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From irrational to nonunitary: Haffnian and Haldane-Rezayi wave functions

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We study the Haffnian and Haldane-Rezayi quantum Hall wave functions and their quasihole excitations by means of their "root configurations," and point out a close connection between these seemingly different states. For both states, we formulate a "generalized Pauli-principle," which makes it possible to count the degeneracies of these states. The connection between these states might elucidate the underlying theory describing the "irrational" Haffnian state.

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The characterization of topological phases of matter is an inherently difficult problem, because nonlocal properties determine in what type of topological phase (if any) a system is. Indeed, having a better understanding of why certain models fail to fully develop such a phase will improve our understanding of topological phases of matter. In this Rapid Communication, we study two particular model states of the fractional quantum Hall effect, the most celebrated experimental system exhibiting topological order. These two model wave functions, the Haffnian and the Haldane-Rezayi (HR) states, do not, in fact, describe a topological phase of matter. Although they have different properties at first glance, we show that these two states are, in fact, closely related to one another. By studying this relationship, one comes closer to answering the question of what precisely constitutes a topological phase of matter.

Although the two above-mentioned wave functions do not describe topological states of matter, we refer to them as "states" in the following. The Haffnian, which is a d-wave paired state of spinless bosons at filling fraction $\nu = 1/2$, was studied in detail in Ref. 1. The wave function reads

$$\Psi_{\mathrm{Hf}} = \mathrm{Hf}[1/(z_i - z_j)^2] \prod_{i<j} (z_i - z_j)^2,$$

where Hf denotes the Haffnian of matrix. It is the unique densest zero-energy ground state of the local three-body Hamiltonian that penalizes any triplet that has relative angular momentum less than 4. It has been argued[1] that the Haffnian describes a phase transition between the incompressible bosonic Laughlin state and a gapped d-wave paired state.

The HR state is a fermionic, spin-singlet d-wave paired state,

$$\Psi_{\mathrm{HR}} = \det[1/(z_i^{\uparrow} - z_j^{\downarrow})^2] \prod_{i<j} (z_i - z_j)^2,$$

where the $z_i^{\uparrow}$'s ($z_i^{\downarrow}$'s) denote the position of particles with spin up (spin down), while the spin index is omitted when the product runs over all coordinates irrespective of the spin. The HR is the unique densest zero-energy ground state of a "hollow-core" two-body Hamiltonian.[2] It was initially proposed as a spin-singlet candidate to explain the physics at $\nu = 5/2$. Read and Green[3] argued, by means of a low-wavelength mapping to a critical d-wave superconductor, that the HR state describes the phase transition between a weak-paring, d-wave spin-singlet phase and a strong-pairing phase.

To explain why the Haffnian and HR states do not qualify as topological phases, we consider the conformal field theory (CFT) descriptions of these states. The CFT describing the HR state is [apart from the U(1) part describing the charge] a nonunitary CFT.[4] It has been argued that wave functions described by nonunitary CFTs do not describe topological phases (see, for instance, Ref. 5), although a microscopic understanding of this failure is lacking. To describe the problems with the Haffnian wave function (see also Ref. 6), we note that the torus degeneracy, and hence the number of types of excitations, grows with the number of particles, which is unphysical.

In this Rapid Communication, we focus on counting, characterizing, and finding a relation between the quasihole excitations of both the Haffnian and the HR states through the means of the underlying generalized Pauli principles (or exclusion statistics[7]). We find that previously defined Pauli principles[8] underestimate the counting of Haffnian quasihole and extra exclusion statistics rules must be imposed to obtain the correct counting. The extra configurations present for the Haffnian, when generalized to the spinful case, also reproduce the correct counting of the HR quasiholes.

Pauli principle: The Haffnian on the sphere. We will start our investigations by examining the Haffnian state on the sphere[9] pierced by N_ϕ flux quanta. In the lowest Landau level, there are $N_\phi + 1$ single particle states that are eigenstates of angular momentum, with l_z values ranging from $-N_\phi/2$ to $N_\phi/2$. Wave functions on the sphere are related to those on the plane using the stereographic projection. The bosonic many-particle wave functions can be expanded on Fock basis $\Psi = \sum_\mu c_\mu m_\mu$. The Fock states m_μ are labeled by their occupation number configuration $(n_{N_\phi/2}, \ldots, n_{-N_\phi/2})$, where n_j is the occupation number of the single particle orbital with angular momentum j. This formalism is valid for both bosons (where m_μ are monomials) and fermions (where m_μ are slater determinants).

Many model quantum Hall states (all the ones that are zero modes of at least one pseudopotential Hamiltonian) have nonzero coefficients for only a part of the Hilbert space,[8] beyond simple symmetry considerations such as their total angular momentum projection L_z. There exists a "root configuration" from which all configurations of the Fock states with *a priori* nonzero coefficients can be obtained. Obtaining these states is done by "squeezing," a procedure in which the relative angular momentum of two particles is decreased by two, relative to its value in the root partition, while the total angular momentum is kept constant. By squeezing repeatedly from the root configuration, one obtains all the Fock states which can have nonzero coefficients in the model states (the "reduced Hilbert space"). Interestingly, the root configuration for model quantum Hall states, corresponds to the state surviving in the Tao-Thouless limit.[10]

To obtain the ground state Ψ, one imposes the condition that Ψ is an $L=0$ state; because for ground states all basis states have $L_z=0$, it suffices to require that $L^+\Psi=0$. Ideally, one would like this procedure to give a *unique* solution for the c_μ and hence a unique state, irrespective of the number of particles. For many model states, including the Haffnian, this is indeed the case. The Haffnian ground state for $N_{\rm e}$ particles, has $N_\phi = 2N_{\rm e}-4$ flux quanta. The root configuration of the Haffnian state $(2,0,0,0,2,0,\ldots,0,2,0,0,0,2)$ is the densest configuration subject to the rule that there are maximally two particles in four orbitals.

Apart from the ground states, one can also obtain the states at higher flux $N_\phi = 2N_{\rm e}-4+n_{\rm qh}/2$, that is, in the presence of $n_{\rm qh}$ quasiholes, which show characteristic degeneracies. The procedure mimics the procedure for the ground state. One first constructs the root configurations corresponding to the lowest weights of the possible L multiplets. For $N_e=6$, and one added flux quantum, the root configurations needed are

$$\begin{aligned}&(2,0,0,0,2,0,0,0,2,0)\ L_z=3,\\&(2,0,0,0,2,0,0,0,1,1)\ L_z=2,\\&(2,0,0,0,2,0,0,0,0,2)\ L_z=1,\\&(2,0,0,0,1,1,0,0,0,2)\ L_z=0,\end{aligned}$$

in which the particles are packed as dense as possible at high angular momentum. The reduced Hilbert spaces are obtained by squeezing, and the number of states at $L=L_z$ is given by the number of solutions for c_μ of $L^+\Psi=0$. We checked that this procedure is in full agreement with the counting of zero energy eigenstates of the model Hamiltonian, as performed in Ref. 1, resulting in the counting formula

$$\sum_b \binom{-2+n_{qh}/2b}{b}\binom{(N_e-b)/2+n_{qh}}{n_{qh}}. \tag{1}$$

From the counting formula one can already see that the Haffnian corresponds to an "irrational" theory, in which the number of excitations grows with the number of electrons. Summing the first factor $\binom{-2+n_{qh}/2b}{b}$ over b, one obtains part of the degeneracy on the torus.[11] The sum over b is only constrained by the number of electrons, $b \leqslant N_e$ (and not by $n_{\rm qh}$ as is the case for many other states), showing that the degeneracy on the torus grows with the number of electrons, instead of being constant for gapped quantum Hall states.[12]

In order to obtain the number of $L=L_z$ multiplets present at a chosen value of N_ϕ, we now introduce a "generalized Pauli principle," which provides a way of counting the number of states for N_e particles, at a given flux N_ϕ. Namely, one writes all the "orbital occupation" configurations, subject to some rules. In the case of the (bosonic) Read-Rezayi states, these rules are simply that no more than k particles can occupy two neighboring orbitals,[8] and the number of multiplets at a certain L can be obtained as the difference between the number of states at L_z, L_z+1 (see also Ref. 13).

For the Haffnian, the main rule is that no more than two particles can occupy four consecutive orbitals. However, such a rule by itself is not consistent with the counting, because it does not capture the irrational behavior described above. We introduce an additional rule, which states that the pattern 0,2,0,0,1,0 is also allowed. One way to view this additional rule is that one allows in a $\nu=1/2$ Laughlin-like root pattern 1,0,1,0,1,0,1 for the squeezing of two "neighboring" particles, that is, $0,1,0,1,0 \rightarrow 0,0,2,0,0$ as long as one does not generate a sequence 0,1,0,0,2. Alternatively, one can think of every configuration 0,2,0,0,1 as having to appear symmetrized with the 0,1,0,0,2 configuration, thereby not counting the latter to avoid double counting. We have checked extensively that by counting the configurations given by these rules, one does indeed generate the right number of states. It is suggestive that the Laughlin patterns are the ones that give rise to patterns capturing the irrational behavior—the Haffnian is at the same filling as the 1/2 Laughlin state.

Dressing with spins: The Haldane-Rezayi case. We now move to the HR state, a *fermonic* d-wave singlet state, also at filling fraction $\nu=1/2$. As we pointed out, the HR state is also the unique zero-energy ground state of a model Hamiltonian, at flux $N_\phi=2N_e-4$. The procedure of defining this state and its quasiholes by generating the Hilbert space from squeezing of a root configuration, and subsequently demanding that one has an $L=0$ state that can be generalized to the spinful case. In this Rapid Communication, we mainly state the appropriate recipe,[14] which we motivate more thoroughly in a different publication,[15] where we will deal with a plethora of spin-singlet states.

To define the HR state, both for the ground state at flux $N_\phi=2N_e-4$, as well as for quasihole states at flux $N_\phi=2N_{\rm e}-4+n_{\rm qh}/2$, one follows this simple recipe: Start with the same root configurations used for the Haffnian state (the HR root configuration was also considered in Ref. 16). From these root configurations, generate the Hilbert space by squeezing, with the constraint that the occupation of each orbital is maximally two, because we are dealing with spin-1/2 fermions. For each configuration obtained, one adds spin to the particles, in all possible ways consistent with the hollow-core Hamiltonian. For the HR state, the relative angular momentum of two electrons with the same spin is at least three. General spinfull fermionic states can be written as $\Psi=\sum_{\mu,\nu} c_{\mu,\nu} m_\mu(z_i^\uparrow) m_\nu(z_i^\downarrow)$, in terms of the coordinates of the spin-up $z_i^\uparrow$ and -down $z_i^\downarrow$ electrons. To obtain all the (L,S) multiplets, one generates the reduced Hilbert spaces for all $S_z \geqslant 0$ and then imposes the highest weight angular momentum and total spin constraints $L^+\Psi = S^+\Psi = 0$ to find all the multiplets. We checked that this indeed yields all the states expected from the character formula, which was obtained by studying the number of zero energy states of the model Hamiltonian for arbitrary flux.[17]

For the HR state, we can also define a generalized Pauli principle: Start from all the configurations satisfying the generalized Pauli principle for the Haffnian and dress them with spin. All orbitals occupied by two particles must harbor a singlet pair, because the particles are fermions. The Hamiltonian implies that the same is true for two nearest-neighbor orbitals, as well as for two next-nearest-neighbor orbitals, which are both singly occupied. Only when both neighbors and both next-nearest neighbors of an occupied orbital are unoccupied is the spin of the particle arbitrary, that is, the densest occupation around a "free" spin is 1,0,0,1,0,0,1, where the middle particle corresponds to the free spin. We note that as a result, configurations of the form 1,0,1,0,1 will be absent for the HR state, as we would need to form singlets between 3 spin-1/2 particles.

The counting of states is now simple: Take all the Haffnian configurations dressed with spins in all possible ways consistent with the spin part of the Hamiltonian. These configurations give rise to several spin multiplets. In general, the Hamiltonian forces several spins to be part of a singlet based on their relative orbital distance. The remaining spins are free, and the only task left is to determine how many different S multiplets (and their degeneracy) can be formed out of these free spin-1/2 particles. This last problem is standard; the number of spin-s multiplets present in the product of n spin-1/2's is given by $\frac{2s+1}{(n+2s)/2+1}\binom{n}{(n-2s)/2}$. We have checked extensively that this Pauli principle indeed gives rise to the same number of (L,S) multiplets as the analytical counting.

Apart from reproducing the correct state counting of the quasihole states, our generalized Pauli principle for the Haffnian and HR states also gives the correct prediction of the orbital entanglement level counting on the sphere (introduced in Ref. 18; see Ref. 19 for the HR case), as well as the particle entanglement spectrum.[20]

The torus geometry. The Pauli principles obtained above are valid on a genus 0 geometry. To further elucidate the connection between the Haffnian and HR states, we study these states on the torus and formulate the correct generalized Pauli principle. We performed calculations using the translation symmetry along the y direction. Thus the states are eigenstate of the momentum along y with values $K_y = (\sum_i n_i)\mathrm{mod}N_\phi$.

It is known that the ground-state degeneracy of the Haffnian, in the absence of quasiholes, grows with system size. In particular, the torus degeneracy is $N_e + 8$ or $N_e + 1$ for N_e even or odd, as obtained by exact diagonalization.[21] In contrast, for the HR state, the degeneracy is 10 (2) for N_e even (odd) (see, for instance, Ref. 4). We now see how this information comes out of the generalized Pauli principle.

The basic rule on which the generalized Pauli principle is based—no more than two particles in four orbitals—gives rise, for N_e even, to ten states based on the configurations

$$\begin{aligned}(2,0,0,0,2,0,0,\ldots,2,0,0,0)K_y &= 0,\frac{N_e}{2},0,\frac{N_e}{2},\\(1,1,0,0,1,1,0,\ldots,1,1,0,0)K_y &= \frac{N_e}{4},\frac{3N_e}{4},\frac{N_e}{4},\frac{3N_e}{4},\\(1,0,1,0,1,0,1,\ldots,1,0,1,0)K_y &= \frac{N_e}{2},0,\end{aligned}\tag{2}$$

and their translations, occurring at the indicated momenta. However, the torus lacks the "shift" of the sphere, and the Haffnian occurs at the same flux as the $\nu = 1/2$ Laughlin state, meaning that our second Pauli rule gives rise to multiple other configurations that are associated with the presence of the Laughlin partitions in Eq. (2).

To obtain all the torus ground states for the Haffnian, we employ the same procedure as we did on the sphere; namely, we allow configurations which contain patterns of the type 0,2,0,0,1 provided they can be squeezed from the $\nu = 1/2$ Laughlin pattern 1,0,1,0,1. Because of the periodic boundary conditions on the torus, we do need to allow one occurrence of the pattern 0,1,0,0,2, if and only if no quasiholes are present. This one occurrence can be "located" in two different positions. Thus, apart from the configurations we listed above, we have the following additional configurations (for $N_e = 8$). Notice the presence of the sequences 1,0,0,2 due to the periodic boundary conditions:

$$\begin{aligned}&(2,0,0,1,0,1,0,1,0,1,0,1,0,1,0,0)\ K_y = 0,\\&(2,0,0,0,2,0,0,1,0,1,0,1,0,1,0,0)\ K_y = 0,\\&(2,0,0,0,2,0,0,0,2,0,0,1,0,1,0,0)\ K_y = 0,\\&(0,2,0,0,1,0,1,0,1,0,1,0,1,0,1,0)\ K_y = 8,\\&(0,2,0,0,0,2,0,0,1,0,1,0,1,0,1,0)\ K_y = 8,\\&(0,2,0,0,0,2,0,0,0,2,0,0,1,0,1,0)\ K_y = 8.\end{aligned}\tag{3}$$

In general, for N_e even, we find $N_e - 2$ additional states, for total of $N_e + 8$ states. For N_e odd, we find a total of $N_e + 1$ states; half of them are at $K_y = 0$, the other half at $K_y = N_\phi/2$.

Turning to the ground-state degeneracy of the HR state, we follow the same procedure as we did for the sphere, namely, by dressing the Haffnian configurations with spin, taking the constraints of the Hamiltonian into account. This means that configurations which contain the pattern 1,0,1,0,1 are excluded, and for all even system sizes, we find ten states, eight of which correspond to the first two lines of Eq. (2), the remaining two, which contain the pattern 2,0,0,1, correspond to lines three and six of Eq. (3). For N_e odd, this procedure gives only two ground states. All ground states have $S = 0$.

We now briefly turn to the quasihole case, starting with the Haffnian. Apart from allowing the configurations which are characterized by allowing maximally two particles in four neighboring orbitals, we also allow configurations which contain 0,2,0,0,1, as long as they can be obtained by squeezing from 1,0,1,0,1 and do not contain the pattern 0,1,0,0,2. The configurations thus obtained are in one-to-one correspondence to the ground states of the model Hamiltonian of the Haffnian state, which we checked by explicit diagonalization. The configurations for the HR state are obtained from those of the Haffnian, by dressing them with spin, in all ways consistent with the Hamiltonian. This excludes patterns 1,0,1,0,1 and forces pairs of particles in patterns of type 0,0,2,0,0, type 0,0,1,1,0,0, and type 0,0,1,0,1,0,0 to form singlets. The remaining spins are free and are allowed to form arbitrary spin multiplets, whose counting we described above. We confirmed the counting of torus states described here by explicit diagonalization of the model Hamiltonian, and found complete agreement. As an example, we give the results for $N_e = 6$ particles, and $N_\phi = 12,\ldots,16$ (i.e., from zero until four added flux quanta) in Table I for both the HR and the Haffnian states.

TABLE I. Number of multiplets for the HR and Haffnian states on the torus with $N_e = 6$ particles and $N_\phi = 12, \ldots, 16$ flux quanta. Dashes indicate repeated degeneracies enforced by symmetry, with momentum K_y period gcd(N_ϕ, N_e).

	Haldane-Rezayi							Haffnian					
N_ϕ	S	$K_y = 0$	1	2	3	4	5	$K_y = 0$	1	2	3	4	5
12	0	3	0	0	2	0	0	5	0	0	2	0	0
13	0	7	–	–	–	–	–	10	–	–	–	–	–
14	0	28	26	–	–	–	–	40	37	–	–	–	–
	1	4	6	–	–	–	–						
15	0	75	72	72	–	–	–	102	99	99	–	–	–
	1	27	27	27	–	–	–						
16	0	165	160	–	–	–	–	214	208	–	–	–	–
	1	83	88	–	–	–	–						
	2	2	2	–	–	–	–						

Discussion. We uncovered a connection between the bosonic, polarized d-wave paired Haffnian state, and the fermionic, spin-singlet d-wave HR state, which both have filling fraction $\nu = 1/2$. The connection between these states is rather indirect, namely, via their root configurations, which encode important (topological) properties of these states. Although this connection could well be "accidental," it might nevertheless shed light on the (irrational) CFT underlying the Haffnian state. The CFT describing the HR state is a nonunitary, $c = -2$ CFT,[22] which is closely related to a $c = 1$ orbifold theory.[23] We leave the details of the CFT description for the Haffnian (based on orbifold CFT's; see also Ref. 24) and its connection with the generalized Pauli principle for future work.

The Haffnian root configuration is part of a general series, namely, $(2,0^{r-1},2,0^{r-1},2,\ldots,0^{r-1},2)$, with $r = 4$. This series contains the Moore-Read state at $r = 2$,[25] and the "Gaffnian"[5] at $r = 3$. The latter correspond to a nonunitary CFT, but one can reinterpret the root configuration as one for spinful fermions and construct a different state. In this case, one finds the "spin-charge separated" state,[26] whose non-Abelian statistics is of Ising type. Interestingly, this state is described by a unitary CFT, in contrast to the Gaffnian. As was the case for the Haffnian, by increasing the internal degree of freedom, it is possible to cure some of the problems plaguing the parent state.

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