Structure of spinful quantum Hall states: A squeezing perspective

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We provide a set of rules to define several spinful quantum Hall model states. The method extends the one that is known for spin-polarized states. It is achieved by specifying an undressed root partition, a squeezing procedure, and rules to dress the configurations with spin. It applies to both the excitationless and the quasihole states. In particular, we show that the naive generalization where one preserves the spin information during the squeezing sequence may fail. We give numerous examples such as the Halperin states, the non-Abelian spin-singlet states, or the spin-charge separated states. The squeezing procedure for the series $(k = 2,r)$ of spinless quantum Hall states, which vanish as r powers when $k + 1$ particles coincide, is generalized to the spinful case. As an application of our method, we show that the counting observed in the particle entanglement spectrum of several spinful states matches the one obtained through the root partitions and our rules. This counting also matches the counting of quasihole states of the corresponding model Hamiltonians, when the latter are available.

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I. INTRODUCTION

The theoretical study of the fractional quantum Hall (FQH) effect has relied on model wave functions since its discovery.[1] They provide an easy way to understand the physical properties of an inherently hard quantum n-body problem. In addition, having knowledge of the wave functions representing different topological phases provides insight to the question as to which topological phases can exist. Trying to fully classify all topological phases is a tremendous task, but progress has been made in the context of topological insulators and superconductors (see, for instance, Refs. 2 and 3).

In the context of the quantum Hall wave functions, progress also has been made in several ways. A popular and successful approach has been to study model Hamiltonians in combination with conformal-field-theory techniques. In this approach, one studies the zero-energy ground states of model electron-electron interactions, which give rise to certain vanishing properties of the model quantum Hall wave functions. The simplest example of this is the Laughlin wave function (say, at filling $\nu = 1/3$), which is the unique, densest zero-energy ground state of the model interaction given by the Haldane pseudopotential.[4] Excitations of quantum Hall states can be created by changing the flux. Upon increasing the flux, one creates quasihole states, which still are zero-energy ground states of the model Hamiltonian. These quasiholes can have fractional charge,[1] fractional statistics,[5] and even non-Abelian statistics, which was pioneered in Ref. 6. For recent developments with regard to the non-Abelian Berry phase, we refer to Refs. 7 and 8.

The model Hamiltonians, for which the quasihole states are the exact ground states, only constrain the behavior of the underlying electrons. Thus, it should be possible to infer the properties of the anyons from the properties of the electrons in the quantum liquid alone. This implies that, for model quantum Hall states, there should be a "duality" between the electrons and excitations. Such a duality has been observed a long time ago already. In the context of the low-energy Chern-Simons description of the Abelian Laughlin states, we refer to Ref. 9 (see, also, Ref. 10 for a detailed account on the edge-state version of this duality). Subsequently, this notion has been extended to non-Abelian quantum Hall states (see, for instance, Ref. 11). More recently, a seemingly related duality was observed between the conformal-field-theory correlators describing the electron and quasihole states.[12]

The notion of this duality is important because it implies that it should be possible to deduce the properties of the excitations from the ground-state wave functions, and therefore restricts the number of wave functions that can describe topological phases. In addition, it puts constraints on the underlying (conformal-) field-theory description of topological phases in the quantum Hall effect setting. Apart from this duality, there is another, more practical, constraint that one can impose, namely, requiring that the wave functions one considers are eigenstates of a local model Hamiltonian. This constraint allows one to effectively study the topological properties of the state, and provides a way of uniquely defining (or specifying) the state, by a small set of rules. We note that, for instance, the successful Jain states[13] do not satisfy this constraint, so this is not a physical but rather a practical requirement to obtain a more tractable, but still very rich and interesting problem. We also stress that even if the state satisfies the duality and is a ground state of a local Hamiltonian, this does not imply that the wave function represents a genuine topological phase of matter.[7]

Recent developments in generating candidate quantum Hall wave functions gave rise to a framework based on root partitions, squeezing, and highest weight conditions that provides an elegant manner to address several candidate wave functions. This includes the ground state, its quasihole,[14,15] and some aspects of quasielectron excitations as well as excitons[16] (see, also, Refs. 17–20 for more details on quasielectrons and excitions).

For the time being, the effort has mostly concentrated on the spin-polarized systems. However, spinful FQH states are relevant in many realistic cases. The additional spin degree of freedom can be the true spin of the electrons, a layer index in bilayer systems, pseudospin to handle valley degeneracy, or spin-1/2 rotating ultracold fermions. With the success of root partitions for spinless (or spin-polarized) systems, it is worth analyzing how this concept can be translated to the spinful case.

The main goal of this paper is to give a set of rules, which can be used to define model quantum Hall states, with spin (or another internal degree of freedom) by specifying a so-called "root partition" and a squeezing procedure, which is used to define a Hilbert space. The model states are then obtained by imposing highest weight conditions on this Hilbert space for both the orbital and spin parts. The model states we are considering in this paper can be uniquely defined in this way in the case when no excitations are present. For such a procedure to be meaningful, this procedure should also work when (quasihole) excitations are present, i.e., in the case when the number of flux quanta is increased, in comparison to the state without excitations.

It is not *a priori* clear how to generalize the squeezing procedure from polarized states to model states with spin (or other "internal" degrees of freedom). There are, in principle, several routes that one might take, but we found that only one of them correctly generates all the ground states of the model Hamiltonians, including the quasiholes states. Prior work[21–24] has mostly focused on the Halperin[25] or Haldane-Rezayi[26] states. We show that this concept can be extended to other known states such as the non-Abelian spin-singlet or spin-charge separation states, but can also provide a way to obtain new interesting states.

The outline of the paper is as follows. In Sec. II, we review the squeezing procedure for the case of spin-polarized quantum Hall states. In Sec. III, we explain how the root partitions and squeezing technique can be extended to the spinful wave functions. We give several examples in Sec. IV. Interesting series of root partitions are described in Sec. V. It generalizes the spinless series $(k=2,r)$, which include the Moore-Read, Gaffnian, and Haffnian states. As an application of our results, we then show in Sec. VI that the counting we have obtained for the quasihole excitations matches the counting deduced from the particle entanglement spectrum.

In Appendix A, we collect the requirements for a state to be a spin-singlet state, and give the spin-raising and -lowering operators explicitly. Appendix B briefly describes how the various highest weight conditions can be implemented on the (reduced) Hilbert spaces. Finally, in Appendix C, we collect the formulas from the literature giving the number of quasihole states for a set of model Hamiltonians that we consider in this paper.

II. OVERVIEW OF SQUEEZING FOR POLARIZED QUANTUM HALL STATES

Because of the importance of the quasihole states in the spinful case, it seems prudent to review the spin-polarized case and pay special attention to the model state in the presence of quasihole excitations. Moreover, many of the spin-polarized states can be viewed as particular spinful states with quasihole excitations.

We focus our attention to those states that have a ground state that can be uniquely defined by a squeezing procedure, including the Laughlin,[1] Moore-Read,[6] and Read-Rezayi[27] states, as well as, for instance, the Gaffnian[28,29] and Haffnian[30] wave functions.

Quantum Hall states in the lowest Landau level are, apart from a geometry-dependent "confining factor," given by (anti)symmetric polynomials in terms of the coordinates of the (fermionic) bosonic constituent particles. For simplicity, we will be mainly considering bosonic states in this paper; fermionic versions can trivially be obtained by multiplying with an additional global Jastrow factor. An exception to this rule will be fermionic states, which do not contain a Jastrow factor of all particles, and these states can thus not be made bosonic by removing an overall Jastrow factor.

Let us now start by reviewing the squeezing procedure for polarized bosonic quantum Hall states. Being symmetric polynomials, these states can be expanded in so-called "symmetrized monomials." Symmetrized monomials are labeled by partitions, or, equivalently, and perhaps more appropriate in the context of quantum Hall states, orbital occupation numbers.

To be explicit, let us consider the orbital occupation $(n_0,n_1,\ldots,n_{N_\phi})$, such that the lth orbital is occupied with n_l particles. The total number of flux quanta is denoted by N_ϕ (we only consider the spherical geometry in this paper), while the total number of particles is given by $N=\sum_{l=0}^{N_\phi} n_l$. The total degree of the symmetrized monomial corresponding to these orbital occupation numbers is $d=\sum_{l=0}^{N_\phi} l\, n_l$. The partition μ partitions the total degree d and has n_l rows of length l. As an example, we take the orbital occupation (2,0,2), which corresponds to a symmetrized monomial of total degree four. The corresponding partition is $\mu=(2,2,0,0)$, where we included the zeros, which indicate that the zeroth orbital is doubly occupied. In addition, including the zeros ensures that the length of the vector describing the partition equals the number of particles. The elements of the partition μ will denoted by μ_i. Now, the symmetrized monomial m_μ corresponding to μ is given by

$$m_\mu = \mathcal{S}\big[z_1^{\mu_1} z_2^{\mu_2},\ldots,z_N^{\mu_N}\big]\,, \tag{1}$$

where z_i is the complex coordinate of the ith particle and $\mathcal{S}$ denotes the symmetrization, which is normalized such that each term in the symmetrization has coefficient one. In particular, in the case of the partitions $\mu=(2,2,0,0)$, $\mu=(2,1,1,0)$, and $\mu=(1,1,1,1)$, corresponding to the orbital occupations (2,0,2), (1,2,1), and (0,4,0), respectively, one obtains

$$\begin{aligned}
m_{(2,2,0,0)} &= z_1^2z_2^2+z_1^2z_3^2+z_1^2z_4^2+z_2^2z_3^2+z_2^2z_4^2+z_3^2z_4^2,\\
m_{(2,1,1,0)} &= z_1^2z_2z_3+z_1z_2^2z_3+z_1z_2z_3^2+z_1^2z_2z_4\\
&\quad +z_1z_2^2z_4+z_1z_2z_4^2+z_1^2z_3z_4+z_1z_3^2z_4\\
&\quad +z_1z_3z_4^2+z_2^2z_3z_4+z_2z_3^2z_4+z_2z_3z_4^2,\\
m_{(1,1,1,1)} &= z_1z_2z_3z_4.
\end{aligned}$$

As stated above, any symmetric polynomial in a certain number of variables can be expressed in terms of symmetrized monomials

$$\Psi_{\rm sym}(\{z_i\}) = \sum_\mu c_\mu m_\mu(\{z_i\})\,. \tag{2}$$

For comparison, antisymmetric wave functions describing fermions can be expanded in antisymmetric monomials (i.e., Slater determinants), which are written as $\mathrm{sl}_\mu=\mathrm{Det}(z_i^{\mu_j})$.

Inspired by quantum Hall states on a spherical geometry,[4] we assign an orbital angular momentum l_z to each orbital. We choose the convention that the l_z quantum numbers of the orbitals are given by $(-N_\phi/2, -N_\phi/2+1, \ldots, N_\phi/2-1, N_\phi/2)$, i.e., the orbital corresponding to z^0 has the lowest angular momentum $-N_\phi/2$. With this convention, we have that the angular momentum operators are given by

$$\begin{aligned} L_- &= \sum_{i=1}^{N} \partial_{z_i}, \quad L_z = N N_\phi/2 - \sum_{i=1}^{N} z_i \partial_{z_i}, \\ L_+ &= \sum_{i=1}^{N} N_\phi z_i - z_i^2 \partial_{z_i}. \end{aligned} \tag{3}$$

With these preliminaries in place, we can now explain how various model states can be completely specified by a few simple rules. First, for all states, there is a unique "highest" symmetrized monomial. The concept of highest can be defined in a few different, but equivalent, ways. In terms of the orbital occupation numbers, all the orbital occupation numbers of the symmetric monomials present in the expansion of the states can be obtained from the highest one by squeezing particles inward (such that all the symmetrized monomials have the same angular momentum). In terms of the orbital occupation numbers, the squeezing process takes the following form. Taking two particles (assumed to be bosons for now) in orbitals i and j, with $i < j-1$, we move these particles to orbitals $i+1$ and $j-1$, respectively. Explicitly, if one starts with the orbital occupation $(n_0, n_1, \ldots, n_i, n_{i+1}, \ldots, n_{j-1}, n_j, \ldots, n_{N_\phi})$, one ends up with $(n_0, n_1, \ldots, n_i - 1, n_{i+1}+1, \ldots, n_{j-1}+1, n_j - 1, \ldots, n_{N_\phi})$ after squeezing particles in orbitals i and j. In the case of spinless fermions, one needs that $i < j-2$, as well as $n_{i+1} = n_{j-1} = 0$, because of the Pauli principle.

It was realized by Haldane and Bernevig[14,31] that many model quantum Hall states can be written as a single Jack polynomial (with negative parameter). Such Jack polynomials had been studied in the literature[32] and indeed have a highest root configuration.

Alternatively, this highest orbital occupation (at least for states in the absence of quasihole excitations) also corresponds to that part of the wave function that survives if one puts the wave function on the cylinder, and takes the thin-cylinder (or Tao-Thoules) limit.[33–35] In this limit, only those states that maximize $\sum_{i=0}^{N_\phi} n_i^2$ survive.

Finally, in mathematical terms, one says that the highest partition dominates all the partitions corresponding to symmetrized monomials present in the wave function. A partition μ dominates a partition λ if λ can be obtained from μ by successive squeezing operations on μ. If there exists a highest dominating partition (which is not completely unsqueezed), one can reduce the sum over partitions μ in the expansion in terms of symmetrized monomials over those partitions μ, which are dominated by the root partition λ, which is denoted as $\mu \leqslant \lambda$:

$$\Psi(\{z_i\}) = \sum_{\mu \leqslant \lambda} c_\mu m_\mu(\{z_i\}) \,. \tag{4}$$

The existence of a dominating partition, which is smaller than the completely unsqueezed partition, means that one can define a reduced Hilbert space by taking this highest partition and obtaining all the states in the reduced Hilbert space by using the squeezing operation successively. In general, this reduced Hilbert space is significantly smaller than the full Hilbert space, which can be exploited in explicit calculations.

We will now explain how one can completely specify a large set of model quantum Hall states, first in the case when no quasihole excitations are present, and then in the presence of quasiholes. The starting point is the model Hamiltonian, which will be used to obtain the highest, or root partition. For the (bosonic) Read-Rezayi states,[27] which we will take as an example throughout this section, the model Hamiltonian simply gives a positive energy any time $k+1$ particles are coincident. In the thin-cylinder limit, this interaction translates to an assignment of a positive energy every time two neighboring angular momentum orbitals have a total occupation that is bigger than k. One can show, for instance, by an explicit calculation using the thin-cylinder limit that the root partition, in the absence of quasihole excitations, is given by $(k,0,k,\ldots,0,k)$ (see Refs. 36 and 14). To obtain the full Read-Rezayi state, one constructs the (reduced) Hilbert space, which contains all those symmetrized monomials, which can be obtained from the one with root partition $(k,0,k,\ldots,0,k)$ by symmetrically squeezing inward. Total angular momentum is a good quantum number, and because we are looking at a state without excitations, we have an $L=0$ state. Note that, indeed, the root partition has $L_z = 0$. To obtain the $L=0$ states in the reduced Hilbert space, one needs to impose the condition $L_+\Psi = 0$. Because the Read-Rezayi state is the unique, highest-density state that vanishes when $k+1$ particles come together, it is ensured that this procedure will completely determine the coefficients of the monomials that form a basis for the reduced Hilbert space.

We will now focus on constructing the Read-Rezayi states in the presence of quasiholes. We will only be interested in those parts of the wave functions that depend on the coordinates of the underlying particles. Increasing the flux will lead to the introduction of quasihole excitations. The ground state of the model interaction will, in general, be degenerate. For the Abelian Laughlin state, there is a so-called orbital degeneracy of the quasihole excitations. In the case of the non-Abelian quantum Hall states, there is an additional intrinsic degeneracy, coming from the non-Abelian nature of the states. This degeneracy is present, even when the quasiholes are completely localized (in which case the orbital degeneracy is absent). We should note that the explicit counting of states in the presence of quasihole excitations has been studied extensively, resulting in an explicit counting formula, from which all the angular momentum multiplets can easily be obtained. For details on these counting formulas for various model states, we refer to Refs. 37–40.

To determine which angular momentum states are present for a given number of quasiholes, and to obtain these states explicitly, one can also use the above squeezing procedure. First, one needs to obtain the root partitions for the various angular momenta L_z. The highest angular momentum is of course obtained by "shoving" all particles as far as possible to the highest angular momentum orbital as possible, i.e., as allowed by the model Hamiltonian. This will, by definition, also be the state that survives in the thin-cylinder limit. The root partitions of the states at lower angular momentum are obtained by successively moving a (or the) particle with the lowest angular momentum to lower angular momenta. Once this particle is in the lowest angular momentum, one takes the next particle that has lowest angular momentum and moves it to lower angular momenta (of course, in such a way that one does not violate the interaction). One stops with this procedure when, in the following step, one would obtain a state with negative L_z. By construction, the states described above will survive the thin-cylinder limit (within the corresponding L_z sector). As an example, we will look at the ($\nu = 1$ bosonic Moore-Read state) for six particles, with $\Delta N_\phi = 2$ added flux quanta, or four quasiholes. In this case, one has the following root partitions for the various angular momenta:

$$\begin{aligned}
&(0,0,2,0,2,0,2) \quad L_z = 6,\\
&(0,1,1,0,2,0,2) \quad L_z = 5,\\
&(1,0,1,0,2,0,2) \quad L_z = 4,\\
&(1,1,0,0,2,0,2) \quad L_z = 3,\\
&(2,0,0,0,2,0,2) \quad L_z = 2,\\
&(2,0,0,1,1,0,2) \quad L_z = 1,\\
&(2,0,1,0,1,0,2) \quad L_z = 0.
\end{aligned}$$

To determine the number of multiplets with $L = l$, one takes the root partition corresponding to this sector and constructs the associated reduced Hilbert space by squeezing. On this Hilbert space, one acts with the constraint $L_+\Psi = 0$. This will give a set of equations on the coefficients of the basis states. The number of nontrivial solutions of this set of equations is the number of $L = l$ angular momentum states. After having obtained these highest weight states, with $L_z = l$, it is a simple matter to obtain the other states in the same multiplet by acting with L_-.

III. SQUEEZING RULES IN THE MULTICOMPONENT CASE

After having reviewed the squeezing rules in the one-component, spin-polarized case, we now turn our attention to the main topic of this paper, the squeezing rules in the multicomponent case. The additional degree of freedom could, for instance, be a layer or valley degree of freedom, but in this paper, we will focus on spin-1/2 particles. Of course, the considerations apply to more general, multicomponent states as well.

As in the spinless, or polarized case, we will be concerned with model Hamiltonians, which have a unique, zero-energy ground state, in the absence of quasihole excitations.

In the case of bosonic states, the quantum Hall states are symmetric in the coordinates of the several components separately, which implies that we can expand them in the following way in terms of symmetrized monomials:

$$\Psi(\{z_i^\uparrow, z_j^\downarrow\}) = \sum_{\mu,\mu'} c_{\mu,\mu'} m_\mu(\{z_i^\uparrow\}) m_{\mu'}(\{z_j^\downarrow\}) , \tag{5}$$

where the $z_i^\uparrow$ and $z_j^\downarrow$ are the coordinates of the spin-up and -down particles, respectively, and the $c_{\mu,\mu'}$ are coefficients. For fermionic states, one writes the states in terms of Slater determinants instead. The main point that we address in this section is how one can use squeezing to reduce the Hilbert space [i.e., to identify a large class of coefficients $c_{\mu,\mu'}$ in the expansion in Eq. (5) which are zero]. Equipped with this reduced Hilbert space, we will again (as in the polarized, one-component case) explain how to explicitly obtain the various quasihole states (and, hence, also the number of quasihole states present at a given flux).

A. Some considerations about root configurations

The main objective will be to find a generalization of the squeezing rules of the polarized case outlined in the preceding section to the spinful case for several model states.

In the preceding section, we explained that, for the ground state [i.e., in the absence of (quasihole) excitations] for several model states, there is a *unique* partition, the root partition, from which all the other basis states could be obtained by successively squeezing particles inward in all possible ways. This root configuration was identical to the unique root configuration of the ground state, which survived in the thin-cylinder limit.

In the spinful cases, there are, in general, several configurations that survive the thin-cylinder limit because this limit is insensitive to the spin (or other internal) degrees of freedom. Some of these configurations might be forced to have zero coefficient due to the explicit form of the model Hamiltonian. Thus, it is not *a priori* clear how to generalize the squeezing procedure to the spinful case. In fact, one can think of several ways. Here, we will discuss the only procedure we found to work for every model state that we considered.

Let us take an explicit example to explain our considerations and focus on a simple spin-singlet state, the Halperin-(221) state[25] [using later the abbreviation (221) state] for spinful bosons, with filling fraction $\nu = 2/3$. This state is written as

$$\Psi_{(221)}(\{z^\uparrow, z^\downarrow\}) = \prod_{i<j}(z_i^\uparrow - z_j^\uparrow)^2 (z_i^\downarrow - z_j^\downarrow)^2 \prod_{k,l}(z_k^\uparrow - z_l^\downarrow), \tag{6}$$

where the $z_i^\uparrow$ and $z_i^\downarrow$ denote the complex coordinates of the ith spin-up and -down particles, respectively.

The (221) state is the ground state of a local Hamiltonian, which can be written in terms of Haldane pseudopotentials. In particular, this Hamiltonian projects onto states in which no two particles of the same spin have angular momentum less than two, and no two particles of opposite spin have relative angular momentum zero. These properties can be read off from the wave function (6).

Let us denote by $P_{i,j}(L,S)$ the projector, which projects onto (i.e., penalizes) the state in which particles i and j have relative momentum L, and have overall spin S. In terms of these projectors, the model Hamiltonian can be written as

$$H_{(221)} = \sum_{i<j} P_{i,j}(0,0) + P_{i,j}(0,1) . \tag{7}$$

The sum here is over all pairs of particles, irrespective of their spin. We remind the reader that we are dealing with bosons, so we do not have to add the projector $P_{i,j}(1,1)$.

For completeness, we quickly introduce the general Halperin-(mmn) states, which take the form

$$\Psi_{(mmn)}(\{z^{\uparrow},z^{\downarrow}\}) = \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})^m (z_i^{\downarrow} - z_j^{\downarrow})^m \prod_{k,l}(z_k^{\uparrow} - z_l^{\downarrow})^n . \tag{8}$$

For $m = n + 1$, these states are singlet states. In general, they are the densest zero-energy ground states of the interaction (note that the projectors now project onto S_z states)

$$H_{(mmn)} = \sum_{i<j}\Bigg[\sum_{0\leqslant p<n} P_{i,j}(p,S_z = 0) + \sum_{0\leqslant q<m} P_{i,j}(q,S_z = 1) + P_{i,j}(q,S_z = -1)\Bigg]. \tag{9}$$

We return to the question of identifying root configurations of spinful wave functions by considering the bosonic (221) state. Because, in this example, no orbital can be occupied by two particles, we will use the following notation. If the ith orbital is occupied by one spin-up particle, we write $n_i = \uparrow$ and $n_i = \downarrow$ for a down particle. An unoccupied orbital simply has $n_i = 0$.

It has been shown that, in the thin-cylinder limit, the states that survive are those that have electrons in neighboring sites that form singlets, separated by an empty site.[41] In particular, there are four configurations of the (221) state of four particles that survive in the thin torus limit, namely,

$$\begin{aligned}&(\downarrow,\uparrow,0,\downarrow,\uparrow), \quad (\downarrow,\uparrow,0,\uparrow,\downarrow),\\ &(\uparrow,\downarrow,0,\downarrow,\uparrow), \quad (\uparrow,\downarrow,0,\uparrow,\downarrow).\end{aligned} \tag{10}$$

The partitions of the form

$$(\uparrow,\uparrow,0,\downarrow,\downarrow), \quad (\downarrow,\downarrow,0,\uparrow,\uparrow) \tag{11}$$

are absent in the (221) state because two particles of equal spin have a minimal relative angular momentum of two, as in the bosonic Laughlin state with $\nu = 1/2$.

We will show in the next section how the configurations (10), which correspond to states that survive in the thin-cylinder limit,[41] can be used as root configurations to obtain the reduced Hilbert space.

B. Squeezing rules for spinful states

Our strategy to uniquely specify spinful states will follow the polarized case as closely as possible, namely, we will try to find a single, or several, root partitions from which the others can be obtained by squeezing. On this restricted Hilbert space, we furthermore impose the highest weight condition $L_+\Psi = 0$. If the state is a spin-singlet state, obeying SU(2) invariance, we will impose the additional condition $S_+\Psi = 0$. As we already pointed out, there are, in principle, several ways of doing this. In the following, we will give a set of rules, which we found to uniquely define a large class of model quantum Hall states, including the spin-singlet Halperin state, the non-Abelian spin-singlet states proposed by Ardonne and Schoutens (AS),[39,42] the Haldane-Rezayi state,[26] and a non-Abelian state exhibiting spin-charge separation,[43] which we will denote by the acronym "SCsep." A lesser-known fermionic spin-singlet state that can be constructed this way is the product of a permanent and a complete Jastrow factor $\Psi_{\mathrm{SFPer}} = \mathrm{Per}(\frac{1}{z_i^{\uparrow}-z_j^{\downarrow}}) \times \Psi_{(111)}$, a state that was studied by Read and Rezayi.[37]

As examples of states that are not SU(2) invariant, we mention the $(pp0)$ states with $p > 1$, and the bosonic $S_z = 0$ state $\Psi_{\mathrm{SBper}} = \mathrm{Per}(\frac{1}{z_i^{\uparrow}-z_j^{\downarrow}}) \times \Psi_{(221)}$. Many of the states we just mentioned turn out to have root configurations that are closely related. We will come back to this interesting issue in Sec. V.

We remark that, although the spin-singlet composite fermion states do obey a squeezing principle, it is not possible to uniquely define these states by imposing constraints on the reduced Hilbert space. The reason behind this is the same as for the polarized composite fermion states: they are not the unique ground states for any local model Hamiltonian.

We will now describe the procedure to generate the model states, which we divide in a few steps.

(i) First, one needs to decide which root configuration to use. This can simply be a choice or derived from a model Hamiltonian. In this root configuration, one completely ignores the spin or internal degree of freedom. For spin-1/2 fermions, the maximal occupation number in the root configuration is two; for spin-1/2 bosons, there is no such constraint.

(ii) To construct the reduced Hilbert space, one starts by constructing all the possible states one can obtain by squeezing from the root configuration obtained in (i). Still, one does not take the spin degree of freedom into account [apart from the restriction in the case of fermions, as in (i)].

(iii) Continue by taking all states obtained in (ii) and distribute the spin degree of freedom in all possible ways.

(iv) Impose the constraints coming from the Hamiltonian, which are not taken into account already.

(v) Impose the applicable highest weight conditions. This always includes $L_+\Psi = 0$. If the total spin is a good quantum number, one also needs to impose $S_+\Psi = 0$.

Some remarks about these steps are in order here. The procedure we employ is to first strip off the internal degrees of freedom, perform the squeezing, and reintroduce the internal degree of freedom. Although we seem to be working in a roundabout way, this procedure is, in fact, necessary to obtain a large enough reduced Hilbert space. By this, we mean that we would like our procedure to work for all known model states with internal degrees of freedom.

An example of a state for which the naive procedure does not work is the Halperin-(332) state. One of the putative root configurations reads as $(\downarrow,0,\uparrow,0,0,\downarrow,0,\uparrow,0,0,\downarrow,0,\uparrow)$, while the other seven are obtained by replacing $\downarrow,0,\uparrow$ with $\uparrow,0,\downarrow$ in the various locations. If one starts to squeeze the up and the down particles from these root configurations, one never obtains a configuration like $(\uparrow,0,0,\uparrow,\downarrow,0,0,\downarrow,0,0,\uparrow,0,\downarrow)$, which is nevertheless present in the expansion of the (332) state. Our procedure overcomes this problem.

The fact that we first drop the internal degree of freedom and later reintroduce them in all possible ways gives sometimes rise to basis states that actually are not allowed by the Hamiltonian. A simple example is the (221) state, in which the basis states in Eq. (11) have zero coefficient. This problem can be dealt with in a simple way by giving these basis states, which are not allowed because of the Hamiltonian zero coefficient by hand. This typically only involves a low number of basis states and only the first few orbitals, depending on how complicated the Hamiltonian is. Typically, the number of constraints coming from the highest weight $L_+\Psi = 0$ condition is much bigger. In fact, explicitly setting coefficients to zero reduces the number of variables one has to solve for. Sometimes, one does not even have to set these coefficients to zero by hand because these constraints are incorporated in the condition $L_+\Psi = 0$. Examples are the (221) state and the AS states. On the other hand, for the Haldane-Rezayi and SCsep states, one has to take additional constraints coming from the Hamiltonian into account explicitly.

The squeezing rules we presented above can be used for states without quasiholes present as well as states with quasiholes. The only difference lies in the root configurations one starts with. One obtains these in the same way as for polarized states with quasiholes present. One considers the root configuration disregarding the spin, with the appropriate number of orbitals, and fills the orbitals such that the particles have as high an angular momentum as possible, taking the Hamiltonian into account. This automatically gives a configuration with the highest L_z possible. The other L_z sectors are obtained by hopping the particles to lower angular momenta, as explained for the polarized case at the end of Sec. II. This gives a set of root configurations, all at different L_z. To obtain the reduced Hilbert spaces in the different L_z sectors, one uses the same squeezing procedure we introduced above. The number of states is then given by the number of solutions to the constraints, namely, $L_+\Psi = 0$, as well as $S_+\Psi = 0$ and the constraints coming from the Hamiltonian, if applicable.

IV. EXPLICIT EXAMPLES OF SPINFUL QUANTUM HALL STATES

In this section, we consider a set of (spinful) states for which we checked that the squeezing procedure we presented in the preceding section works, and gives the right number of multiplets as given by the counting formulas. For singlet states, this means that we obtain the right number of (L,S) multiplets, while for states where total spin is not a good quantum number, but S_z is, we obtain the correct number of L multiplets at each possible value of S_z.

Underlying these counting formulas lies an exclusion[44] (or generalized Pauli) principle, which limits the number of particles that can occupy a certain number of adjacent orbitals. In the polarized cases, the orbital occupations that satisfy the exclusion principle are precisely those orbitals that are used in the construction of the states using the squeezing principle. We will show that in the spinful or multicomponent case, we have exactly the same result. Namely, one can obtain the right number of states from an exclusion principle, but to make the correspondence work, one needs a procedure where one first ignores the spin to generate a set of orbital occupations. Then, one has to dress these orbital occupations with the spin degrees of freedom, taking constraints coming from the Hamiltonian into account. The amount of states obtained in this way is in one-to-one correspondence to the number of states present for the number of particles and quasiholes under consideration. Below, we will go over the different states in more detail, and state in detail the constraints one has to impose on the configurations to obtain the correct counting. We checked this in each case for a considerable number of particles and quasiholes, but a proof for the claims made will be left for another occasion.

A. (221) singlet states

As we pointed out in the preceding section, the bosonic (221) state is the ground state of a local Hamiltonian that can be written in terms of Haldane pseudopotentials. We repeat the wave function here for convenience, and refer to the preceding section for the model interaction [Eq. (7)]

$$\Psi_{(221)}(\{z^{\uparrow},z^{\downarrow}\}) = \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})^2(z_i^{\downarrow} - z_j^{\downarrow})^2 \prod_{l,m}(z_l^{\uparrow} - z_m^{\downarrow}) \,. \tag{12}$$

The root configuration, which one should use to generate this state, is closely related to the configurations that survive in the Tao-Thouless limit (see Ref. 41 for this state). We already discussed these configurations in the preceding section, where we described the squeezing procedure in detail. In particular, the configurations needed are the Tao-Thouless configurations, but with the spin degrees of freedom removed, which leads to configurations of the form $(1,1,0,1,1,0,\ldots,0,1,1)$ in the case of the ground states (i.e., states without additional quasiholes). The various configurations needed for states with quasiholes are obtained in exactly the same way as the configurations of polarized states in the presence of quasiholes, which we explained in detail in Sec. II.

The number of states generated in this way indeed form all the ground states of the pseudopotential Hamiltonian described above. The counting of the number of states has been described in detail in the literature. Here, we will formulate this counting in terms of an exclusion (or generalized Pauli) principle.[44]

To describe this exclusion principle, which can be used to count the number of ground states for an arbitrary number of particles and flux, we start by noting that the filling fraction of the (221) state is $\nu = 2/3$. So, we will be considering orbital occupations in which no three neighboring orbitals contain more than two particles. In addition, no orbital can be occupied by two particles. By enumerating all the configurations that satisfy these criteria, we obtain a set of configurations, which can be grouped into a set of angular momentum multiplets. We will now turn our attention to the question of how to "introduce spin" to these multiplets.

We thus consider all possible ways to distribute spin over the orbital configurations obtained from the rules above. Distributing the spins over the orbital configurations is subjected to a constraint, namely, two neighboring orbitals can not contain two particles with the same spin (or better, can not form an $S = 1$ multiplet), which follows from the pseudopotential Hamiltonian. Because the (221) state is SU(2) symmetric, this implies that two particles occupying neighboring orbitals must form a singlet pair.

The particles that are not forced to be part of a singlet pair by this rule are free and can be part of an arbitrary spin multiplet. To complete the counting, we thus need to know the number

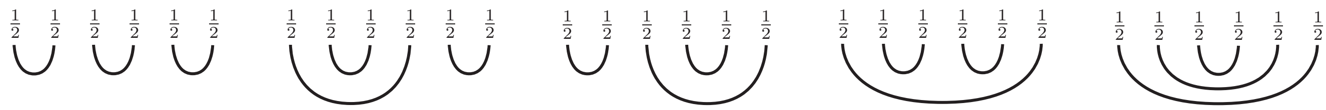

FIG. 1. The configurations enumerating the number of $S = 0$ states in the tensor product of six spin-$1/2$ particles.

of different s multiplets that the free spins can form. This is a standard problem. If one has n spin-$1/2$ particles, the number of s multiplets is given by

$$\#(n,s) = \frac{2s+1}{n/2+s+1}\binom{n}{n/2+s}. \tag{13}$$

This completes the counting of the ground states of the model Hamiltonian of the (221) state in terms of the exclusion principle outlined above.

We checked that the above is in accordance with the counting formula for the number of (quasihole) states given the number of particles N and the total number of flux quanta N_ϕ on the sphere. The number of flux quanta is given by $N_\phi = \frac{3N}{2} - 2 + \frac{n}{2}$, where $n = n_\uparrow + n_\downarrow$ is the total number of quasiholes, and $N = N_\uparrow + N_\downarrow$. Then, the number of states is given by

$$\#_{(221)}(N,n) = \sum_{\substack{N_\uparrow + N_\downarrow = N \\ n_\uparrow + n_\downarrow = n}}{}' \binom{N_\uparrow + n_\uparrow}{N_\uparrow}\binom{N_\downarrow + n_\downarrow}{N_\downarrow}, \tag{14}$$

where the sum is over all possible ways of dividing N (and n) into up and down particles. In addition, the sum is constrained by the relation $N_\uparrow + n_\uparrow = N_\downarrow + n_\downarrow$, which guarantees that both spin species see the same amount of flux. Finally, the total S_z quantum number of particular contribution to the number of states is given by $2S_z = N_\uparrow - N_\downarrow$.

It will be useful in the following to give an alternative description of the number of spin-s multiplets in the tensor product of n spin-$1/2$ representations. One of the simpler ways, out of the many ways possible, to show that this number is given by Eq. (13) is as follows. The number of states with a fixed, total value s_z is given by $\binom{n}{(n+2s_z)/2}$. The number of spin-s multiplets is then given by the number of states with $s_z = s$ minus the number of states with $s_z = s+1$, or $\binom{n}{n/2+s} - \binom{n}{n/2+s+1} = \frac{2s+1}{n/2+s+1}\binom{n}{n/2+s}$.

For the non-Abelian generalization of the (221) state, we will need a more graphical description of the number of spin-s multiplets present in the tensor product of n spin-$1/2$ particles, which goes under the name of the Rumer-Pauling rules.[45,46] In this representation, all the n spin-$1/2$ particles are depicted by lines, which carry the SU(2) $s = 1/2$ representation. For convenience, we order the lines next to each other. Joining two lines, as depicted in Fig. 1, means that the two spin-$1/2$ representations form a singlet (or valence bond). The total number of spin singlets one can form out of n spin-$1/2$ particles is given by the number of ways one can connect the n spins pairwise, such that the connecting lines do not cross. The number of such diagrams can easily be shown to be a Catalan number, in accordance with Eq. (13). The total number of spin-1 states can be found in a similar way, but this time, one should leave two of the spin-$1/2$ particles unpaired, and pair up the remaining ones.[69] Again, the lines representing the spin-$1/2$ representations can not cross one another. In Fig. 2, we display the diagrams enumerating the spin-1 diagrams. Analogously, there are five spin-2 configurations and only one spin-3 configuration, with all spins unpaired.

B. Non-Abelian spin-singlet states

One can construct non-Abelian analogs of the (221) spin-singlet states in the same way as one can generalize the Laughlin $\nu = \frac{1}{2}$ state to the Moore-Read and Read-Rezayi states. The Read-Rezayi (RR) states are labeled by a parameter k, which characterizes the vanishing properties of the states when one clusters the constituent particles. By concentrating on the simplest bosonic state, one has that the RR-k state does not vanish when k particles coincide, while the wave function vanishes quadratically when $k+1$ particles coincide. It turns out that there is a unique, densest state with these properties.

The non-Abelian spin-singlet states[42] are the spin-singlet analogs of the Read-Rezayi states. The AS ground states also have the property that they do not vanish when k particles coincide (irrespective of their spin), while the wave function vanishes quadratically (linearly) when $k+1$ particles of the same (mixed) type coincide. An easy explicit form of the wave function uses the Cappelli form[47] of the Read-Rezayi wave functions, which is a symmetrized product of k bosonic Laughlin $1/2$ states. Similarly,[48] one can write the AS states as a symmetrized product of k (221) states

$$\Psi_{\mathrm{AS},k}(\{z^\uparrow,z^\downarrow\}) = \mathcal{S}_{z^\uparrow,z^\downarrow}[\Psi_{(221)}(\{z_a^\uparrow,z_a^\downarrow\}) \times \Psi_{(221)}(\{z_b^\uparrow,z_b^\downarrow\})\ldots\Psi_{(221)}(\{z_k^\uparrow,z_k^\downarrow\})], \tag{15}$$

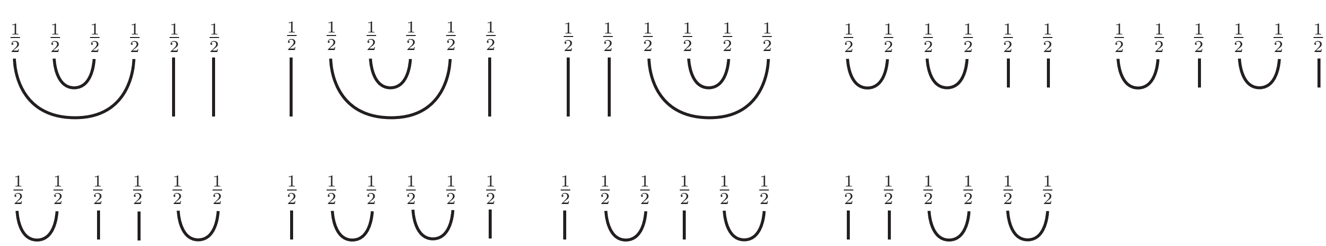

FIG. 2. The configurations enumerating the number of $S = 1$ states in the tensor product of six spin-$1/2$ particles.

where the (221) wave function is given in Eq. (6) and $\mathcal{S}_{z\uparrow,z\downarrow}$ denotes the separate symmetrization of the spin-up particles on the one hand and the spin-down particles on the other. The filling fraction of these simplest bosonic AS states is given by $\nu = \frac{2k}{3}$, which changes to $\nu = \frac{2k}{2kM+3}$ upon multiplication of a complete Jastrow factor for spin-up and -down particles. For future reference, we will write this factor as $\prod_{i<j}(x_i - x_j)^M$, where x can denote the position of either a spin-up or -down particle.

For $k = 2$, it is rather straightforward to write down an interaction for which the (simplest bosonic) AS states are the unique ground states. We will concentrate on the simplest bosonic case $M = 0$. For $k = 2$, the interaction is a three-body interaction, which does not depend on the spin of the interacting particles, and is identical to the model interaction having the (spinless) Moore-Read state as its ground state. In particular, we can write

$$H_{\mathrm{AS},k=2} = \sum_{i<j<k} P_{i,j,k}\left(0,\frac{1}{2}\right) + P_{i,j,k}\left(0,\frac{3}{2}\right) . \qquad (16)$$

We do not need the term $P_{i,j,k}(1,\frac{3}{2})$, because this term will not give a contribution to the energy because we are dealing with bosons. For arbitrary k, the interactions will be a $k+1$ body interaction, penalizing the coincidence of $k+1$ particles.

After this short overview of the AS states, we turn our attention to the root configurations, which survive in the Tao-Thouless limit, and which are the configurations to be used in generating the states (with or without quasihole excitations) by using the squeezing procedure we presented in this paper. Because the states can be written as a symmetrized product over k (221) states, it naturally follows that the root configurations (after stripping the spin degrees of freedom) can be written as $(k,k,0,k,k,0,\ldots,0,k,k)$. We have checked extensively that the number of states [or better, (L,S) multiplets] generated from the root configurations via our procedure to construct model states as explained in the preceding section corresponds one-to-one with the counting formula obtained from the underlying conformal field theory. This counting formula precisely gives the number of (L,S) multiplets, given the number of particles and flux quanta. In Appendix C, we collect counting formulas for several model quantum Hall states.

The number of states can also be obtained from an exclusion principle, analogously to the RR and Halperin states. This exclusion principle makes use of the structure of the root configurations. As we did for the (221) state, we describe the counting in the case $M = 0$; multiplication of the wave functions by an overall Jastrow factor does not change the counting, although the precise form of the root configurations changes.

From the symmetrized expression for the AS states in Eq. (15), one observes that every orbital can at most be occupied by k particles, while every set of three consecutive orbitals can at most be occupied by $2k$ particles. These rules are enough to determine the possible angular momentum multiplets for a given number of particles and number of orbitals. The more interesting part of this problem lies in how one has to "introduce" the spin degrees of freedom to the obtained configurations.

In the Halperin states, no two up particles can occupy neighboring orbitals, which forces two particles occupying two neighboring orbitals to form a singlet. In the case of the AS states, we instead have that two neighboring orbitals can occupy at most k up particles. This means that if two neighboring orbitals are occupied by $k+1$ particles or more, some of these particles will have to form singlets. Two particles forming such a singlet have to occupy neighboring orbitals. This follows from the fact that the AS states are symmetrized products of (221) states, which allow, in their root configurations, for maximally one particle per orbital. Upon symmetrization, no singlets are formed in a single orbital. As a result, we find that some particles occupying neighboring orbitals are forced to form singlets.

We focus now on the remaining particles. If these particles were free to form arbitrary multiplets, we could use Eq. (13) to obtain the number of S multiplets for each L multiplet we obtained earlier. However, the free spins, which are not bound to form singlets, can not form arbitrary S multiplets because we have the additional constraint that no singlet can be formed on a single site. As such, the amount of S multiplets actually depends on the precise distribution of the free spins over the orbitals. To complete the description of the exclusion principle for the AS states, we therefore make use of the explicit diagrams enumerating the number of S multiplets, given a number of (free) spin-1/2 particles, which we outlined in the preceding section. Given these diagrams, in which all the singlets are completely explicit, we can simply check if they give rise to singlets on a single site for a particular orbital occupation of the free spins. If so, the diagram does not contribute to the number of (L,S) multiplets. By making use of the rather simple exclusion principle for the (221) state, and the fact that AS states are symmetrized products of these, we were able to obtain an exclusion principle for the AS states. We checked the results from this method against the known counting formula derived from the underlying conformal field theory (which also makes use of an exclusion principle) and found complete agreement.

V. ROOT CONFIGURATIONS $(2,0^{r-1},2,0^{r-1},\ldots,0^{r-1},2)$

In the following sections, we concentrate on a set of fermionic spin-singlet states for which the root configurations are of the form $(2,0^{r-1},2,0^{r-1},\ldots,0^{r-1},2)$, where 0^{r-1} denotes a sequence of $r-1$ zeros. These states are interesting because they are closely related to a set of spinless (or spin-polarized), bosonic quantum Hall states at the same filling fraction. In a recent paper,[23] we explained this connection in detail for the fermionic spin-singlet Haldane-Rezayi state and the bosonic polarized Haffnian state. Both these states can be obtained from the root configuration $(2,0^{r-1},2,0^{r-1},\ldots,0^{r-1},2)$ with $r = 4$. In the following section, we will consider $r = 3$, giving rise to the bosonic, spin-polarized Gaffnian wave function, while if one considers the same root configuration for spinful fermions, one obtains a non-Abelian spin-singlet state, showing spin-charge separated excitations. Finally, for $r = 2$, the root configuration gives rise to the Moore-Read state as well as a spin-singlet, fermionic permanent state.

A. Haldane-Rezayi case

Let us start with the Haldane-Rezayi wave function,[26] which is a fermionic, spin-singlet d-wave paired state, which takes the form

$$\Psi_{\rm HR}(\{z^{\uparrow},z^{\downarrow}\}) = \mathrm{Det}\left(\frac{1}{(z_i^{\uparrow}-z_j^{\downarrow})^2}\right)\prod_{i<j}(x_i-x_j)^2\,, \qquad (17)$$

using the convention that the variables x_i can stand for either spin-up or -down particles. The filling of the Haldane-Rezayi (HR) wave function is $\nu = 1/2$, and originally, this wave function was proposed to describe the $\nu = 5/2$ quantum Hall effect. Nowadays, we know that this wave function describes the transition between a gapped strong paring phase and a weak pairing d-wave singlet phase.[49] A lot more is known about the HR wave function, which we will not dwell on here, but instead refer the reader to the literature.[26,49–52]

One property we would like to point out is that the wave function does not vanish when a spin-up and a spin-down particle coincide. The wave function vanishes, however, as a fourth power when any three particles come together (when two particles of the same spin coincide, the wave function vanishes as a third power).

In the following, we will focus on the connection between the HR wave function and the so-called Haffnian wave function, first pointed out in Ref. 23. This connection has its origin in the root configurations needed to generate both states, as well as in the the generalized Pauli (or exclusion) principle, which can be used to count the number of states.

Let us start by giving the interaction, for which the HR state with filling fraction $\nu = \frac{1}{2}$ is the exact ground state.[26] The interaction assigns a nonzero energy to any two particles with relative angular momentum 1. If one changes the exponent of the Jastrow factor in Eq. (17) to q, with $q \geqslant 2$, the interaction that will have the Haldane-Rezayi wave function as its unique ground state at flux $N_\phi = qN - (q+2)$ gives nonzero energy to any two particles with relative angular momentum $q-1$ or $q \leqslant 3$. We will, however, mostly be concerned with the (fermionic) case $q = 2$. In terms of two-body projectors $P_{i,j}(L,S)$, the interaction for $q = 2$ can be written as

$$H_{\rm HR} = \sum_{i<j} P_{i,j}(1,0) + P_{i,j}(1,1). \qquad (18)$$

To generate the HR wave function via our squeezing procedure, one has to specify the root configuration (without spin!), which for the case at hand can be described, for $q = 2$, as follows. Each orbital is occupied by at most two particles (this follows of course from the Pauli principle), and any sequence of four consecutive orbitals can also at most be occupied by two particles. This leads to the following most densely packed root configuration $(2,0,0,0,2,0,0,0,2,0,\ldots,0,2,0,0,0,2)$, corresponding to filling $\nu = \frac{1}{2}$ and shift $\delta = 4$ (the shift being defined as $N_\phi = \nu^{-1}N - \delta$). To obtain the wave function, we use the method outlined in the preceding section. The only things we need to specify are the additional constraints coming from the Hamiltonian. Two particles with combined spin-1 can not have relative angular momentum 1. Indeed, from the wave function, one sees that the minimal relative angular momentum of two up (or down) particles is two. For the squeezing rules, this implies that all configurations with $n_0^{\uparrow} = n_1^{\uparrow} = 1$ or $n_0^{\downarrow} = n_1^{\downarrow} = 1$ get zero coefficient. With this rule in place, we have specified all the rules necessary to generate the zero-energy ground states of the model interaction for the HR state, at any flux. We have verified that the amount of zero-energy ground states corresponds exactly to the counting of such states as performed on the sphere originally in Ref. 37.

To formulate an exclusion principle, which can be used to count the number of (quasihole) states for the Haldane-Rezayi case, one has to follow the same strategy as for the Halperin-(221) and AS states. One takes the root configurations with the spin degrees of freedom removed and adds spin in all possible ways consistent with the Hamiltonian. We will follow the discussion of this as given in Ref. 23. In that paper, it was shown that it does not suffice to start from the configurations that satisfy the basic principle that each four consecutive orbitals can be occupied by a maximum of two particles, as is the case for the root configurations used to construct the state. In addition, one needs to consider configurations of the form $(0,2,0,0,1)$ as well. The presence of these configurations was confirmed by the results for the HR state on the thin-cylinder limit.[24] This latter paper also provided a counting formula for (nonlocalized) quasihole states on the torus.

Following Ref. 23, it was found that to formulate an exclusion principle for the Haffnian state, it was necessary to consider these additional configurations. They take care of the fact that the Haffnian is a so-called irrational state, with a ground-state degeneracy that grows linearly with the number of particles. For the results on the torus, we refer to Ref. 23 (see, also, Ref. 24) and focus on the spherical geometry here. The additional configurations can be described as follows. Every time one has a $\nu = 1/2$ Laughlin-type root pattern, namely, $1,0,1,0,1,0,1,0,1$, one allows squeezing of two neighboring particles, i.e., $0,1,0,1,0 \to 0,0,2,0,0$, as long as one does not generate a sequence $0,1,0,0,2$. Alternatively, one can think of the configurations $0,2,0,0,1$ as appearing symmetrized with a $0,1,0,0,2$ configuration (but not separately counting the latter). The basic configurations, combined with the additional ones, do account for all the ground states of the model Hamiltonian having the Haffnian as its densest ground state. This counting was performed in Ref. 30.

To obtain the exclusion principle for the HR state, one takes the configurations we just described for the Haffnian and dresses them with spin in all possible ways consistent with the model Hamiltonian. The Pauli principle implies that an orbital occupied by two particles harbors a singlet. The Hamiltonian implies in addition that the same is true for two neighboring orbitals that are singly occupied, and even for two next-nearest-neighbor orbitals that are singly occupied. Thus, for a spin to be free, meaning that it could be part of an arbitrary big spin multiplet, both its two nearest-neighbor and its two next-nearest-neighbor orbitals have to be unoccupied. Thus, the spin of the particle occupying the middle orbital in the configuration $1,0,0,1,0,0,1$ is free to be part of an arbitrary large spin multiplet.

The rules given above suffice to count the number of ground states of the HR model Hamiltonian at arbitrary number of fluxes on the sphere. Namely, one takes all the configurations allowed for the Haffnian state, and dresses them with spin, in all possible ways consistent with the rules above. One determines which of the spins are forced to be part of a singlet. The remaining spins form arbitrary big spin multiplets, with a degeneracy given by, as explained in the preceding section, $\frac{2s+1}{n/2+s+1}\binom{n}{n/2+s}$, where n is the number of (free) spins and s the spin multiplet.

In Ref. 23, it was explained that a similar reasoning indeed gives the right ground-state degeneracy on the torus for both the Haffnian and HR states. For the HR state, a conformal-field-theory description has been worked out in Refs. 50–52. The quasihole states can be counted by employing the same exclusion principle. The generalized Pauli principle we described here can also be used to count the number of states for the Haffnian and HR state on the torus (see Ref. 23). Explicit counting formulas for these cases were given in Ref. 24.

B. Spin-charge separated states

By considering the root configuration $(2,0,0,2,0,0,2,0,\ldots,0,2,0,0,2)$, which for spin-polarized bosons gives rise to the Gaffnian wave function,[29] one can also construct a fermionic spin-singlet state. The state one obtains in this way has been considered in the literature before and goes under the name of the spin-charge separated state because the state exhibits minimal quasihole excitations without spin.[43] The relevance of this state in the realm of cold atomic gases was studied in Ref. 53. Interestingly, while the Gaffnian state is described by a nonunitary conformal field theory, the spin-charge separated state is obtained from a unitary conformal field theory, which is a necessary condition for a well-behaved, unitary theory describing the edge excitations of the bulk, gapped phase.

The wave function of this state takes the form

$$\Psi_{\mathrm{SCsep}}(\{z^{\uparrow},z^{\downarrow}\}) = Pf\left(\frac{1}{x_i - x_j}\right)\Psi_{(221)}(\{z^{\uparrow},z^{\downarrow}\}), \quad (19)$$

where the Pfaffian factor is with respect to all particles. This state has filling $\nu = \frac{2}{3}$, and the shift on the sphere is given by 3. The interaction for which this state is the unique, zero-energy ground state was worked out in Ref. 54, and can be written in terms of three-body projectors[55] $P_{i,j,k}(L,S)$, assigning energy according to the relative angular momentum and the overall spin of the particles:

$$H_{\mathrm{SCsep}} = \sum_{i<j<k} P_{i,j,k}\left(3,\frac{3}{2}\right) + P_{i,j,k}\left(1,\frac{1}{2}\right) + P_{i,j,k}\left(2,\frac{1}{2}\right), \quad (20)$$

where we choose to set the coefficients of the projectors to one. This Hamiltonian penalizes the closest approach, allowed by the Pauli principle, of three up particles (say). In addition, the two closest approaches allowed by the Pauli principle of three particles that form a doublet $S = 1/2$ are also penalized.

To describe how we can construct this state by means of our squeezing procedure, we have to specify the additional constraints coming from the Hamiltonian. In this case, it turns out we have to set the coefficients of all basis states that obey $n_0^{\uparrow} = n_1^{\uparrow} = n_2^{\uparrow} = 1$ or $n_0^{\downarrow} = n_1^{\downarrow} = n_2^{\downarrow} = 1$ to zero. In this way, we can generate all states by squeezing from the appropriate root configuration, which satisfies the rule that every three consecutive orbitals are occupied by at most three particles. Solving the highest weight conditions for L and S gives, with the additional constraints just given, the ground states of the Hamiltonian (20).

We checked that the number of states generated by our squeezing procedure indeed gives the correct number of ground states. This counting was performed in Ref. 54; the resulting counting formula will be reproduced in Appendix C. As we did for the Haldane-Rezayi state, we will also give an exclusion principle in this case, based on the root configurations we employ to generate the (quasihole) states, which can also be used to count the number of ground states of the model Hamiltonian (20).

In contrast to the Haldane-Rezayi case, in the case at hand, no additional patterns are required to reproduce the counting. The procedure to arrive at the exclusion principle will be equivalent to the HR case, namely, we take the patterns from the related, polarized bosonic state and dress them with spins, taking the constraints from the Hamiltonian into account. The related polarized bosonic state is the Gaffnian. The exclusion principle for the Gaffnian wave function is simply that one allows all configurations, which satisfy the basic rule that no three consecutive orbitals are occupied by three particles or more. Taking these configurations, we assign spins in all possible ways to each configuration. Each site occupied by two particles will have to host a singlet. In addition, there is an additional constraint originating in the Hamiltonian and Hilbert space constraints. In particular, all configurations with three particles of the same spin in any four consecutive orbitals are to be discarded in the exclusion principle. This puts a constraint on configurations such as $(1,1,0,1)$ and $(1,0,1,1)$, which dictates that two of the three particles in these configurations have to form a singlet. With these rules, one can convince oneself that one indeed reproduces the number of ground states of the model Hamiltonian.

C. Overview

In the previous sections, we pointed out that various states can be related to each other via the root configurations that are used to generate these states. This gave a relation between the nonunitary Gaffnian and a unitary spin-charge separated state, as well as a relation between the irrational Haffnian and the nonunitary Haldane-Rezayi wave function. Here, we will give a broader perspective by considering the root configurations $(2,0^{r-1},2,0^{r-1},\ldots,0^{r-1},2)$, with r an integer. These root configurations can be used to generate spinless bosonic states, spin-singlet fermionic states, as well as spinful bosonic states.

In Table I, we give an overview of the states one can construct for $r = 1,2,3,4$. To generate the spinless boson states, one simply uses squeezing to generate the reduced Hilbert space from the appropriate root configuration, and demands that the state is an $L = 0$ state. For the singlet fermionic states, one in addition requires the states to be $S = 0$ states as well. Finally, to define some of the spinful bosonic or fermionic states, one needs to impose that some of the states in the reduced Hilbert space have zero coefficient. We list these additional constraints separately below:

(i) $S = 1/2$ fermions, $r = 3$ [SCsep state]. Partitions with $n_0^{\uparrow} = n_1^{\uparrow} = n_2^{\uparrow} = 1$ or with $n_0^{\downarrow} = n_1^{\downarrow} = n_2^{\downarrow} = 1$ have zero coefficient.

TABLE I. Table with the various states that one can define starting from the $(2,0^{r-1},2,0^{r-1},\ldots,2)$ root configurations. A dash indicates that there is no $L=0$ state for a general number of particles. We remind the reader that $\Psi_{(mmn)}$ denotes the (mmn) state, while Ψ_m denotes the Laughlin state with filling $\nu=\frac{1}{m}$.

	(2,2,2)	(2,0,2,0,2)	(2,0,0,2,0,0,2)	(2,0,0,0,2,0,0,0,2)
Spinless bosons	–	$\mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)\times\Psi_1$ (MR)	Gaffnian	$\mathrm{Hf}\left(\frac{1}{(z_i-z_j)^2}\right)\times\Psi_2$ (Haffnian)
$S=1/2$ fermions ($S=0$ GS)	$\Psi_{(110)}$	$\mathrm{Per}\left(\frac{1}{z_i^\uparrow-z_j^\downarrow}\right)\times\Psi_{(111)}$	$\mathrm{Pf}\left(\frac{1}{x_i-x_j}\right)\times\Psi_{(221)}$ (SCsep)	$\mathrm{Det}\left(\frac{1}{(z_i^\uparrow-z_j^\downarrow)^2}\right)\times\Psi_{(222)}$(HR)
Two-component bosons ($S_z=0$ GS)	–	$\Psi_{(220)}$	$\mathrm{Per}\left(\frac{1}{z_i^\uparrow-z_j^\downarrow}\right)\times\Psi_{(221)}$	$\Psi_{(440)}$

(ii) $S=1/2$ fermions, $r=4$ [HR state]. Partitions with $n_0^\uparrow=n_1^\uparrow=1$ or with $n_0^\downarrow=n_1^\downarrow=1$ have zero coefficient.

(iii) Two-component bosons, $r=2$ [H(220) state]. Partitions with $n_0^\uparrow=2$ or with $n_0^\downarrow=2$ have zero coefficient.

(iv) Two-component bosons, $r=3$ [$\mathrm{Per}(\frac{1}{z_i^\uparrow-z_j^\downarrow})\times\Psi_{(221)}$ state]. Partitions with $n_0^\uparrow=2$ or with $n_0^\downarrow=2$ have zero coefficient.

(v) Two-component bosons, $r=4$ [H(440) state]. Partitions with $n_0^\uparrow=2, n_0^\downarrow=2, n_0^\uparrow=n_2^\uparrow=1$ or with $n_0^\downarrow=n_2^\downarrow=1$ have zero coefficient.

We note in passing that it is possible to construct another two-component bosonic state with $r=2$, namely, $\Psi=\mathrm{Pf}(\frac{1}{z_i^\uparrow-z_j^\uparrow})\times\mathrm{Pf}(\frac{1}{z_i^\downarrow-z_j^\downarrow})\times\Psi_{(111)}$. This state was considered in the context of cold atomic gases in Ref. 56. This state can be obtained from our squeezing procedure with root partition $(2,0,2,\ldots)$, but now one needs the additional constraint that partitions with $n_0^\uparrow=n_0^\downarrow=1$ have zero coefficient. This gives rise to the state Ψ, but only when the number of particles is a multiple of four. If the number of particles satisfies $N_e=4p+2$, with p an integer, then the equations obtained from the construction above do not have a nontrivial solution, in agreement with the fact that one can not write down the state above in this case (at least in the absence of quasiholes). Similarly, we can construct a state at $r=4$ of the form $\Psi=\mathrm{Hf}(\frac{1}{(z_i^\uparrow-z_j^\uparrow)^2})\times\mathrm{Hf}(\frac{1}{(z_i^\downarrow-z_j^\downarrow)^2})\times\Psi_{(222)}$ by squeezing from $(2,0,0,0,2,\ldots)$, and requiring that partitions with $n_0^\uparrow=n_0^\downarrow=1$, or $n_0^\uparrow=n_1^\downarrow=1$ or $n_1^\uparrow=n_0^\downarrow=1$ have zero coefficient.

In the next section, we will consider the state $\Psi_{\mathrm{SBper}}=\mathrm{Per}(\frac{1}{z_i^\uparrow-z_j^\downarrow})\times\Psi_{(221)}$ in some more detail. We do not currently have a Hamiltonian for which this state is the unique zero-energy ground state. Thus, to find for instance the number of quasihole states, we have to rely on our squeezing method to obtain these states. What we will show in the next section is that the numbers we obtain are in accordance with the numbers obtained from the so-called particle entanglement spectrum calculated for the state in the absence of quasihole excitations.

VI. SOME APPLICATIONS: PARTICLE ENTANGLEMENT

As one possible application of our root-configuration analysis, we can compare the results that we have obtained for the quasiholes (namely, the number of quasihole states for a given number of flux quanta) with the one provided through the entanglement spectrum (ES).[57,58] For a single nondegenerate ground state $|\Psi\rangle$, the entanglement spectrum can be defined through the density matrix $\rho=|\Psi\rangle\langle\Psi|$ and the decomposition of $|\Psi\rangle$ in two regions A, B. By tracing out the degrees of freedom of B, one obtains the reduced density matrix $\rho_A=\mathrm{Tr}_B\rho$. Its spectrum is called the entanglement spectrum and it unveils a rich structure of the state $|\Psi\rangle$. The key idea is to focus on one block of ρ_A, fixing all but one of the quantum numbers that are conserved within this operation. Then, one plots the ξ_i as a function of this quantum number, where $\exp(-\xi_i)$ are the non-negative eigenvalues of ρ_A. Depending on the space in which the system is split into two parts, be it real, momentum, orbital, or particle space, different aspects of the system excitations will be revealed through the ES.

It was shown that, if the regions A, B are regions of particles,[58] the particle entanglement spectrum (PES) hence obtained by tracing over the positions of a set of B particles gives information about the number of quasiholes of the system of N_A particles and number of orbitals identical to that of the untraced system. In the case of the many model FQH states, the particle entanglement spectrum contains an identical number of levels as those of the quasihole states with a reduced number of particles. This property seems to be valid even when no local Hamiltonian is known (such as the composite fermion wave functions[13]).

In Fig. 3, we show a typical ES, namely, the particle ES for the Haffnian wave function. All the entanglement levels are plotted against the total projected angular momentum of part A, $L_{z,A}$. From the figure, it is immediately clear that indeed the total angular momentum of part A, L_A^2 is also a good quantum number. For comparison, we show the same spectrum but with only the highest L_z state of every multiplet in Fig. 4.

It is interesting to note that the particle ES in Fig. 4 shows a great deal of resemblance to the real-energy spectrum of a typical quantum Hall state on the sphere, with a lowest lying $L=0$ state, separated by a gap from a continuum. In addition, even a feature resembling the typical roton mode present in the energy spectrum seems to be present in the particle ES. The particle ES of the Haffnian state shown in Fig. 4 was obtained by tracing out half of the particles. The Haffnian can be seen as a symmetrized product of two Laughlin $\nu=\frac{1}{4}$ states. It is, thus, perhaps not so surprising that a state such as the Laughlin $\nu=\frac{1}{4}$ state should have a large contribution to the density matrix after tracing out half of the particles. Indeed, the overlap between the state corresponding to the lowest $L=0$ entanglement level has a very large overlap with the Laughlin $\nu=\frac{1}{4}$, namely, $\langle\Psi_4|\rho_0\rangle^2\approx 0.999\,860$. Such a feature has also been observed for the Moore-Read state.[58]

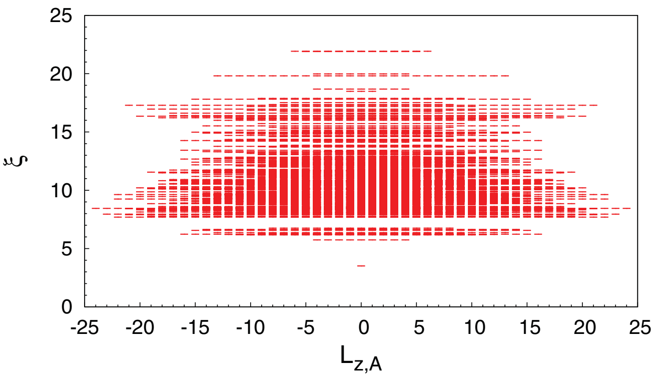

FIG. 3. (Color online) The particle entanglement spectrum for the bosonic Haffnian state with $N = 10$ particles, keeping $N_A = 5$ particles. The $L_{z,A}$ degeneracy is due to the multiplet structure associated with L_A^2. The counting per value $L_{z,A}$ sector exactly matches the corresponding number of quasihole states for 5 particles and 10 added flux quanta.

For the spin-polarized case on the sphere geometry, we can rely on two quantum numbers: the total angular momentum L_A^2 and its projection $L_{z,A}$. The additional L_A^2 quantum number compared to the orbital ES explains the multiplet degeneracy observed in Fig. 3. The PES can be trivially extended to the spinful case. There, we have up to two additional quantum numbers that we can use, namely, the total spin S_A^2 if the state is a spin singlet and its projection $S_{z,A}$, which is always available. The orbital entanglement spectrum was already calculated for a spinful quantum Hall wave function, namely, the Haldane-Rezayi case.[59]

As an example of the particle ES for a singlet state, we use the AS state, for $N = 12$ particles, and trace out half of them. The spectrum is shown in Fig. 5, where we plot the highest L_z and S_z levels for each (L,S) multiplet. The lowest entanglement level, i.e., the state contributing the most to the reduced density matrix, is an $L = 0$, $S = 0$ multiplet. The $k = 2$ AS state can be thought of as a symmetrized product of two Halperin (221) states. This fact is reflected in the overlap between the $N = 6$ particle (221) state and the state $|\rho_0\rangle$ corresponding to the lowest lying $L = 0$, $S = 0$ multiplet, which is $\langle\Psi_{(221)}|\rho_0\rangle \approx 0.997\,878$.

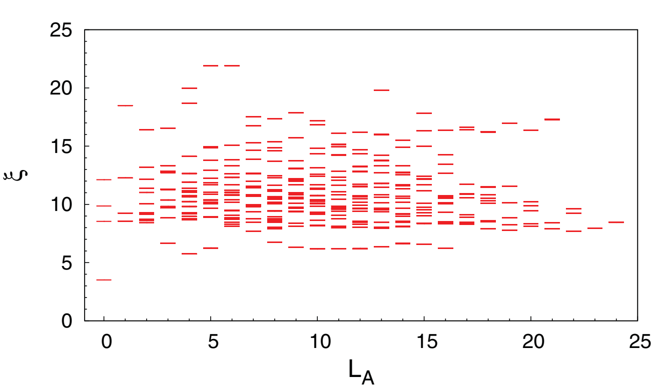

FIG. 4. (Color online) The particle entanglement spectrum for the bosonic Haffnian state with $N = 10$ particles, keeping $N_A = 5$ particles. Only the entanglement levels of the highest weights are shown.

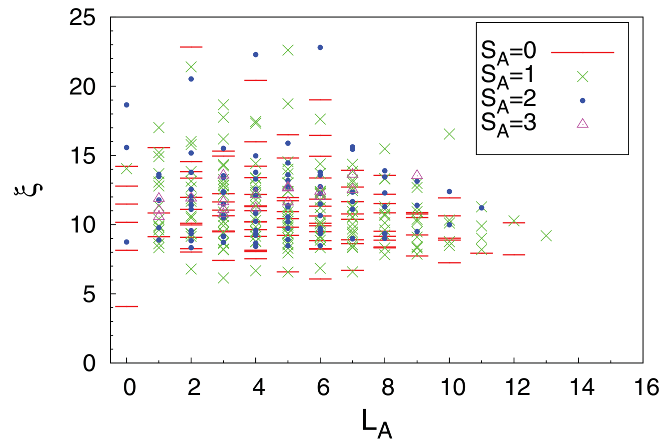

FIG. 5. (Color online) The particle ES for the $N = 12$ bosonic AS state, with $N_A = 6$.

We will now employ the particle ES to obtain some knowledge about the spinful $S_z = 0$ bosonic permanent state $\Psi_{\rm SBper} = {\rm Per}(\frac{1}{z_i^\uparrow - z_j^\downarrow}) \times \Psi_{(221)}$. This state can be obtained by squeezing from the root configuration $(2,0,0,2,0,\ldots,0,2,0,0,2)$. Then, requiring a state to be an $L = 0$ state, and that no two particles with the same spin have relative angular momentum smaller than two (and thus that configurations with $n_0^\uparrow = 2$ or $n_0^\downarrow = 2$ have coefficient zero), leads to a unique state, the $S_z = 0$ bosonic permanent state. We checked this statement for small particle numbers. One way to analyze this state would be to find a model Hamiltonian for which this state is the unique ground state. With this model Hamiltonian, one can check the number of quasihole states upon adding flux in comparison to the state without quasiholes. These numbers can then be compared to the number of states one obtains from the squeezing procedure we presented in this paper. Another way of comparing the number of quasihole states is to make use of the connection between the level counting of the particle ES, and the number of quasihole states, which has been shown to hold for all model states so far. To this end, we calculated the particle entanglement spectrum for the state $\Psi_{\rm SBper} = {\rm Per}(\frac{1}{z_i^\uparrow - z_j^\downarrow}) \times \Psi_{(221)}$ with six particles. Figure 6 shows the particle ES for system A consisting of two and three particles in parts (a) and (b), respectively.

In these particle entanglement spectra, we only plot the maximum $L_{z,A}$ state of each multiplet for clarity. The red lines indicate $S_{z,A} = 0$ ($S_{z,A} = 1/2$) states, the green crosses $S_{z,A} = 1$ ($S_{z,A} = 3/2$) state, for $N_A = 2$ ($N_A = 3$). This state is not a spin-singlet state, so S_A^2 is not a good quantum number. The number of (L,S_z) multiplets for the two cases are given in Table II. In Fig. 7, we show the particle ES in the case of eight particles, and $N_A = 4$.

The total number of flux quanta for the bosonic spin-permanent state is $N_\phi = \frac{3N_e}{2} - 3$. For six particles, $N_\phi = 6$. So, if we want to compare the number of levels in the particle ES, we should compare with the number of states obtained from the squeezing procedure, with 2 and 3 particles, for $N_\phi = 6$. The root configurations that one should use as a

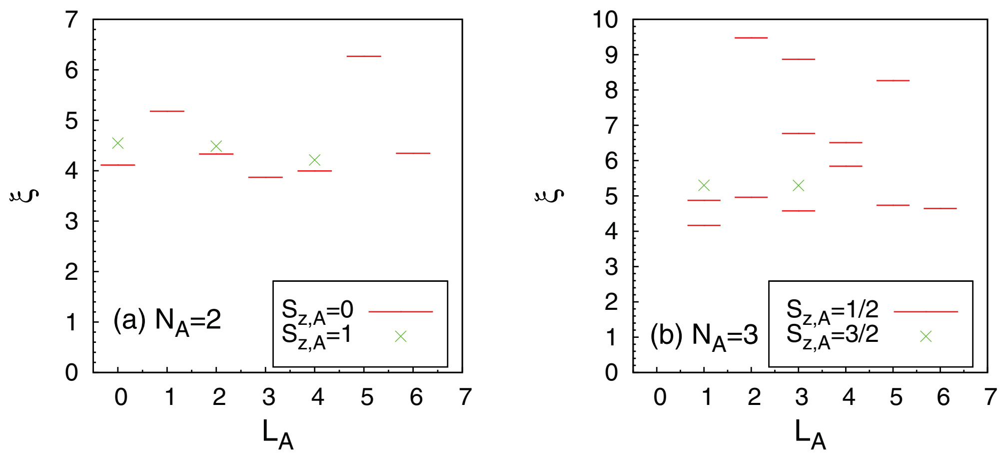

FIG. 6. (Color online) The particle entanglement spectrum for the bosonic spin-permanent state SBper, with $N = 6$ particles, keeping (a) $N_A = 2$ and (b) $N_A = 3$ particles.

starting point for the squeezing procedure are

	$N = 2$	$N = 3$
$L_z = 6$	(0,0,0,0,0,0,2)	(0,0,0,1,0,0,2)
$L_z = 5$	(0,0,0,0,0,1,1)	(0,0,1,0,0,0,2)
$L_z = 4$	(0,0,0,0,1,0,1)	(0,1,0,0,0,0,2)
$L_z = 3$	(0,0,0,1,0,0,1)	(1,0,0,0,0,0,2)
$L_z = 2$	(0,0,1,0,0,0,1)	(1,0,0,0,0,1,1)
$L_z = 1$	(0,1,0,0,0,0,1)	(1,0,0,0,1,0,1)
$L_z = 0$	(1,0,0,0,0,0,1)	(1,0,0,1,0,0,1).

By performing the procedure we outlined above, we obtained a number of states for each possible value of L and S_z, which is in complete accordance to the number of multiplets obtained from the particle entanglement spectrum. We checked this for both $N_e = 6$ and 8, which gives us a nontrivial consistency check on the squeezing procedure we proposed, where we used a state for which (at the moment) no other approaches, such as a conformal-field-theory approach, are available. It seems likely, however, that a conformal-field-theory description is possible. Most likely, such a description would rely on a nonunitary conformal field theory, which could serve as a check on the results obtained above.

VII. CONCLUSION AND OUTLOOK

In this paper, we have generalized the concept of root partitions and squeezing, known for spinless states, to the case of spinful quantum Hall states. We have checked for several model states that our procedure leads to the right wave function both for the ground state and the quasihole states. In particular, we have stressed that the naive generalization, i.e., keeping the spin information during the squeezing procedure, may fail. Thus, one has to rely on an undressed root partition, proceed with the squeezing, and then dress the configurations with spin in a way that is compatible with the Hamiltonian.

We have looked at several model states, such as the Halperin states and non-Abelian spin-singlet states, to test the validity of our set of rules. Using these spinful root partitions, we have provided a spin-1/2 generalization of the spinless $(k = 2,r)$ sequence, which includes the Moore-Read, Gaffnian, and Haffnian states. As an application, we have shown that the counting observed when performing the particle entanglement spectrum on the ground state exactly matches the counting of the quasihole states relying on our rules. In addition, this counting also matches the counting results obtained by counting the number of zero-energy states of the model Hamiltonian for the state under consideration in the cases when such a Hamiltonian is available.

We hope that our method will provide a way to study topological phases with internal degrees of freedom, and shed light on some poorly understood quantum Hall wave

TABLE II. Number of particle ES (L,S_z) multiplets for the bosonic spin-permanent state with $N_A = 2$ (top) and $N_A = 3$ (bottom).

$N_a = 2$	$L = 0$	1	2	3	4	5	6
$S_{z,A} = 0$	1	1	1	1	1	1	1
1	1	0	1	0	1	0	0
$N_a = 3$	$L = 0$	1	2	3	4	5	6
$S_{z,A} = 1/2$	0	2	2	3	2	2	1
3/2	0	1	0	1	0	0	0

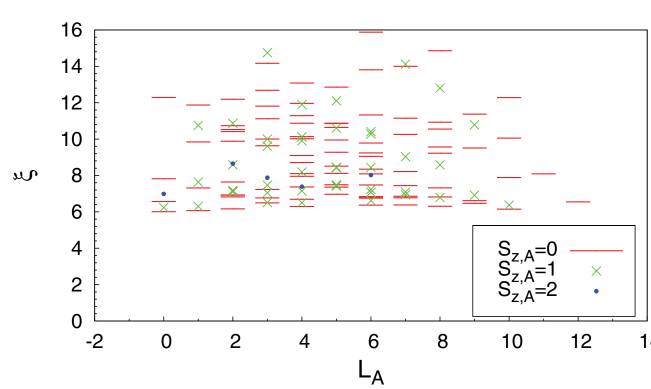

FIG. 7. (Color online) The particle entanglement spectrum for the bosonic spin-permanent state SBper, with $N = 8$ particles, keeping $N_A = 4$.

functions, such as the irrational Haffnian wave function, via the connection with better understood wave functions such as the nonunitary Haldane-Rezayi wave function. In addition, it would be interesting to compare our method in detail with other methods (inspired by the question of classifying the possible topological phases), such as the pattern of zeros approach[60–63] (see, also, Ref. 64) and generalization of the Jack polynomials.[65] Another interesting question is the generalization of the series $(k=2,r)$ for $r>4$ since the polarized case already displays a rich structure, such as its connection to the $N=1$ superconformal theories[66] for $r=6$.

ACKNOWLEDGMENTS

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APPENDIX A: SPIN-SINGLET STATES

We will start by a brief description of spin-singlet states. In the following, the coordinates $z_i^{\uparrow}$ will denote the spin-up particles, while the $z_j^{\downarrow}$ denote the spin-down particles. The wave functions are composed of the orbital part $\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\})$, which is a polynomial in the $z_i^{\uparrow}$'s and $z_j^{\downarrow}$'s, as well as a spin part, which we usually omit. The spin part has the first $N_{\uparrow}$ spins up, and the following $N_{\downarrow}$ spins down. We will omit the usual exponential factors.

We will state the condition on $\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\})$ in order for the state to be a spin singlet. We assume that we are dealing with either bosons or fermions states. We will be concerned with the symmetry properties under the exchange of $z^{\uparrow}$'s with $z^{\downarrow}$'s. For the state to be a singlet, acting with both spin-raising and -lowering operators should give zero. Acting with the spin-raising operator has the following effect on the orbital part $\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\})$ of the wave function. A spin-down particle, say $z_{N_{\downarrow}}^{\downarrow}$, has to be raised to become a spin-up particle, which means it has to be symmetrized (antisymmetrized) with all spin-up particles:

$$S^{+}\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) = \Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) \pm \sum_{i=1}^{N_{\uparrow}} \Psi(z_i^{\uparrow} \leftrightarrow z_{N_{\downarrow}}^{\downarrow}), \quad \text{(A1)}$$

where in the bosonic (fermionic) case, one needs the plus (minus) sign. We will implicitly assume that the variable $z_{N_{\downarrow}}^{\downarrow}$ will be renamed to $z_{N_{\uparrow}+1}^{\uparrow}$ to incorporate the effect that the number of spin-up (-down) particles was increased (decreased) by one. Similarly, we have the spin-lowering operator

$$S^{-}\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) = \Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) \pm \sum_{j=1}^{N_{\downarrow}} \Psi(z_{N_{\uparrow}}^{\uparrow} \leftrightarrow z_j^{\downarrow}), \quad \text{(A2)}$$

where we assume that $z_{N_{\uparrow}}^{\uparrow}$ is renamed to $z_{N_{\downarrow}+1}^{\downarrow}$. The condition for the state to be a spin singlet is now easily written. First, to have $S_z=0$, we need to have $N_{\uparrow}=N_{\downarrow}$. Second, both spin-raising and -lowering operators should give zero:

$$\begin{aligned} S^{+}\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) &= 0,\\ S^{-}\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) &= 0. \end{aligned} \quad \text{(A3)}$$

The conditions (A3) go under the name of the Fock-cyclic conditions, and were spelled out in detail in Ref. 67. Note that, in this paper, we will not be concerned with the Young "symmetrization" procedure. In the case of quasihole states, we will have to consider multiplets of both spin and angular momentum. In that case, to obtain the highest spin state, we only need to consider the action of the spin-raising operator. This is actually also true for the spin-singlet case because the polynomials we will consider will be (anti)symmetric under exchange of all spin-up particles with all the spin-down particles.

For completeness, we recall that angular momentum raising and lowering operators (on the sphere) take the following form:

$$L^{-}\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) = \left(\sum_{i=1}^{N_{\uparrow}} \partial_{z_i^{\uparrow}} + \sum_{j=1}^{N_{\downarrow}} \partial_{z_j^{\downarrow}}\right)\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}), \quad \text{(A4)}$$

$$L_z\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) = \left(NN_{\phi}/2 - \sum_{i=1}^{N_{\uparrow}} z_i^{\uparrow}\partial_{z_i^{\uparrow}} - \sum_{j=1}^{N_{\downarrow}} z_j^{\downarrow}\partial_{z_j^{\downarrow}}\right)\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}), \quad \text{(A5)}$$

$$L^{+}\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}) = \left(N_{\phi}\sum_{i=1}^{N_{\uparrow}} z_i^{\uparrow} + N_{\phi}\sum_{j=1}^{N_{\downarrow}} z_i^{\downarrow} - \sum_{i=1}^{N_{\uparrow}} (z_i^{\uparrow})^2\partial_{z_i^{\uparrow}} - \sum_{j=1}^{N_{\downarrow}} (z_j^{\downarrow})^2\partial_{z_j^{\downarrow}}\right)\Psi(\{z_i^{\uparrow},z_j^{\downarrow}\}), \quad \text{(A6)}$$

where N_{ϕ} is the number of flux quanta, and the total number of particles is given by $N=N_{\uparrow}+N_{\downarrow}$.

APPENDIX B: NUMERICAL IMPLEMENTATION

Having the explicit form of the raising and lowering operators (see Appendix A), it is now a straightforward matter to implement the squeezing procedure we introduced in this paper numerically. From the form of raising and lowering operators L^{+} and L^{-}, it is clear that it is easiest to implement L^{-}, and demand that the states are lowest weight states, which is of course completely equivalent with demanding states to be highest weight.

In practice, one has to implement the form of L^{-} and S^{+} on arbitrary symmetric or antisymmetric monomials, depending on whether one is considering bosons or fermions. We have implemented these routines, as well as some others, in a MATHEMATICA package, which is available for download.[68] These routines include solving routines, which find the solutions for the highest weight constraints.

APPENDIX C: A COLLECTION OF COUNTING FORMULAS

In this appendix, we will collect, for convenience, the counting formulas for the number of states of the various model quantum Hall states we considered in this paper. After introducing some notation, we will start with some polarized states, in particular, the Read-Rezayi states for arbitrary k (including the Laughlin and Moore-Read cases), followed by the characters for the (polarized states obtained from the root configurations $(2,0^{r-1},2,\ldots,0^{r-1},2)$ for $r=2,3,4$, i.e., the Moore-Read, Gaffnian, and Haffnian states.

We will continue with some spin-singlet states, first the AS states for arbitrary k [including for $k=1$ the Halperin-(221) state], again followed by states obtained from the root configurations $(2,0^{r-1},2,\ldots,0^{r-1},2)$, in this case the fermionic spin-singlet states, i.e., the permanent state for $r=2$, the spin-charge separated state for $r=3$, and the HR state for $r=4$

1. Some notation

The character formulas are stated in terms of q binomials, which are q deformations of the ordinary binomials, and keep track of the L_z angular momentum. We will first introduce the notation $(q)_m$, for m a positive integer, $(q)_m=\prod_{i=1}^{m}(1-q^i)$. In addition, we define $(q)_0=1$. The q binomial is defined as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} \frac{(q)_a}{(q)_{a-b}(q)_b} & \text{for} \quad a,b\in\mathbb{N} \quad \text{and} \quad 0\leqslant b\leqslant a, \\ 0 & \text{otherwise}. \end{cases} \tag{C1}$$

For instance, the number of states with f fermions in $N_\phi+1$ orbitals is given by $\binom{N_\phi+1}{f}$. Assigning the l_z angular momenta $-N_\phi/2,N_\phi/2+1,\ldots,N_\phi/2$ to the orbitals, as is applicable for quantum Hall states on the sphere with N_ϕ flux quanta, one finds that the number of states is generated by $q^{-(N_\phi+1-f)f/2}\left[\begin{smallmatrix} N_\phi+1 \\ f\end{smallmatrix}\right]$. Namely, this expression can be expanded as $\sum_{l=-(N_\phi+1-f)f/2}^{(N_\phi+1-f)f/2} c_l\, q^l$, where l runs over half-integers if both N_ϕ and f are odd. Otherwise, l runs over the integers. The numbers c_l are equal to the number of states with $L_z=l$. In addition, these states can be organized into angular momentum multiplets because, for $l\geqslant 0$, one has $c_l\geqslant c_{l+1}$ and $c_l=c_{-l}$. As an example, the number of states for two fermions in six orbitals is given by $q^{-4}\left[\begin{smallmatrix} 6 \\ 2\end{smallmatrix}\right]=q^{-4}+q^{-3}+2q^{-2}+2q^{-1}+3+2q+2q^2+q^3+q^4$. This gives rise to one $L=4$, one $L=2$, and one $L=0$ multiplet.

The number of states of b bosons with N_ϕ flux, or in $N_\phi+1$ orbitals, is similarly given by $q^{-N_\phi b/2}\left[\begin{smallmatrix} N_\phi+b \\ b\end{smallmatrix}\right]$.

Finally, we will make use of the following notation in the subsequent sections. The matrix $\mathcal{M}_k$ has dimensions $k\times k$, and elements $(\mathcal{M}_k)_{i,j}=\min(i,j)$. The matrix $\mathcal{O}_k$ has dimensions $k\times k$, and elements $(\mathcal{O}_k)_{i,j}=\max(0,i+j-k)$.

2. Read-Rezayi state

We will start out with the character for the Read-Rezayi states, with parameter k. The basic bosonic RR states, i.e., those without any overall Jastrow factor, have filling fraction $\nu=\frac{k}{2}$. The number of flux quanta for these states is given by $N_\phi=\frac{2}{k}N-2+\frac{n}{k}$, with N the number of particles and n the number of quasiholes. We note that, for N not a multiple of k, n has to be nonzero in order that the number of flux quanta is an integer. We note that the counting of quasihole states remains unchanged if the wave function is multiplied by an overall Jastrow factor. We therefore write the formulas in terms of N and n, the number of quasiholes, and not in terms of the number of flux quanta because the latter will change upon multiplying the wave function by an overall Jastrow factor.

The counting formula for the number of (quasihole) state in the RR case is given by[40]

$$\#_{\mathrm{RR}}(N,n,k) = q^{-\frac{(2N+n)N}{2k}} \sum_{\substack{a_1,\ldots,a_k\geqslant 0 \\ \sum_{i=1}^{k} i a_i = N}} q^{\mathbf{a}\cdot\mathcal{M}_k\cdot\mathbf{a}} \times \prod_{j=1}^{k}\begin{bmatrix} j\frac{2N+n}{k}-2(\mathcal{M}_k\cdot\mathbf{a})_j+a_j \\ a_j\end{bmatrix}. \tag{C2}$$

Here, the vector $\mathbf{a}$ is given by $\mathbf{a}=(a_1,a_2,\ldots,a_k)$.

3. Root configurations $(2,0^{r-1},2,0^{r-1},\ldots,0^{r-1},2)$: Polarized bosonic states

For $r=2$, this case equals the Moore-Read cases, which in turn can be thought of as the RR state for $k=2$. Here, we will display the (q-deformed version of the) form of the counting formula as it originally appeared in Ref. 37. Equation (C2) with $k=2$ yields a different, but equivalent, expression

$$\#_{\mathrm{MR}}(N,n) = q^{-\frac{Nn}{4}}\sum_f q^{\frac{1}{2}f^2}\begin{bmatrix} \frac{n}{2} \\ f\end{bmatrix}\begin{bmatrix} \frac{N-f}{2}+n \\ n\end{bmatrix}. \tag{C3}$$

The sum over f runs over even (odd) integers for N even (odd), and the number of quasiholes n is always even.

For $r=3$, we obtain the Gaffnian wave function for which the the number of flux quanta is given by $N_\phi=\frac{3N}{2}-3+n$. The counting formula was derived in Ref. 29 and we will display its q-deformed version here, which is valid for $n>0$:

$$\#_{\mathrm{Gaffnian}}(N,n) = q^{-\frac{Nn}{4}}\sum_f q^{\frac{f}{2}(\frac{f}{2}+1)} \times \begin{bmatrix} \frac{n+f}{2}-1 \\ f\end{bmatrix}\begin{bmatrix} \frac{N-f}{2}+n \\ n\end{bmatrix}. \tag{C4}$$

For $n=0$, there is only one state for N even and no states otherwise. This case can be included in the formula, if we define $\left[\begin{smallmatrix} a \\ 0\end{smallmatrix}\right]=1$ for all integers a, even when $a<0$. For N odd, the minimal number of quasiholes required to have a state is three. We note that N and n have the same parity.

The case $r=4$ corresponds to the Haffnian wave function, which was considered in detail in Ref. 30, where the counting was performed. The number of flux quanta is given by $N_\phi=2N-4+\frac{n}{2}$. The counting formula is given by

$$\#_{\mathrm{Haffnian}}(N,n) = q^{-\frac{Nn}{4}}\sum_b q^b\begin{bmatrix} b+\frac{n}{2}-2 \\ b\end{bmatrix}\begin{bmatrix} \frac{N-b}{2}+n \\ n\end{bmatrix}. \tag{C5}$$

In order for this formula to be valid in all cases, we again have to define $\left[{a \atop 0}\right] = 1$ for all integers a. For N odd, n has to be at least four.

4. Non-Abelian spin-singlet states

We will continue with a set of spin-singlet states, which, analogously to the Read-Rezayi states, can be defined for an arbitrary integer k. For $k = 1$, they reduce to the Halperin-(221) states. The filling fraction of these states is $\nu = \frac{2k}{3}$ (in their simplest bosonic version). The flux is given by $N_\phi = \frac{3}{2k}N - 2 + \frac{n}{2k}$, with N the total number of particles $N = N_\uparrow + N_\downarrow$, and n the total number of quasiholes $n = n_\uparrow + n_\downarrow$. There is a constraint on these numbers, namely, $N_\uparrow + n_\uparrow = N_\downarrow + n_\downarrow$, which implies that the flux seen by the spin-up particles is the same as the flux seen by the spin-down particles. The counting formula for the number of states is given by[40]

$$
\begin{aligned}
&\#_{\mathrm{AS}}(N,n,k)\\
&= q^{-\frac{(3N+n)N}{4k}} \sideset{}{'}\sum_{\substack{N_\uparrow+N_\downarrow=N\\ n_\uparrow+n_\downarrow=n\\ a_1,\ldots,a_k\geqslant 0\\ b_1,\ldots,b_k\geqslant 0}} s^{\frac{N_\uparrow-N_\downarrow}{2}} q^{\mathbf{a}\cdot\mathcal{M}_k\mathbf{a}+\mathbf{b}\cdot\mathcal{M}_k\mathbf{b}-\mathbf{a}\cdot\mathcal{O}_k\mathbf{b}}\\
&\times\prod_{j=1}^{k}\left[{j\frac{2N_\uparrow+N_\downarrow+n_\uparrow}{k}-(2\mathcal{M}_k\cdot\mathbf{a}+\mathcal{O}_k\cdot\mathbf{b})_j+a_i \atop a_i}\right]\\
&\times\left[{j\frac{N_\uparrow+2N_\downarrow+n_\downarrow}{k}-(2\mathcal{M}_k\cdot\mathbf{b}+\mathcal{O}_k\cdot\mathbf{a})_j+b_i \atop b_i}\right], \qquad \text{(C6)}
\end{aligned}
$$

where the prime denotes the constraints $\sum_{i=1}^{k} i a_i = N_\uparrow$, $\sum_{i=1}^{k} i b_i = N_\downarrow$, and $N_\uparrow + n_\uparrow = N_\downarrow + n_\downarrow$. The vectors $\mathbf{a}$ and $\mathbf{b}$ are given by $\mathbf{a} = (a_1,\ldots,a_k)$ and $\mathbf{b} = (b_1,\ldots,b_k)$. The exponent of s gives the S_z quantum number of the particular contribution to the number of states. Having access to both the L_z and S_z quantum numbers, one can extract the number of (L,S) multiplets present for an arbitrary number of flux quanta.

5. Root configurations $(2,0^{r-1},2,0^{r-1},\ldots,0^{r-1},2)$: $S = 0$ fermionic states

As in Sec. C3, we will define $\left[{a \atop 0}\right] = 1$ for all integers a. We will start with the case $r = 2$, which corresponds to the $\nu = 1$ fermionic singlet permanent state $\Psi_{\mathrm{SFper}} = \mathrm{Per}\left(\frac{1}{z_i^\uparrow - z_j^\downarrow}\right) \times \Psi_{(111)}$. The number of flux quanta is given by $N_\phi = N - 2 + \frac{n}{2}$. The model Hamiltonian as well as the counting formula were given in Ref. 37. Here, we give the q-deformed version in a slightly different form

$$
\begin{aligned}
\#_{\mathrm{SFper}}(N,n) &= q^{-\frac{Nn}{4}} \sum_{b_\uparrow,b_\downarrow\geqslant 0} s^{\frac{b_\uparrow-b_\downarrow}{2}} q^{\frac{b_\uparrow+b_\downarrow}{2}} \left[{b_\uparrow+\frac{n}{2}-1 \atop b_\uparrow}\right]\\
&\times\left[{b_\downarrow+\frac{n}{2}-1 \atop b_\downarrow}\right]\left[{\frac{N-b_\uparrow-b_\downarrow}{2}+n \atop n}\right]. \qquad \text{(C7)}
\end{aligned}
$$

The structure resembles the structure of the counting formula for the Haffnian. In particular, it is expected that the number of states without quasiholes on the sphere grows linearly with the number of particles, indicating that this state is also irrational.

The case $r = 3$ corresponds to the (unitary) spin-charge separated state of Ref. 43. The number of flux quanta for this state is given by $N_\phi = \frac{3}{2}N - 3 + \frac{n_\uparrow+n_\downarrow+n_h}{2}$, where $N = N_\uparrow + N_\downarrow$ is the number of particles, while $n_\uparrow, n_\downarrow$, and n_h are the number of up, down, and charged but spinless quasiholes. The total number of quasiholes $n = n_\uparrow + n_\downarrow + n_h$ has the same parity as N. The counting was worked out in Ref. 54, with the following result:

$$
\begin{aligned}
\#_{\mathrm{SCsep}}(N,n) &= q^{-\frac{Nn}{4}} \sideset{}{'}\sum_{\substack{N_\uparrow+N_\downarrow=N\\ n_\uparrow+n_\downarrow+n_h=n\\ f\geqslant 0}} s^{\frac{N_\uparrow-N_\downarrow}{2}} q^{\frac{f^2}{2}+\frac{(n_\uparrow+n_\downarrow)^2}{4}}\\
&\times\left[{\frac{n_h}{2} \atop f}\right]\left[{N_\uparrow-n_\downarrow+n_\uparrow \atop n_\uparrow}\right]\\
&\times\left[{N_\downarrow-n_\uparrow+n_\downarrow \atop n_\downarrow}\right]\left[{\frac{N-f}{2}+n_h \atop n_h}\right], \qquad \text{(C8)}
\end{aligned}
$$

where the prime indicates the constraint $N_\uparrow + n_\uparrow = N_\downarrow + n_\downarrow$.

Finally, we come to $r = 4$, namely, the Haldane-Rezayi case. The counting for this state was worked out in Ref. 37. The number of flux quanta is given by $N_\phi = 2N - 4 + \frac{n}{2}$, with n the number of quasiholes. The counting formula reads as

$$
\begin{aligned}
\#_{\mathrm{HR}}(N,n) &= q^{-\frac{Nn}{4}} \sum_{f_\uparrow,f_\downarrow\geqslant 0} s^{\frac{f_\uparrow-f_\downarrow}{2}} q^{\frac{f_\uparrow^2+f_\downarrow^2+f_\uparrow+f_\downarrow}{2}}\\
&\times\left[{\frac{n}{2}-1 \atop f_\uparrow}\right]\left[{\frac{n}{2}-1 \atop f_\downarrow}\right]\left[{\frac{N-f_\uparrow-f_\downarrow}{2}+n \atop n}\right]. \qquad \text{(C9)}
\end{aligned}
$$

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