**, 120405 (2001).
[21] Consider joining $\mathrm{su}(2)_k$ and $\mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1$ Hall liquids along a line, letting first *vacuum* intervene, see Fig. 2(a). The, say, left moving $[\mathrm{su}(2)_{k-1} \times \mathrm{su}(2)_1]_L$ edge state between the latter fluid and vacuum is the same as the left moving CFT $[\mathcal{M}_k \times \mathrm{su}(2)_k]_L$; this is the content of the coset construction [8]. The right moving edge state between the $\mathrm{su}(2)_k$ fluid and the intervening vacuum is $[\mathrm{su}(2)_k]_R$. Right and left $\mathrm{su}(2)_k$ edges interact through the vacuum by a coupling of the two right and left su(2) currents which is known to gap out these gapless modes. What remains is a left moving minimal model $[\mathcal{M}_k]_L$.
[22] We can also rewrite this statement by bringing the Abelian factor $\mathrm{su}(2)_1$, a semion, from the fluid X to the edge. We then arrive at the statement that juxtaposition of the two topological fluids $\mathrm{su}(2)_k$ and $\mathrm{su}(2)_{k-1}$, Read-Rezayi states, along a line yields an edge consisting of the (neutral) $\mathcal{M}_k$ theory and a charged semion $\mathrm{su}(2)_1$ moving in the opposite direction. For corresponding fermionic quantum Hall fluids the edge state between parafermionic Z_k and Z_{k-1} Read-Rezayi states is the neutral CFT $\mathcal{M}_k$ and an oppositely moving Abelian $u(1)$ edge state compensating for the difference in $u(1)$ anomalies (fillings). For $k = 3$, a tricritical Ising model $\mathcal{M}_3$ neutral edge appears between the Z_3 Read-Rezayi and Z_2 Moore-Read states.
[23] The wave function of X is the product of the $\mathrm{su}(2)_{k-1}$ and $\mathrm{su}(2)_1$ wave functions. Symmetrization over the two species of particles gives the $\mathrm{su}(2)_k$ Read-Rezayi state. [Cf., A. Cappelli, L. S. Georgiev, and I. T. Todorov, Nucl. Phys. **B599**, 499 (2001).]
[24] N. Read and A. W. W. Ludwig, Phys. Rev. B **63**, 024404 (2000).
[25] Only the maximally Abelian subtheory $\mathrm{su}(3)_1$ of $\mathrm{su}(2)_4$ enters.
[26] E. Grosfeld and K. Schoutens, arXiv:0810.1955 [Phys. Rev. Lett. (to be published)].
[27] P. Bonderson and J. K. Slingerland, Phys. Rev. B **78**, 125323 (2008).
[28] A. F. Albuquerque *et al.*, J. Magn. Magn. Mater. **310**, 1187 (2007).
[29] F. A. Bais, J. K. Slingerland, and S. M. Haaker, Phys. Rev. Lett. **102**, 220403 (2009).