# Domain Walls, Fusion Rules, and Conformal Field Theory in the Quantum Hall Regime

Eddy Ardonne

*Nordita, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden*

(Received 4 September 2008; revised manuscript received 3 March 2009; published 4 May 2009)

We provide a simple way to obtain the fusion rules associated with elementary quasiholes over quantum Hall wave functions, in terms of domain walls. The knowledge of the fusion rules is helpful in the identification of the underlying conformal field theory describing the wave functions. We show that, for a certain two-parameter family $(k, r)$ of wave functions, the fusion rules are those of $su(r)_k$. In addition, we give an explicit conformal field theory construction of these states, based on the $\mathcal{M}_k(k+1, k+r)$ "minimal" theories. For $r = 2$, these states reduce to the Read-Rezayi states. The "Gaffnian" wave function is the prototypical example for $r > 2$, in which case the conformal field theory is nonunitary.

DOI: 10.1103/PhysRevLett.102.180401 PACS numbers: 73.43.−f, 05.30.Pr

Wave functions have played an instrumental role in the theoretical development of the quantum Hall effect. The Laughlin wave function [1] predicted excitations with fractional charge, which has been observed experimentally [2]. In the seminal work of Moore and Read [3], the connection between conformal field theory (CFT) and wave functions was made, and a state in which the excitations obey so-called non-Abelian statistics was proposed (by now, many quantum Hall states are written in terms of CFT correlators [4–6]). There is ample numerical evidence (see [7] for recent results) that some states observed in the second Landau level harbor particles which obey non-Abelian statistics. The possibility of non-Abelian statistics has recently spurred a tremendous amount of experimental effort [8], with encouraging results, although a direct observation of non-Abelian statistics is lacking so far.

The fundamental property underlying non-Abelian statistics is fusion, which describes the possible outcomes of bringing two particles together. We will denote the different types of particles (or quasiholes) by $a$, $b$, $c$, etc. The fusion of $a$ and $b$ is characterized by the non-negative integers $N_{abc}$, which encode which particle types $c$ are present in the fusion of the particles of type $a$ and $b$:

$$a \times b = \sum_c N_{abc} c. \tag{1}$$

Particles for which there can appear more than one particle after fusion with another particle are called non-Abelian, because the Hilbert space associated with several such particles is higher dimensional, opening up the possibility of non-Abelian braid statistics.

In this Letter, we will focus on a two-parameter family of (bosonic) states, labeled by integers $(k, r)$, at filling fraction $\nu = k/r$, which have the property that they do not vanish when $k$ particles are at the same position but do vanish with power $r$ when $k+1$ particles coincide. In particular, we consider the states written in terms of Jack polynomials in Refs. [9,10]. We note that these clustering conditions (which do not, in general, uniquely specify states) were first studied in Ref. [11]. The $(k, r)$ states under consideration reduce to the Read-Rezayi states [4] in the case $r = 2$. For $r > 2$ (the typical example is the Gaffnian wave function [12]), the CFT is nonunitary, which indicates, according to the arguments put forward in Ref. [13], that these wave functions describe a critical phase rather than a topological state. The new results in this Letter are twofold. First, we find that the fusion rules obeyed by the excitations are those of $su(r)_k$. This result is obtained without invoking CFT but by examining the domain wall structure in the thin-torus limit. Building on this result, we provide an explicit CFT description of the $(k, r)$ wave functions, based on the "minimal" models $\mathcal{M}_k(k+1, k+r)$.

*Orbital occupation numbers.*—To obtain the fusion rules, it is useful to examine the thin-torus limit [14]. On the torus with dimensions $(L_x, L_y)$, one has a basis of one-particle orbitals $\psi_j \propto \sum_m e^{ix[(2\pi j/L_x)+mL_y]-[y+(2\pi j/L_x)+mL_y]^2/2}$, where $j = 0, 1, \ldots, N_\phi - 1$ and $N_\phi = L_x L_y/(2\pi)$; the $\psi_j$ are centered around the line $y = -2\pi j/L_x$. The many-body wave functions can be written in terms of the orbital occupation basis states $|l_0, l_1, l_2, \ldots\rangle$. From the structure of the one-particle orbitals, it follows that in the thin-torus limit $L_x \to \infty$ (with $N_\phi$ constant), only those components of the many-body wave function which maximize $\sum_j l_j^2$ survive. We will show below that these thin-torus states suffice to determine the fusion rules of the excitations, following the results of Ref. [15]. In Refs. [9,10], it was shown that the full wave functions can be written as Jack polynomials, which are labeled by a partition $\lambda$, which is related to the (thin-limit) orbital occupation numbers of the bosons (see below), and a parameter $\alpha = -\frac{k+1}{r-1}$ (for $k+1$ and $r-1$ relative prime). The orbital occupations are characterized by the rule that each set of $r$ neighboring orbitals contains exactly $k$ bosons, giving

$$\binom{k+r-1}{k}$$

different (bulk) patterns or sectors, which also naturally arise in the thin-torus limit [14]. To be explicit, we will give the sectors of the Gaffnian [12] with $(k, r) = (2, 3)$ as an example (see also [15] for the $r = 2$ case). In this case we have six sectors characterized by the following patterns of orbital occupation numbers $|l_0, l_1, l_2, \ldots\rangle$:

$$|200\,200\ldots\rangle, \qquad |020\,020\ldots\rangle, \qquad |002\,002\ldots\rangle,$$
$$|110\,110\ldots\rangle, \qquad |101\,101\ldots\rangle, \qquad |011\,011\ldots\rangle,$$

i.e., by the unit cells (200) and (110) and their translations.

*Excitations and domain walls.*—One can consider quasiholes by allowing configurations in which $r$ neighboring orbitals contain less than $k$ bosons. The fundamental quasiholes, with the smallest possible charge, correspond to configurations in which there is only one set of $r$ neighboring orbitals which contains $k - 1$ bosons. It follows from the Su-Schrieffer counting argument [16] that these fundamental quasiholes have charge $-e/r$, where $e$ is the charge of the constituent bosons. The quasiholes can be viewed as domain walls between different sectors. The general structure is explained by using the Gaffnian as an example. In particular, we will look at the possible fundamental quasiholes starting from the (110) sector:

$$|110\,11\mathbf{0}\,\mathbf{10}1\,101\rangle, \qquad |110\,1\mathbf{10}\,\mathbf{0}20\,020\rangle. \tag{2}$$

The boldface shows the location of the quasihole. Starting from the (110) sector, inserting a quasihole corresponds to a domain wall to either the (020) or the (101) sector. These sectors are obtained from the (110) sector by allowing one boson to hop one orbital "to the right." This is the general structure: Fundamental quasiholes correspond to domain walls between two sectors, where the unit cell of the resulting sector is obtained from the initial one by hopping one boson one place to the right (assuming periodic "boundary conditions" on the unit cell). In general, the different sectors are in one-to-one correspondence with the quasihole types. Starting from one sector, create a quasihole-particle pair, which will have a new, different sector in between them. Move, say, the quasihole around the torus. After annihilating the quasihole-particle pair, the ground state will be the new sector. Each different type of quasihole-particle pair will lead to a different ground state sector.

*Fusion rules.*—Because the domain walls discussed above correspond to the lowest charged quasihole, and the sectors correspond to all possible types of quasiholes, we can interpret the domain walls in terms of the fusion rules. Let us denote the lowest charged quasihole as a particle of type $a$. If there is one fundamental domain wall connecting two sectors, say, $b$ and $c$, we interpret this by saying that sector $c$ is present in the fusion of $b$ with the elementary quasihole $a$, i.e., $N_{abc} = 1$. The possible fundamental domain walls completely specify the fusion rules of the particle type $a$. In the quantum Hall case, we can obtain all types of particles by repeated fusion of this fundamental quasihole. This implies that all of the fusion rules can be obtained from the fusion rules of $a$ by associativity. In general, it might not be obvious which sector corresponds to the fundamental quasihole, but for the $(k, r)$ wave functions, we can explicitly identify the fusion rules.

Before describing this general result, we first note that, for $(k, r) = (2, 3)$, we can identify the six sectors in terms of the $su(3)_2$ representations (see below): $(200) = \mathbf{1}$, $(110) = \mathbf{3}$, $(101) = \bar{\mathbf{3}}$, $(020) = \mathbf{6}$, $(011) = \mathbf{8}$, and $(002) = \bar{\mathbf{6}}$, where the last two numbers in each unit cell correspond to the $su(3)$ Dynkin labels. We interpret the two domain walls in (2) as the fusion rule $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6}$.

*Identification with $su(r)_k$.*—To identify the fusion rules we obtained for the $(k, r)$ states above, we will map the ground state patterns to the labels of the irreducible representations of the affine Lie algebra $su(r)_k$ (see [17] for an introduction). The irreducible representations of $su(r)_k$ can be labeled by $r$ non-negative integers $(l_0; l_1, \ldots, l_{r-1})$, such that $\lambda = \sum_{i=1}^{r-1} l_i \omega_i$ is an $su(r)$ representation ($\omega_i$ are the fundamental weights), and $l_0$ is fixed by $\sum_{i=0}^{r-1} l_i = k$. If this results in $l_0 < 0$, $\lambda$ does not correspond to an irreducible representation of $su(r)_k$. This establishes a one-to-one correspondence between the particle types of the $(k, r)$ states and the representations of $su(r)_k$.

To obtain the fusion rules, we will make use of the Littlewood-Richardson (LR) rule [17] for tensor products of $su(r)$ representations, which is stated in terms of the associated Young diagrams. A Young diagram is a set of rows of boxes, which are weakly decreasing in length. In general, the $j$th row has length $\sum_{i=j}^{r-1} l_i$. For our purposes, it will be useful to add a line of length $\sum_{i=0}^{r-1} l_i$ to the top of the diagram; see Fig. 1 for an $su(4)_7$ example.

We will consider only tensor products of arbitrary representations with one of the fundamental weights $\lambda = \omega_i$, whose Young diagram is a single column of $i$ boxes. We can obtain the associated fusion rules of $su(r)_k$ from the rule that, in the fusion rules, diagrams whose top row contains $k + 1$ boxes are absent, as follows from the Kac-Walton formula [18] relating tensor and fusion products.

The LR rule specifies in which ways the boxes of the Young diagram of (in our case) $\omega_i$ can be added to the arbitrary representation $\lambda$, to obtain the representations in the tensor product. The $i$ boxes of $\omega_i$ have to be added in such a way that the resulting diagram is a Young diagram (i.e., the length of the rows does not increase from top to bottom), and no two boxes can be placed in the same row. It

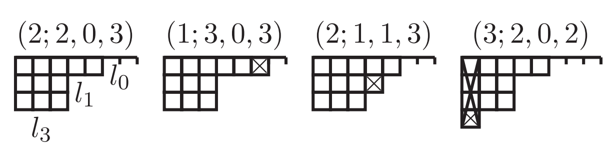


FIG. 1. Leftmost diagram: The "Young diagram" associated to the $(2; 2, 0, 3)$ representation of $su(4)_7$. Remaining diagrams: The Young diagrams obtained after fusing $(2; 2, 0, 3)$ with $\omega_1 = (6; 1, 0, 0)$. The crosses indicate the positions where the box of $\omega_1$ was added and columns of height four are removed.

is allowed to place a box under the leftmost column. If this generates a column of height $r$, the whole column is to be removed and $l_0$ adjusted if necessary. To obtain the $su(r)_k$ fusion rules, we discard all resulting diagrams whose top row contains $k+1$ boxes.

In Fig. 1, we give the Young diagram corresponding to $\lambda = (2;2,0,3)$, as well as the diagrams resulting from fusion with $\omega_1 = (6;1,0,0)$. The crosses denote the positions where the box of $\omega_1$ was added to the diagram of $\lambda$. It is not hard to convince oneself that the LR rules presented above lead to the picture that fusing an arbitrary representation $(l_0; l_1, \ldots, l_{r-1})$ with $\omega_1$ gives maximally $r$ representations, characterized by $l_i \to l_i - 1$ and $l_{i+1} \to l_{i+1} + 1$, for fixed $i$, such that $l_i > 0$ ($l_0$ and $l_r$ are identified). These rules are exactly the ones we found from the domain wall structure of the $(k, r)$ wave functions, which shows that the fusion rules associated with the $(k, r)$ wave functions are the $su(r)_k$ fusion rules. We can go one step further and identify the fusion with an arbitrary $\omega_i$ in terms of domain walls. In Fig. 2, we give the four possible fusion outcomes when one fuses $(2;2,0,3)$ with $\omega_2$. We find that one can interpret fusing with $\omega_i$ in terms of the occupation numbers as follows. The states in the fusion of a general representation are obtained by hopping $i$ bosons one place to the right, with the constraint that from each position one can hop only one boson (which can be the one just hopped to that position). In terms of the domain walls, this precisely corresponds to the situation in which there are $i$ strings of $r$ neighboring orbitals which have a deficit of one boson. A deficit of more than one boson in a string of $r$ neighboring orbitals is not allowed, because in the LR rule this would correspond to placing two boxes in the same row. We clarify this by using the Gaffnian as an example, we find the following "double" domain walls starting from the (110) sector:

$$|11\mathbf{0}\ \mathbf{100}\ 200\ 200\rangle, \qquad |1\mathbf{10}\ \mathbf{01}1\ 011\ 011\rangle. \qquad (3)$$

In both cases, there are two strings of 3 orbitals which have a deficit of one boson, as indicated by the boldface. Like (2), we can interpret (3) in terms of an $su(3)_2$ fusion rule, namely, $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8}$.

*CFT construction.*—Having identified the fusion rules, we will continue by giving an explicit conformal field theory description of the $(k, r)$ wave functions. This construction reduces to the known results for the Gaffnian [12] and corroborates the results obtained from the study of Jack polynomials [10,19]. We will make use of the knowledge of the fusion rules, as well as constraints coming from the explicit wave functions. To get started, we will start by splitting off the $u(1)$-charge part of the theory and consider the remainder, containing the non-Abelian structure. We will reinsert the charge part again in the end. We are after a two-parameter $(k, r)$ family of CFTs which for $r = 2$ reduces to the $Z_k$ parafermion CFT, describing the Read-Rezayi states [4].

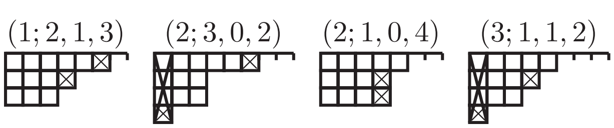


FIG. 2. The four possible fusion outcomes of fusing $(2;2,0,3)$ with $\omega_2$. The crosses denote the positions where the boxes were added. Columns of height four are removed.

As was anticipated in Ref. [19], the CFTs needed are the minimal series $(k+1, k+r)$ related to the $W_k$ algebra [20], which for $k = 2$ is the Virasoro algebra [21]. Here we will explicitly give the operators creating the particles and quasiholes and argue that they have the right properties to generate the $(k, r)$ wave functions. To do this, we write the minimal models in terms of the coset [22]

$$\frac{su(k)_1 \times su(k)_{-\alpha-k}}{su(k)_{1-\alpha-k}}, \qquad \alpha = -\frac{k+1}{r-1}. \qquad (4)$$

We note that, even though $\alpha$ is fractional for $r > 2$, these cosets are well defined but nonunitary. These models are special cases of a more general set of minimal models $\mathcal{M}_k(p, p')$ (where $p$ and $p'$ are coprime), which reduce to the Virasoro minimal models for $k = 2$. In our case, we have $p = k+1$ and $p' = k+r$, and the central charge is given by $c = \frac{r(k-1)}{k+r}[1 - k(r-2)]$. For $r = 2$, the resulting coset is $su(k)_1 \times su(k)_1 / su(k)_2$, which indeed corresponds to the $Z_k$ parafermion CFT.

We will refer to Refs. [22,23] to obtain the field-content of the $\mathcal{M}_k(k+1, k+r)$ models. As usual, the coset fields carry labels of the constituent algebras. In the case at hand, one can restrict oneself (by making use of field identifications [22]) to the labels of $su(k)_r$. Thus, we write the fields as $\Phi_{\boldsymbol{l}}$, where $\boldsymbol{l}$ is vector of $k-1$ non-negative integers whose sum does not exceed $r$. The number of fields in this theory is given by

$$\binom{k+r-1}{r},$$

and the fusion rules are identical to the fusion rules of $su(k)_r$. We will show later that if one includes the charge sector, one indeed obtains the correct $su(r)_k$ fusion rules for the full theory, in agreement with the domain wall picture. The scaling dimensions of the fields $\Phi_{\boldsymbol{l}}$ are given by [24]

$$h_{\boldsymbol{l}} = \frac{k+1}{2(k+r)} \boldsymbol{l} \cdot A_{k-1}^{-1} \cdot \boldsymbol{l} - \frac{r-1}{2(k+r)} \boldsymbol{l} \cdot A_{k-1}^{-1} \cdot (2\boldsymbol{\rho}), \quad (5)$$

where $(A_{k-1}^{-1})_{i,j} = \min(i, j) - ij/k$ are the elements of the inverse Cartan matrix of $su(k)$ and $\boldsymbol{\rho} = (1, 1, \ldots, 1)$.

To establish that these models can be used to obtain the $(k, r)$ states, we will identify a class of fields within these theories, which can be used as the creation operators for the bosons and quasiholes. The first set reduces to the $Z_k$ parafermion fields $\psi_i$ when $r = 2$. These fields $\psi_i^{(r)} = \Phi_{r\omega_i}$

have the same fusion rules as the $\psi_i$'s, namely, $\psi_i^{(r)} \times \psi_j^{(r)} = \psi_{i+j \bmod k}^{(r)}$. Their scaling dimension is $\Delta_{\psi_i} = \frac{r}{2} \times \frac{i(k-i)}{k}$. By making use of the same operator product expansion arguments as those presented in Refs. [4,25] (see also [10]), one can show that the conformal correlator of $N$ operators $\psi_1^{(r)} e^{i\phi\sqrt{r/k}}(z)$ (of dimension $r/2$), and a suitable background charge, gives rise to the lowest degree symmetric polynomial which does not vanish when $k$ particles are brought at the same location but vanishes with power $r$ when $k+1$ particle positions coincide. One can also identify the generalization of the $Z_k$ "spin-field" operators, namely, $\sigma_i^{(r)} = \Phi_{\omega_i}$, which have scaling dimensions $\Delta_{\sigma_i} = \frac{i(k-i)[1-k(r-2)]}{2k(k+r)}$. When these fields are combined with the appropriate vertex operator, they can be thought of as quasihole operators: $V_{\rm qh}(w) = \sigma_1^{(r)} e^{i\phi/\sqrt{rk}}(w)$. One can show that these are the quasiholes with the smallest possible charge, such that the wave functions for the bosons and quasiholes are analytic in the boson coordinates. It is in fact these operators which generate, upon fusion, all of the sectors of the $su(r)_k$ states, which are in one-to-one correspondence to the sectors of the $(k, r)$ states.

We will now argue that if one combines the $\mathcal{M}_k(k+1, k+r)$ theory with the $u(1)_{rk}$ chiral boson describing the charge, one obtains the fusion rules of $su(r)_k$. This is a consequence of rank-level duality [26]. In particular, the modular $S$ matrix of $su(r)_k$ can be written in terms of the modular $S$ matrices of $su(k)_r$ and $u(1)_{rk}$ [17], which relates the fusion rules of $su(k)_r$ and $su(r)_k$. We will demonstrate this by considering the example of the Gaffnian, which is described by the $\mathcal{M}_2(3,5)$ theory [which, using the notation of [12], contains the fields $\mathbf{1}$, $\sigma$, $\varphi$, and $\psi$, with dimensions 0, $-1/20$, $1/5$, and $3/4$, respectively, and which obey $su(2)_3$ fusion rules]. This theory is to be combined with the chiral boson $u(1)_6$. The six particle sectors of the full theory describing the Gaffnian are obtained by first constructing the boson creation operator $\psi e^{3i\phi/\sqrt{6}}$, which belongs to the identity sector, and the smallest charged quasihole $\sigma e^{i\phi/\sqrt{6}}$. By subsequently fusing the quasihole, one obtains the six sectors of the theory, which indeed satisfy $su(3)_2$ fusion rules.

We note that the rank-level duality also occurs in the Read-Rezayi states. The $k+1$ fields in the full theory obey $su(2)_k$ fusion rules, but the $\frac{1}{2}k(k+1)$ fields of the associated $Z_k$ parafermion theory obey $su(k)_2$ fusion rules.

In conclusion, we presented a simple picture of the fusion rules of non-Abelian states in terms of the occupation numbers. Elementary domain walls between regular patterns correspond to the fusion rules of elementary quasiholes. After having identified the fusion rules as those of $su(r)_k$, we presented an explicit CFT construction (following the conjecture in Ref. [19] and Ref. [10]) of the wave functions, based on the $\mathcal{M}_k(k+1, k+r)$ minimal models, which are representations of the $W_k$ algebra. The described method to obtain the fusion rules is general and can help identifying the CFT, for instance, for the wave functions considered in Ref. [27].

The author thanks E. Bergholtz, J. Kailasvuori, and E. Wikberg for numerous discussions and collaboration on a related project [15]. Discussions with B. A. Bernevig, H. Hansson, A. Karlhede, N. Read, S. Simon, J. Slingerland, and Z. Wang are gratefully acknowledged.

[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2] R. de Piccioto *et al.*, Nature (London) **389**, 162 (1997); L. Saminadayar *et al.*, Phys. Rev. Lett. **79**, 2526 (1997).
[3] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[4] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[5] E. Ardonne and K. Schoutens, Phys. Rev. Lett. **82**, 5096 (1999); E. Ardonne *et al.*, Nucl. Phys. **B607**, 549 (2001).
[6] T. H. Hansson *et al.*, Phys. Rev. Lett. **98**, 076801 (2007); Phys. Rev. B **76**, 075347 (2007).
[7] M. R. Peterson *et al.*, Phys. Rev. Lett. **101**, 016807 (2008); A. E. Feiguin *et al.*, Phys. Rev. B **79**, 115322 (2009).
[8] M. Dolev *et al.*, Nature (London) **452**, 829 (2008); I. P. Radu *et al.*, Science **320**, 899 (2008); R. L. Willet *et al.*, arXiv:0807.0221.
[9] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **100**, 246802 (2008).
[10] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **101**, 246806 (2008).
[11] S. Simon *et al.*, Phys. Rev. B **75**, 075318 (2007).
[12] S. Simon *et al.*, Phys. Rev. B **75**, 075317 (2007).
[13] N. Read, Phys. Rev. B **79**, 045308 (2009); arXiv:0807.3107.
[14] E. J. Bergholtz and A. Karlhede, Phys. Rev. Lett. **94**, 026802 (2005); E. J. Bergholtz *et al.*, Phys. Rev. B **74**, 081308(R) (2006); A. Seidel and D.-H. Lee, Phys. Rev. Lett. **97**, 056804 (2006).
[15] E. Ardonne *et al.*, J. Stat. Mech. (2008) P04016.
[16] W. P. Su and J. R. Schrieffer, Phys. Rev. Lett. **46**, 738 (1981).
[17] See, for instance, P. di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997).
[18] M. A. Walton, Nucl. Phys. **B340**, 777 (1990).
[19] B. Feigin *et al.*, Int. Math. Res. Not. 23 (**2002**) 1223.
[20] P. Bouwknegt and K. Schoutens, Phys. Rep. **223**, 183 (1993).
[21] A. A. Belavin *et al.*, Nucl. Phys. **B241**, 333 (1984).
[22] P. Mathieu and M. A. Walton, Prog. Theor. Phys. Suppl. **102**, 229 (1990); P. Mathieu *et al.*, Int. J. Mod. Phys. A **7**, 731 (1992).
[23] V. A. Fateev and S. L. Lykyanov, Int. J. Mod. Phys. A **3**, 507 (1988).
[24] J. Soda and H. Yoshii, Prog. Theor. Phys. **80**, 941 (1988).
[25] V. Gurarie and E. Rezayi, Phys. Rev. B **61**, 5473 (2000).
[26] D. Altschuler *et al.*, Commun. Math. Phys. **132**, 349 (1990).
[27] X.-G. Wen and Z. Wang, Phys. Rev. B **77**, 235108 (2008); **78**, 155109 (2008); M. Barkeshli and X.-G. Wen, arXiv:0807.2789 [Phys. Rev. B (to be published)].