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K-matrices for 2D conformal field theories

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Abstract

In this paper we examine fermionic type characters (Universal Chiral Partition Functions) for general 2D conformal field theories with a bilinear form given by a matrix of the form $\mathbb{K} \oplus \mathbb{K}^{-1}$. We provide various techniques for determining these K-matrices, and apply these to a variety of examples including (higher level) WZW and coset conformal field theories. Applications of our results to fractional quantum Hall systems and (level restricted) Kostka polynomials are discussed.

1. Introduction

Two-dimensional conformal field theories can be studied in a variety of ways. In this paper, we will pursue the quasiparticle description, which has attracted a lot of attention recently. In a quasiparticle description, the characters of the conformal field theories are of the fermionic sum type. It has been conjectured that all these fermionic sums are of a form which goes under the name of the ‘Universal Chiral Partition Function’ (UCPF), see, for instance, [6,9,12], and references therein. In general, the statistics of the quasiparticles is fractional and interpolates between Fermi and Bose statistics. Moreover, to describe general CFTs, we need to be able to incorporate the effect of the non-trivial fusion rules of the fields, which can be done by allowing for so-called pseudoparticles. These

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pseudoparticles do not carry any energy and are essential in describing the non-Abelian statistics which is found in the CFTs with non-trivial fusion rules.

Fractional statistics can be described in terms of the Haldane 'exclusion statistics' [27]. If we allow for new types of particles, such as the pseudoparticles, the same is true for the non-Abelian statistics, see [26] and [9]. The exclusion statistics is defined in terms of the exclusion statistics parameters of the particles. The parameters are intimately related to the Universal Chiral Partition Functions, as it is these parameters which lie at heart of the UCPF, via the so-called K-matrix, which contains all the (mutual) statistics parameters. In this paper, we will determine the K-matrices related to the affine Lie algebra CFTs, in a particular basis. This basis was first proposed in the context of the fractional quantum Hall states.

The topological properties of (fractional) quantum Hall states are also encoded in matrices, which turned out to be the same as the K-matrices alluded to in the above. In the Abelian states, the entries correspond to the coupling parameters of the Chern–Simons fields which appear in the effective action of the quantum Hall system (see, in particular [47], and references therein). The Chern–Simons term effectively changes the statistics of the matter fields, making the relation between with the exclusion statistics plausible. More details on this relation can be found in [3]. The basis used in the description of certain classes of non-Abelian quantum Hall states is found to be useful in the context of general affine Lie algebra CFTs as well.

One of the reasons that this basis is useful relates to the presence of a duality, which relates the 'electron-like' particles to the quasiparticles (the notion of electron-like and quasiparticles will be explained in Section 2.1.4). Moreover, there is no mutual statistics between these two types of particles. As this structure simplifies the study of the conformal field theories, we will use this type of basis throughout this paper.

One of the main themes in this paper will be the determination of the K-matrices for the affine Lie algebra CFTs. We will develop a scheme which is used to find the general K-matrices. The main idea is to use 'Abelian coverings' of the (in general non-Abelian) CFTs, and project out some degrees of freedom (see also [13]). Having obtained the K-matrices, we will propose a scheme to obtain the K-matrices for conformal field theories which are of the coset form. We will address the diagonal cosets, as well the parafermion CFTs, related to the affine Lie algebra CFTs. Another application are the Kostka-polynomials (see, e.g., [34,35], and references therein), which can also be described in terms of the K-matrices.

In more detail, the outline of this paper is as follows. We start with a general introduction to the role of the K-matrix in 2D conformal field theories in Section 2. We will review some results concerning the Universal Chiral Partition Function and the relation with exclusion statistics. The structure of the basis of quasiparticles which will be used throughout this paper is explained. We will end Section 2 by explaining the relation between the pseudoparticles and the fusion rules of CFTs. In Section 3 we will explain the tools we will use in determining the K-matrices for a general affine Lie algebra. The idea is to embed the level-k affine Lie algebra in k copies of the level-1 version, and project out certain degrees of freedom, by using what we call a P-transformation. In Section 4, we will explicitly give the K-matrices for all the simple (untwisted) affine Lie algebras. We will apply these results to obtain K-matrices for cosets in Section 5. Finally, in Section 6, we will present some new results on level restricted Kostka polynomials related to affine Lie algebras. Some of the details are presented in the appendices. Appendix A deals with some notational issues, and explicitly gives all the Cartan matrices and there inverses. Appendix B and Appendix C deal with the K-matrices for $\mathfrak{so}(5)_1$ and $G_{2,1}$, respectively, while Appendix D relates two different bases for $\mathfrak{sl}(3)_k$.

2. K-matrices for 2D conformal field theories

2.1. The UCPF and exclusion statistics

Quasiparticles play an important role in the description of 2-dimensional conformal field theories (CFTs). The exclusion statistics of these particles is closely related to characters for CFTs, or more precisely, the 'Universal Chiral Partition Function' (UCPF).

2.1.1. Quasiparticle basis

We will start the discussion by introducing quasiparticle bases for two-dimensional conformal field theories, and in particular (truncated) partition functions based on these bases. In CFTs, the quasiparticles take the form of chiral vertex operators $\phi^{(i)}(z)$ ($i = 1,\ldots,n$), which intertwine between irreducible representations of the chiral algebra. By applying the modes of these operators on a set of vacua $|\omega\rangle$, one finds (in general) an over complete basis, which, by using suitable restrictions on the modes $(s_1,\ldots,s_N)$, can be turned into a maximal, linearly independent set of states

$$\phi^{i_N}_{-s_N}\cdots\phi^{i_2}_{-s_2}\phi^{i_1}_{-s_1}|\omega\rangle. \tag{2.1}$$

The grand canonical partition function is obtained by taking the trace over this basis

$$P(\mathbf{z};q) = \mathrm{Tr}\left(\left(\prod_i z_i^{N_i}\right)q^{L_0}\right). \tag{2.2}$$

N_i is the number operator for the quasiparticles $\phi^{(i)}$ and $L_0 = \sum_i s_i$. Furthermore, $z_i = e^{\beta\mu_i}$ is a (generalized) fugacity and $q = e^{-\beta\varepsilon}$. To find the 'one particle grand canonical partition functions' λ_i, we will use truncated partition functions, see [44]. In particular, one defines the truncated partition function $P_{\mathbf{L}}(\mathbf{z};q)$ by restricting the trace over the states (2.2) in such a way that the modes s of the quasiparticles of species i satisfy $s \leqslant L_i$ ($\mathbf{L} = (L_1,\ldots,L_n)$). In the limit of large $\mathbf{L}$ one has

$$\frac{P_{\mathbf{L}+\mathbf{e}_i}(\mathbf{z};q)}{P_{\mathbf{L}}(\mathbf{z};q)} \sim \lambda_i\big(z_i q^{L_i}\big), \tag{2.3}$$

where $\mathbf{e}_i$ is the unit vector in the i-direction. By using a recursion relation for the truncated partition function $P_{\mathbf{L}}(\mathbf{z};q)$ (which can be obtained from the basis (2.1)) and the limit (2.3), one finds relations for the one particle partition functions λ_i (for more details, see [9,12]). For all the CFTs which were investigated by means of a quasiparticle basis as discussed in this section, the equations determining λ_i are of the form (2.14), and thus the quasiparticles satisfy so-called 'exclusion statistics', see Section 2.1.3.

2.1.2. The universal chiral partition function

It has been conjectured (see [6], and references therein) that the characters of all the irreducible representations of (rational) conformal field theories can be written in the form

$$P(\mathbf{z};q)=\sum_{\mathbf{m}}{}'\left(\prod_i z_i^{m_i}\right)q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m}+\mathbf{Q}\cdot\mathbf{m}}\prod_i\left[\begin{pmatrix}((\mathbb{I}-\mathbb{K})\cdot\mathbf{m}+\mathbf{u})_i\\ m_i\end{pmatrix}\right],\tag{2.4}$$

which goes under the name of the 'Universal Chiral Partition Function' (UCPF) (or 'fermionic-type character'). The matrix $\mathbb{K}$ is a symmetric $n\times n$ matrix, $\mathbb{I}$ is the $n\times n$ identity matrix and $\mathbf{Q}$ and $\mathbf{u}$ are n-vectors. The sum is over the non-negative integers $m_1,\dots,m_n$. The restrictions denoted by the prime are (in general) such that the coefficients of the q-binomials are integers. These q-binomials are defined by

$$\begin{bmatrix}M\\ m\end{bmatrix}=\frac{(q)_M}{(q)_{M-m}(q)_m},\qquad (q)_m=\prod_{k=1}^m\left(1-q^k\right).\tag{2.5}$$

Depending on the parameters u_i, the associated particles are of certain type. For *physical particles* $u_i=\infty$, while *pseudoparticles* have $u_i<\infty$. Note that in the limit $u_i\to\infty$ the ith q-binomial reduces to $1/(q)_{m_i}$ due to

$$\lim_{M\to\infty}\begin{bmatrix}M\\ m\end{bmatrix}=\frac{1}{(q)_m}.\tag{2.6}$$

As will become clear below, pseudoparticles do not carry energy. They come about in theories with a non-Abelian symmetry, and in a sense they serve as bookkeeping devices for the internal structure of the theory.

It was conjectured in [9,26] that the UCPF (2.4) is the partition function of a set of particles satisfying exclusion statistics. To be able to make this connection with exclusion statistics, we will take a closer look at truncated versions of the UCPF, and continue with a discussion on exclusion statistics and the relation between the two.

Suppose that the truncated partition function $P_{\mathbf{L}}(\mathbf{z};q)$ takes the form of a 'finitized' UCPF[2]

$$P_{\mathbf{L}}(\mathbf{z};q)=\sum_{\mathbf{m}}{}'\left(\prod_i z_i^{m_i}\right)q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m}+\mathbf{Q}\cdot\mathbf{m}}\prod_i\begin{bmatrix}(\mathbf{L}+(\mathbb{I}-\mathbb{K})\cdot\mathbf{m}+\mathbf{u})_i\\ m_i\end{bmatrix}.\tag{2.7}$$

One can then derive recursion relations for these truncated characters by using the q-binomial relation

$$\begin{bmatrix}M\\ m\end{bmatrix}=\begin{bmatrix}M-1\\ m\end{bmatrix}+q^{M-m}\begin{bmatrix}M-1\\ m-1\end{bmatrix}.\tag{2.8}$$

This leads to the recursion relations [3,8]

$$P_{\mathbf{L}}(\mathbf{z};q)=P_{\mathbf{L}-\mathbf{e}_i}(\mathbf{z};q)+z_iq^{-\frac{1}{2}\mathbb{K}_{ii}+\mathbf{Q}_i+\mathbf{u}_i+\mathbf{L}_i}P_{\mathbf{L}-\mathbb{K}\cdot\mathbf{e}_i}(\mathbf{z};q).\tag{2.9}$$

[2] While this is the case for many examples, in general the finitized UCPF corresponding to a set of (quasi)particles may differ from (2.7) by terms q^n with $n=\mathcal{O}(L_i)$. This will, however, not affect the conclusion.

After dividing by $P_{\mathbf{L}}(\mathbf{z}; q)$, setting $q = 1$, taking the large $\mathbf{L}$ limit and using relation (2.3), one finds

$$1 = \lambda_i^{-1} + z_i \prod_j \lambda_j^{-\mathbb{K}_{ji}}, \tag{2.10}$$

or equivalently,

$$\frac{\lambda_i - 1}{\lambda_i} \prod_j \lambda_j^{\mathbb{K}_{ij}} = z_i. \tag{2.11}$$

These relations are known as the Isakov–Ouvry–Wu (IOW) (2.14) equations, which give the one particle partition functions for a system of particles which obey exclusion statistics; this will be addressed in the next section. For more details on this issue, we refer to [3] and references therein.

In the case of WZW Conformal Field Theories, i.e., CFTs with affine Lie algebra symmetry, it is known that in many cases (see [10,28,39,48], and references therein) the (chiral) partition function can be written in the form

$$P(\mathbf{z}; q) = \sum M_{\lambda\mu}^{(k)}(q) M_{\mu}^{(\infty)}(\mathbf{z}; q), \tag{2.12}$$

where $M_{\lambda\mu}^{(k)}(q)$ are the so-called level-k truncated Kostka polynomials, $M_{\mu}^{(\infty)}(\mathbf{z}; q)$ their $k \to \infty$ limit (with fugacity parameter $\mathbf{z}$). Having found an expression for the K-matrices of these CFTs will thus give a natural guess for an explicit expression of these level-k truncated Kostka polynomials. We will explore this further in Section 6.

For completeness, let us recall the value of the central charge c_{ALA}, of a CFT with affine Lie algebra symmetry $\hat{\mathfrak{g}}$ at level k,

$$c_{\mathrm{ALA}} = \frac{k \dim \mathfrak{g}}{k + \mathrm{h}^\vee}, \tag{2.13}$$

where $\mathrm{h}^\vee$ is the dual Coxeter number corresponding to $\mathfrak{g}$.

For convenience, throughout this paper we will denote the (untwisted) affine Lie algebra at level k, corresponding to a finite-dimensional Lie algebra X_n, by $X_{n,k}$, rather than by $(X_n^{(1)})_k$ which is more common in the literature.

2.1.3. Exclusion statistics

The starting point of the discussion on exclusion statistics will be an ideal gas of particles which satisfy 'fractional (exclusion) statistics' [27].

The one particle grand canonical partition functions λ_i for a set of quasiparticles obeying fractional exclusion statistics can be obtained from the IOW equations [31]

$$\frac{\lambda_i - 1}{\lambda_i} \prod_j \lambda_j^{\mathbb{K}_{ij}^{\mathrm{st}}} = x_i, \tag{2.14}$$

where $\mathbb{K}^{\mathrm{st}}$ is the 'statistics matrix' and $x_i = z_i q = e^{\beta\mu_i} e^{-\beta\varepsilon}$ the fugacity. Here, μ_i is the chemical potential of species i and ε the energy. Under the assumption of a symmetric matrix $\mathbb{K}^{\mathrm{st}}$, the one particle distribution functions follow:

$$n_i(\varepsilon) = x_i \frac{\partial}{\partial x_i} \log \prod_j \lambda_j \bigg|_{x_i = e^{\beta(\mu_i - \varepsilon)}} = \sum_j x_j \frac{\partial}{\partial x_j} \log \lambda_i \bigg|_{x_i = e^{\beta(\mu_i - \varepsilon)}}. \tag{2.15}$$

These distribution functions are in general interpolations between the Bose–Einstein and Fermi–Dirac distribution functions.

The discussion above holds in the case of Abelian statistics, but can be generalized to the non-Abelian case [2,3]. Non-Abelian statistics arises when quasiparticle operators (chiral vertex operators, see below) in the underlying CFT have non-trivial fusion rules. The effect of these fusion rules can be taken into account via so-called ‘pseudoparticles’, which do not carry any energy (i.e., $q = 1$). Note that for all the cases we consider, a formulation in which the pseudoparticles have $x = 1$ is possible. In fact, we only consider formulations in which $x = 1$ for the pseudoparticles. More on the relation between fusion rules and pseudoparticles can be found in Section 2.2.

We will now turn to the question of how to calculate the central charge of a system of quasiparticles satisfying exclusion statistics with statistics matrix $\mathbb{K}^{\mathrm{st}}$ (and speak of the central charge associated to the matrix $\mathbb{K}^{\mathrm{st}}$). First, we consider an Abelian system, i.e., a system without pseudoparticles. In that case, the central charge is given by

$$c_{\mathrm{CFT}} = \frac{6}{\pi^2} \int_0^1 \frac{dx}{x} \lambda_{\mathrm{tot}}(x), \tag{2.16}$$

where $\lambda_{\mathrm{tot}}(z)$ denotes the product

$$\lambda_{\mathrm{tot}}(x) = \prod_i \lambda_i(x_j = x). \tag{2.17}$$

By using the IOW-equations the central charge of Eq. (2.16) can be rewritten in the form (see, for instance, [12])

$$c_{\mathrm{CFT}} = \frac{6}{\pi^2} \sum_i L(\xi_i), \tag{2.18}$$

where the ξ_i’s are solutions of the ‘central charge equations’

$$\xi_i = \prod_j (1 - \xi_j)^{\mathbb{K}^{\mathrm{st}}_{ij}}, \tag{2.19}$$

and $L(z)$ is Rogers’ dilogarithm

$$L(z) = -\frac{1}{2} \int_0^z dy \left(\frac{\log y}{1 - y} + \frac{\log(1 - y)}{y} \right). \tag{2.20}$$

The presence of pseudoparticles gives rise to a reduction of the central charge. This reduction can be calculated in a similar way, by considering the central charge equations restricted to the pseudoparticles. For future convenience, we will denote the statistics matrix restricted to the pseudoparticles by $\mathbb{K}_{\psi\psi}$. The central charge equations become

(the prime denotes the restriction to the pseudoparticles)

$$\xi_i' = \prod_j{}' \bigl(1 - \xi_j'\bigr)^{(\mathbb{K}_{\psi\psi})_{ij}}, \tag{2.21}$$

giving rise to a reduction $\frac{6}{\pi^2} \sum_j L(\xi_j')$. The central charge becomes

$$c_{\mathrm{CFT}} = \frac{6}{\pi^2} \left(\sum_i L(\xi_i) - \sum_j{}' L\bigl(\xi_j'\bigr) \right). \tag{2.22}$$

This formula agrees with the central charge calculated from the asymptotics of the UCPF (2.4) (see, e.g., the discussion in [3]).

To summarize the above, we note that the truncated UCPFs in the large $\mathbf{L}$ limit give rise to one particle partition functions (2.11), which are of the form of the IOW-equations (2.14), with statistics matrix $\mathbb{K}^{\mathrm{st}} = \mathbb{K}$. Thus the K-matrix of the UCPF can be interpreted as a matrix which describes the statistical interactions between the (quasi)particles.

The other important point was that in all the cases where conformal field theories were studied by means of quasiparticle bases, Eqs. (2.3) which determine λ_i were shown to be of the form of the IOW-equations.

We end this section by discussing the so-called quantum Hall basis, which turns out to be very convenient for determining and studying K-matrices for conformal field theories.

2.1.4. The quantum Hall basis

A convenient basis for WZW conformal field theories was first proposed in the context of the quantum Hall effect [18]. (This basis is also very natural from the mathematical point of view as it is closely related to the existence of generalizations of the Durfee square formula in combinatorics [8].) The 'electron-like' particles (with unit charge and spin-$\frac{1}{2}$ and (fractionally) charged quasiparticles (sometimes called quasiholes) are chosen to form a basis. It was found that a basis could be chosen in such a way that the statistics matrix $\mathbb{K}_{\mathrm{e}}$ for the electron-like particles, and the matrix $\mathbb{K}_{\mathrm{qp}}$ for the quasiparticles are each others inverse

$$\mathbb{K}_{\mathrm{qp}} = \mathbb{K}_{\mathrm{e}}^{-1}, \tag{2.23}$$

while, furthermore, there is no mutual statistics between the quasiparticles and electrons, i.e.,

$$\mathbb{K} = \mathbb{K}_{\mathrm{e}} \oplus \mathbb{K}_{\mathrm{qp}}. \tag{2.24}$$

This is a very important observation, which will have many consequences. Though this basis was first proposed in the context of the Laughlin and Jain states [18], it was soon realized that a basis with a similar structure could be constructed for the non-Abelian generalizations of the Abelian quantum Hall states [2,3,26]. These non-Abelian generalizations are based upon Wess–Zumino–Witten conformal field theories. In this paper, we will determine bases for general WZW conformal field theories. In the next section, we will review and develop some techniques which are needed to perform this task. Here, we will first explore some consequences of the 'duality' between the electron and quasiparticle sector.

In the description of the quantum Hall effect, the quantum numbers of the particles play an important role, as they are used to calculate physical properties. The most important are the charge and spin quantum numbers, which are usually grouped in the so-called charge and spin vectors, **t** and **s**, respectively (see, for instance, [47]). Denoting a general vector for the electron (quasiparticle) sector by $\mathbf{q}_{\rm e}$ ($\mathbf{q}_{\rm qp}$) we have

$$\mathbf{q}_{\rm qp} = -\mathbb{K}_{\rm e}^{-1} \cdot \mathbf{q}_{\rm e}. \tag{2.25}$$

The filling fraction ν and the spin filling σ are given by the expressions

$$\nu = \mathbf{t}_{\rm e}^{\rm T} \cdot \mathbb{K}_{\rm e}^{-1} \cdot \mathbf{t}_{\rm e} = \mathbf{t}_{\rm qp}^{\rm T} \cdot \mathbb{K}_{\rm qp}^{-1} \cdot \mathbf{t}_{\rm qp}, \qquad \sigma = \mathbf{s}_{\rm e}^{\rm T} \cdot \mathbb{K}_{\rm e}^{-1} \cdot \mathbf{s}_{\rm e} = \mathbf{s}_{\rm qp}^{\rm T} \cdot \mathbb{K}_{\rm qp}^{-1} \cdot \mathbf{s}_{\rm qp}. \tag{2.26}$$

These quantities are important physically; from a mathematical point of view they are interesting, as they are conserved by the W- and P-transformations of Section 3. In a sense, these transformations are constructed in such a way that they have this property.

Let us explore some consequences of the duality, in particular Eqs. (2.23) and (2.24). We will focus on the thermodynamic properties first and have a closer look at the IOW-equations (2.14). We will denote the one particle distribution functions for the electron-like particles and quasiparticles by μ_i and λ_i, respectively. The corresponding fugacities are given by y_i and x_i. Thus, the μ_i and λ_i are the solutions to the equations

$$\frac{\mu_i - 1}{\mu_i} \prod_j \mu_j^{(\mathbb{K}_{\rm e})_{ij}} = y_i, \qquad \frac{\lambda_i - 1}{\lambda_i} \prod_j \lambda_j^{(\mathbb{K}_{\rm qp})_{ij}} = x_i. \tag{2.27}$$

Now Eq. (2.23) leads to the following relations

$$\lambda_i = \frac{\mu_i - 1}{\mu_i}, \qquad x_i = \prod_j y_i^{-(\mathbb{K}_{\rm e})_{ij}^{-1}}. \tag{2.28}$$

Another important feature of the basis described in this section is that the presence of pseudoparticles in the quasiparticle matrix $\mathbb{K}_{\rm qp}$ is accompanied by the presence of so-called 'composite' particles in the electron matrix $\mathbb{K}_{\rm e}$. The reason for this will become clear in Section 3. In general, the matrix $\mathbb{K}_{\rm e}$ contains a few 'electrons' (particles with unit charge and spin up or down), with fugacities y. In addition, there are composite particles, with fugacities y^{l_i}, where the l_i are positive integers. The quantum numbers of the composites in the electron sector are integer multiples of the quantum numbers of the electrons. In the presence of composites in the electron sector, there will be pseudoparticles in the quasiparticle sector. Pseudoparticles have $x = 1$, and as a consequence, pseudoparticles will have all quantum numbers equal to zero. In principle, the fugacity of pseudoparticles might be of the more general form x_i/x_j (compare Eqs. (3.19) and (3.20)), but in all cases we will consider, this will not be the case. Also, physical particles with all quantum numbers trivial might occur, but again, we will not encounter such a situation in this paper.

In the following, we will only encounter the situation where the electron sector has composites, but no pseudoparticles, while the quasiparticle sector does contain pseudoparticles, but no composites. Thus, we will assume that the quasiparticle matrix has the following form:

$$\mathbb{K}_{\mathrm{qp}} = \left(\begin{array}{c|c} \mathbb{K}_{\psi\psi} & \mathbb{K}_{\psi\phi} \\ \hline \mathbb{K}_{\phi\psi} & \mathbb{K}_{\phi\phi} \end{array}\right),$$
$$\mathbb{K}^{\mathrm{T}}_{\psi\psi} = \mathbb{K}_{\psi\psi}, \qquad \mathbb{K}^{\mathrm{T}}_{\phi\phi} = \mathbb{K}_{\phi\phi}, \qquad \mathbb{K}^{\mathrm{T}}_{\psi\phi} = \mathbb{K}_{\phi\psi}, \tag{2.29}$$

where $\mathbb{K}_{\phi\phi}$ denotes the statistic matrix for the physical (as opposed to pseudo) quasi-particles and $\mathbb{K}_{\psi\phi}$ the mutual statistics between the pseudo- and physical particles.

In the presence of composites and pseudoparticles, we have to generalize the definition of λ_{tot} (see Eq. (2.17)) to

$$\lambda_{\mathrm{tot}}(x) = \prod_i \left[\lambda_i\left(x_j = x^{l_j}\right)\right]^{l_i}. \tag{2.30}$$

With this definition, the central charge is still given by Eq. (2.16). In the absence of pseudoparticles, the central charge associated to the system $\mathbb{K}_{\mathrm{e}} \oplus \mathbb{K}_{\mathrm{qp}}$, is simply given by the rank n of the matrix $\mathbb{K}_{\mathrm{e}}$ (see, for instance, [3]). To show this, we take a look at the central charge equations

$$\zeta_i = \prod_j (1-\zeta_j)^{\mathbb{K}_{\mathrm{e}ij}}, \qquad \xi_i = \prod_j (1-\xi_j)^{\mathbb{K}_{\mathrm{qp}ij}}. \tag{2.31}$$

Now because of the fact that $\mathbb{K}_{\mathrm{qp}} = \mathbb{K}_{\mathrm{e}}^{-1}$, the solutions to these equations ζ_i and ξ_i are simply related by $\xi_i = 1 - \zeta_i$. We find the central charge to be

$$c_{\mathrm{CFT}} = \frac{6}{\pi^2}\sum_i \bigl(L(\xi_i) + L(1-\xi_i)\bigr) = \frac{6}{\pi^2} n L(1) = n, \tag{2.32}$$

by using the dilogarithm relation

$$L(z) + L(1-z) = L(1) = \frac{\pi^2}{6}. \tag{2.33}$$

In the case pseudoparticles are present, we again have a simple subtraction (see Eq. (2.22), the prime denotes the restriction to the pseudoparticles)

$$c_{\mathrm{CFT}} = n - \frac{6}{\pi^2}\sum_j{}' L(\xi'_j). \tag{2.34}$$

It is important to note that the knowledge of the K-matrix is not enough to specify the theory completely. In addition, one has to know, or rather specify, which particles are pseudoparticles. So two theories can have the same K-matrix, but differ in the 'particle content' and thereby (for instance) have different central charge. We will encounter this situation frequently, namely as we discuss the K-matrices for CFTs with affine Lie algebra symmetry, in cases the Lie algebra is non-simply-laced.

2.2. Pseudoparticles and fusion rules

There is an intimate connection between the pseudoparticle K-matrix $\mathbb{K}_{\psi\psi}$ and the fusion rules of a CFT, which can be used as a consistency check or guiding principle on the construction of K-matrices. To explain this connection, consider a CFT with fusion rules ${N_{ij}}^k$, $i, j, k = 1, \ldots, \ell$. The incidence matrix of the fusion graph Γ_i, corresponding to taking consecutive fusions with the field i, is given by the matrix N_i with components ${(N_i)_j}^k = {N_{ij}}^k$. Hence, if ${P_{ij}}^k(M)$ denotes the number of paths of length M on the fusion graph Γ_i beginning at j and ending at k we have

$$ {P_{ij}}^k(M) = {\big((N_i)^M\big)_j}^k. \tag{2.35} $$

Thus we find a recursion relation

$$ {P_{ij}}^k(M) = \sum_l {P_{ij}}^l(N)\,{P_{il}}^k(M-N), \tag{2.36} $$

for each $0 \leqslant N \leqslant M$, with initial condition ${P_{ij}}^k(0) = {\delta_j}^k$. These recursion relations, however, involve paths beginning and ending at arbitrary points. To derive a recursion relation for fixed j and k we apply the characteristic equation of N_i, i.e., the ℓth order polynomial equation for N_i arising from the eigenvalue equation, to ${P_{ij}}^k(M)$. If the characteristic equation is given as

$$ \sum_{n=0}^{\ell} a_n (N_i)^{\ell-n} = 0, \quad a_0 \equiv 1, \tag{2.37} $$

then, by using (2.36) for $N = 1$, we find the recursion relation

$$ \sum_{n=0}^{\ell} a_n {P_{ij}}^k(M-n) = 0. \tag{2.38} $$

That is, a recursion relation for fixed j and k and with coefficients independent of j and k. Different solutions of (2.38), determined by different initial conditions, correspond to different choices of j and k.[3] In particular, asymptotically the number of paths is given by $(\lambda_{\max})^M$, where $\lambda_{\max}$ is the largest eigenvalue of N_i.

On the other hand, according to the UCPF assumption, the number of paths $P(M)$ of length M on the fusion graph Γ_i is given in terms of the $q \to 1$ limit of the UCPF (2.4), i.e.,

$$ P_{\mathbf{L}} = \sum_{m_i} \prod_i \begin{pmatrix} ((\mathbb{I} - \mathbb{K}_{\psi\psi}) \cdot \mathbf{m})_i + L_i \\ m_i \end{pmatrix}, \tag{2.39} $$

where $L_i = a_i M + u_i$ and $\mathbb{K}_{\psi\psi}$ the pseudoparticle K-matrix. The numbers a_i are fixed (only depend on the sector i), in fact they arise as the part of the K-matrix describing the coupling of the pseudoparticles to physical particles, while u_i is determined by begin and end point of the path. (The q-analogue of Eq. (2.39) is related to (level restricted) Kostka polynomials and will be discussed in Section 6.) The numbers $P_{\mathbf{L}}$ satisfy the recursion relations (cf. (2.9))

$$ P_{\mathbf{L}} = P_{\mathbf{L}-\mathbf{e}_i} + P_{\mathbf{L}-\mathbb{K}_{\psi\psi}\cdot\mathbf{e}_i}, \tag{2.40} $$

[3] In fact, for specific initial conditions, the solution might actually satisfy simpler recursion relations obtained by factorizing the characteristic equation and taking a subset of the factors.

where $\mathbf{e}_i$ is the unit vector in the ith direction. In principle, the recursion relations (2.40) can be manipulated to yield a recursion relation for $P(M) \equiv P_{a_i M}$, the quantity of interest. Ideally, this recursion relation should be the same as (2.38). In practice, however, one finds that one corresponds to a factor of the other due to the fact we are dealing with specific initial conditions. In practice, it is easier to study the recursion relations (2.40) in the large M limit, where they reduce to the IOW-equations (2.11). These can then be used to derive an equation for $\mu = \prod_i \lambda_i^{a_i}$ which should correspond to the characteristic equation for the eigenvalues of ${N_{ij}}^k$, i.e., Eq. (2.37). In particular, the largest root of the equation determining μ should be equal to $\lambda_{\max}$.

Moreover, note that while the recursion relations corresponding to graphs on Γ_i depend on the sector i, they should all derive from one and the same pseudoparticle matrix $\mathbb{K}_{\psi\psi}$ (they just differ in the choice of a_i). This puts extra constraints on the possible choices of $\mathbb{K}_{\psi\psi}$, given a set of fusion rules ${N_{ij}}^k$. Unfortunately, this still does not suffice to uniquely associate a pseudoparticle $\mathbb{K}_{\psi\psi}$ with a set of fusion rules ${N_{ij}}^k$ as is illustrated, for instance, by the matrix

$$\mathbb{K}_{\psi\psi} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}, \tag{2.41}$$

which arises both in $A_{2,2}$ and $F_{4,1}$ (see Sections 4.3.1 and 4.3.6), while these two theories clearly have different fusion rules. This is because additional information is present in the coupling of pseudoparticles to the physical particles (i.e., the numbers a_i). Conversely, given a pseudoparticle K-matrix leading to the correct fusion rules, one can always construct other K-matrices giving rise to the same recursion relations by extending the matrix 'symmetrically'. An example of this will be given in Section 2.3.

Finally, given a set of fusion rules ${N_{ij}}^k$, we can compute the modular S-matrix, since this is the matrix which simultaneously diagonalizes all matrices N_i [46]. Since the T-matrix acts diagonally on the characters of the CFT with values $\exp(2\pi i(h_i - c/24))$, we can find constraints on the conformal dimensions h_i and the central charge c from the condition $(ST)^3 = 1$ (when $S^2 = 1$) or $(ST)^6 = 1$ (when $S^4 = 1$).

The central charge constraint in particular can be compared to the central charge (2.34) arising from a particular choice of pseudoparticle K-matrix. Obviously, the constraints on which fusion rules correspond to which pseudoparticle K-matrix derived this way are much weaker than those arising from the comparison of the above recursion relations.

2.3. Simple examples

Let us illustrate the considerations of the previous section in a few examples.

Consider a CFT with two primary fields 1 and ϕ and non-trivial fusion rule $\phi \times \phi = 1$, i.e.,

$$N_\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.42}$$

which has eigenvalues $\lambda = \pm 1$ and is diagonalized by

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.43}$$

which satisfies $S^2 = 1$. We find that $(ST)^3 = 1$ yields the condition

$$h_\phi = \frac{1}{4} \bmod \frac{1}{2}, \tag{2.44}$$

while

$$c = \begin{cases} 1 \bmod 8 & \text{for } h_\phi = \dfrac{1}{4} \bmod 1, \\ 7 \bmod 8 & \text{for } h_\phi = \dfrac{3}{4} \bmod 1. \end{cases} \tag{2.45}$$

Clearly, $A_{1,1}$ is an example of the first possibility, while $E_{7,1}$ is an example of the second.

Since c is necessarily an integer, one would conclude that as far as this calculation is concerned no pseudoparticles are necessary. The characteristic equation for N_ϕ is given by $\lambda^2 - 1 = 0$ and leads to the recursion

$$P(M) = P(M-2), \tag{2.46}$$

which is trivially solved by $P(2M) = P(0)$ and $P(2M+1) = P(1)$. Again, this does not require pseudoparticles, since the fusion paths are obviously unique.

Now consider $A_{1,k}$ for generic level k. The fusion matrix of the generating field ϕ_2 is given by the incidence matrix of the Dynkin diagram of A_{k+1} (see, for example, [23]). The characteristic equation is thus given by

$$\sum_{j=1}^{[(k+1)/2]} (-1)^j \binom{k+1-j}{j} \lambda^{k+1-2j} = 0, \tag{2.47}$$

and has roots (see, e.g., [23])

$$\lambda_j = 2\cos\left(\frac{\pi j}{k+2}\right), \quad j = 1, \ldots, k+1. \tag{2.48}$$

For example, the characteristic equation at the first few levels is given by

$$\begin{aligned} &k = 1, \quad \lambda^2 - 1 = 0, \\ &k = 2, \quad \lambda(\lambda^2 - 1) = 0, \\ &k = 3, \quad \lambda^4 - 3\lambda^2 + 1 = (\lambda^2 + \lambda - 1)(\lambda^2 - \lambda - 1) = 0, \\ &k = 4, \quad \lambda^5 - 4\lambda^3 + 3\lambda = \lambda(\lambda^2 - 3)(\lambda^2 - 1) = 0. \end{aligned} \tag{2.49}$$

On the other hand, the pseudoparticle K-matrix for $A_{1,k}$, is known to be $\mathbb{K}_{\psi\psi} = \frac{1}{2}\mathbf{A}_{k-1}$, while $\mathbf{a} = (\frac{1}{2}, 0, \ldots, 0)$. This leads to, e.g.,

$$\begin{aligned} &k = 2, \quad \mu^2 - 1 = 0, \\ &k = 3, \quad \mu^2 - \mu - 1 = 0. \end{aligned} \tag{2.50}$$

which, in general, corresponds to a factor of (2.49) as discussed in Section 2.2.

As a final example consider a CFT with two primary fields 1 and ϕ and fusion rule $\phi \times \phi = 1 + \phi$, i.e.,

$$N_\phi = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.51}$$

The characteristic equation is given by

$$\lambda^2 - \lambda - 1 = 0, \tag{2.52}$$

with roots $\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{5})$. The constraints on h and c, arising from the modular matrices, are (see, e.g., [23], Exercise 10.16)

$$c - 12h = -2 \bmod 8, \tag{2.53}$$

while

$$h = \frac{m}{5} \bmod 1, \quad m = 1, 2, 3, 4. \tag{2.54}$$

$G_{2,1}$ is an example of a solution for $m = 2$ ($c = 14/5$, $h = 2/5$), while $F_{4,1}$ is an example of a solution for $m = 3$ ($c = 26/5$, $h = 3/5$). Examples of $m = 1, 4$ solutions can be found among the minimal (non-unitary) models.

The characteristic equation (2.52) leads to the recursion relation

$$P(M) = P(M-1) + P(M-2), \tag{2.55}$$

the solutions of which are (generalized) Fibonacci numbers. Clearly, the recursion relation (2.55) arises from the pseudoparticle matrix (cf. (2.40))

$$\mathbb{K} = (2), \tag{2.56}$$

with $\mathbf{a} = (1)$.

The central charge subtraction corresponding to (2.56) is, according to (2.34), given by

$$\frac{6}{\pi^2} L\left(\frac{3}{2} - \frac{1}{2}\sqrt{5}\right) = \frac{2}{5}, \tag{2.57}$$

which is not the correct subtraction for either G_2 or F_4. We can however double the subtraction while, at the same time, keeping the recursion relation, by a 'symmetric doubling' of (2.56), i.e., by making a 2×2 matrix with entries that sum to 2 in all columns and rows and which is such that the solution to the IOW-equation is identical for all components, e.g.,

$$\mathbb{K} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}, \tag{2.58}$$

with, $\mathbf{a} = (a_1, a_2)$ where $a_1 + a_2 = 1$. This case is relevant for $(F_4^{(1)})_{k=1}$ (see Section 4.3.6). To get a subtraction of $6/5$, as needed for $(G_2^{(1)})_{k=1}$, we need to do a 'symmetric tripling' such as

$$\mathbb{K} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}. \tag{2.59}$$

Cf. Section 4.3.7.

3. Composite and dual composite construction

As is well known in the context of the quantum Hall effect, the K-matrices describing the Abelian quantum Hall states are not unique, but are in fact determined up to similarity transformations. These similarity transformations can be thought of as changing the basis for the description. Moreover, the physical properties such as the filling fraction are not changed by this transformation. Also the central charge is left unchanged.

A similar situation occurs when we want to view the K-matrices as the data for a general (i.e., non-Abelian) CFT. There exist transformations of the K-matrices, which leave the corresponding characters unchanged. Therefore, the K-matrices related by such a transformation correspond to the same theory. A prime example will be described in Section 3.2 and the dual version in Section 3.3. At first sight, this might be a disturbing observation because we would like to have a unique description of the theory. However, the situation can be used in our advantage, for instance, in the construction of the K-matrices for general affine Lie algebra CFTs, as will be pointed out in Section 3.4.

3.1. W-transformations

To describe the well-known W-transformations (see, for instance, [47]), we will use the notation of the fqH basis (as we will do in the rest of this section). Of course, it is applicable to all Abelian quantum Hall systems. So we have a K-matrix $\mathbb{K}_{\mathrm{e}}$ and the quantum number vectors $\mathbf{q}_{\mathrm{e}}$ (the dual data is obtained by applying Eqs. (2.23) and (2.25)). Let $\mathbb{W}$ be an $SL(n,\mathbb{Z})$ matrix, where n is the rank of $\mathbb{K}$. The W-transformation takes the form

$$\widetilde{\mathbb{K}}_{\mathrm{e}} = \mathbb{W}\cdot\mathbb{K}_{\mathrm{e}}\cdot\mathbb{W}^{\mathrm{T}}, \qquad \widetilde{\mathbb{K}}_{\mathrm{qp}} = \left(\mathbb{W}^{-1}\right)^{\mathrm{T}}\cdot\mathbb{K}_{\mathrm{qp}}\cdot\mathbb{W}^{-1}, \tag{3.1}$$

while

$$\tilde{\mathbf{q}}_{\mathrm{e}} = \mathbb{W}\cdot\mathbf{q}_{\mathrm{e}}, \qquad \tilde{\mathbf{q}}_{\mathrm{qp}} = \left(\mathbb{W}^{-1}\right)^{\mathrm{T}}\cdot\mathbf{q}_{\mathrm{qp}}. \tag{3.2}$$

Indeed, physical quantities of the form $\mathbf{q}_{\mathrm{e}}^{\mathrm{T}}\cdot\mathbb{K}_{\mathrm{e}}^{-1}\cdot\mathbf{q}_{\mathrm{e}}$, such as the filling fraction are invariant under this transformation. Also, the central charge, which is given by n for the Abelian states, is not changed. In the non-Abelian case, we can also apply these W-transformations, however, to conserve the central charge, we can only use those transformations which do not change the pseudoparticle part of the K-matrix.

In the following, we will concentrate on constructions based on character identities (so we view the K-matrices as matrices containing CFT data). In addition, we will show that extended matrices obtained in this way can be used to make a reduction of the theory, which turns out to be closely related to the W-transformations described above. We will use the results of this section extensively in the remainder of this paper, in particular in Section 4, where we will obtain the K-matrices for general affine Lie algebra CFTs.

3.2. Composite construction

The basic 'transformation' one can do on a K-matrix, leaving the theory invariant, is the composite construction [3]. The effect of this transformation is to add a particle, which is the composite of two particles already present in the theory. The quantum numbers of this composite particle are just the sum of the quantum numbers of the two constituent particles. In order to keep the theory unchanged, one has to increase the mutual exclusion statistics of the two constituent particles. In a sense, they avoid one another more, while the gap is filled by the composite particle.

To make this more precise, consider the IOW-equations (2.14) with a symmetric matrix $\mathbb{K}_\mathrm{e}$ (i.e., $a_{12}=a_{21}$ and $\overline{\mathbb{K}}_\mathrm{e}=\overline{\mathbb{K}}_\mathrm{e}^\mathrm{T}$), fugacities $\mathbf{y}$ and quantum numbers $\mathbf{q}_\mathrm{e}$

$$\mathbb{K}_\mathrm{e}=\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn}\end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & \mathbf{a}_1^\mathrm{T} \\ a_{21} & a_{22} & \mathbf{a}_2^\mathrm{T} \\ \mathbf{a}_1 & \mathbf{a}_2 & \overline{\mathbb{K}}_\mathrm{e}\end{pmatrix},$$
$$\mathbf{y}=\begin{pmatrix} y_1 \\ y_2 \\ \bar{\mathbf{y}}\end{pmatrix},\qquad \mathbf{q}_\mathrm{e}=\begin{pmatrix} q_{\mathrm{e},1} \\ q_{\mathrm{e},2} \\ \bar{\mathbf{q}}_\mathrm{e}\end{pmatrix}. \tag{3.3}$$

If we define the operation $\mathcal{C}_{12}$, corresponding to adding a composite of the quasiparticles 1 and 2 to the system, by

$$\mathcal{C}_{12}\mathbb{K}_\mathrm{e}=\begin{pmatrix} a_{11} & a_{12}+1 & a_{11}+a_{12} & \mathbf{a}_1^\mathrm{T} \\ a_{21}+1 & a_{22} & a_{21}+a_{22} & \mathbf{a}_2^\mathrm{T} \\ a_{11}+a_{21} & a_{12}+a_{22} & a_{11}+2a_{12}+a_{22} & \mathbf{a}_1^\mathrm{T}+\mathbf{a}_2^\mathrm{T} \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_1+\mathbf{a}_2 & \overline{\mathbb{K}}\end{pmatrix}, \tag{3.4}$$

and

$$\mathcal{C}_{12}\mathbf{y}=\begin{pmatrix} y_1 \\ y_2 \\ y_1y_2 \\ \bar{\mathbf{y}}\end{pmatrix},\qquad \mathcal{C}_{12}\mathbf{q}_\mathrm{e}=\begin{pmatrix} q_{\mathrm{e},1} \\ q_{\mathrm{e},2} \\ q_{\mathrm{e},1}+q_{\mathrm{e},2} \\ \bar{\mathbf{q}}_\mathrm{e}\end{pmatrix}, \tag{3.5}$$

then the two systems are equivalent, at least at the level of thermodynamics. The action of the general $\mathcal{C}_{ij}$ is defined, as above, by a suitable permutation of the rows and columns. The solutions $\{\mu_i\}$ to the IOW-equations defined by $(\mathbb{K}_\mathrm{e},\mathbf{y})$ and $\{\mu'_i\}$ defined by $(\mathbb{K}'_\mathrm{e},\mathbf{y}')=(\mathcal{C}_{ij}\mathbb{K}_\mathrm{e},\mathcal{C}_{ij}\mathbf{y})$ are simply related by

$$\mu'_i=\frac{\mu_i+\mu_j-1}{\mu_j},\qquad \mu'_j=\frac{\mu_i+\mu_j-1}{\mu_i},$$
$$\mu'_{n+1}=\frac{\mu_i\mu_j}{\mu_i+\mu_j-1},\qquad \mu'_k=\mu_k,\quad k\neq i,j,n+1. \tag{3.6}$$

Note that, in particular, it follows $\mu_i=\mu'_i\mu'_{n+1}$ and $\mu_j=\mu'_j\mu'_{n+1}$ such that $\mu_\mathrm{tot}=\mu'_\mathrm{tot}$. Also, from $\mu_i=\mu'_i\mu'_{n+1}$ and $\mu_j=\mu'_j\mu'_{n+1}$ one sees that the original one particle partition functions for i and j, receive contributions from the new particles i and j, respectively, as well as from the composite particle $n+1$. The operation $\mathcal{C}_{ij}$ has the effect that states in the spectrum containing both particles i and j get less dense (their mutual exclusion statistics is bumped up by 1), while the resulting 'gaps' are now filled by the new composite particle.

A consistency check on the equivalence of the systems described by $(\mathbb{K}_\mathrm{e},\mathbf{y})$ and $(\mathbb{K}'_\mathrm{e},\mathbf{y}')=(\mathcal{C}_{ij}\mathbb{K}_\mathrm{e},\mathcal{C}_{ij}\mathbf{y})$ is the fact that both lead to the same central charge. It was shown in [9] that this is in fact a consequence of the five-term identity for Rogers' dilogarithm.

For completeness, we repeat the argument here. It is not hard to check that the solutions to the Eqs. (2.19), with $\mathbb{K}_{\mathrm{e}}$ and $\mathcal{C}_{ij}\mathbb{K}_{\mathrm{e}}$, which we will denote by ζ_i and ζ_i', respectively, are related by

$$\zeta_i' = \frac{\zeta_i(1-\zeta_j)}{1-\zeta_i\zeta_j}, \qquad \zeta_j' = \frac{\zeta_j(1-\zeta_i)}{1-\zeta_i\zeta_j},$$
$$\zeta_{n+1}' = \zeta_i\zeta_j, \quad \zeta_k' = \zeta_k, \quad k \neq i, j, n+1. \tag{3.7}$$

The equivalence of the central charge for both matrices follows from

$$L(x) + L(y) = L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right) + L(xy), \tag{3.8}$$

which is the five-term identity for Rogers' dilogarithm.

Finally, we note that the composite transformation (3.4) can be derived from the following character identity, which is a special case of the q-Pfaff–Saalschütz sum (see [24])

$$\begin{bmatrix} M_1 \\ m_1 \end{bmatrix}\begin{bmatrix} M_2 \\ m_2 \end{bmatrix} = \sum_{m\geqslant 0} q^{(m_1-m)(m_2-m)} \begin{bmatrix} M_1 - m_2 \\ m_1 - m \end{bmatrix}\begin{bmatrix} M_2 - m_1 \\ m_2 - m \end{bmatrix} \times \begin{bmatrix} M_1 + M_2 - (m_1+m_2) + m \\ m \end{bmatrix}. \tag{3.9}$$

If one inserts this identity at the (i,j)th entry in the UCPF of Eq. (2.4), one finds, after shifting the summation variables $m_i \mapsto m_i - m$ and $m_i \mapsto m_i - m$, another UCPF, based on the data $(\mathcal{C}_{ij}\mathbb{K}, \mathcal{C}_{ij}\mathbf{y})$.

The form (3.9) is used for the composite construction on two pseudoparticles. Taking the limit $M_1 \to \infty$ ($M_1, M_2 \to \infty$) by using Eq. (2.6), gives the appropriate identity for the composite construction applied to a physical and a pseudoparticle (two physical particles), respectively.

3.3. Dual composite construction

Using the logic of the fqH basis, one might expect that upon inverting the extended matrix $\mathcal{C}_{ij}\mathbb{K}_{\mathrm{e}}$, one should find a matrix, which is related to $\mathbb{K}_{\mathrm{qp}} = \mathbb{K}_{\mathrm{e}}^{-1}$ by a character identity as well. This turns out to be the case.

We will denote this transformation by $\mathcal{D}_{ij}$, thus we define $\mathcal{D}_{ij}\mathbb{K}_{\mathrm{qp}} = (\mathcal{C}_{ij}\mathbb{K}_{\mathrm{qp}}^{-1})^{-1}$. After performing this transformation, the quasiparticles corresponding to i and j have become pseudoparticles. This is necessary, because otherwise the central charge of the transformed system $\widetilde{\mathbb{K}} = \mathcal{C}_{ij}\mathbb{K}_{\mathrm{e}} \oplus \mathcal{D}_{ij}\mathbb{K}_{\mathrm{qp}}$ would have been increased by one with respect to $\mathbb{K}_{\mathrm{e}} \oplus \mathbb{K}_{\mathrm{qp}}$, because the rank of the K-matrices is increased by one. The presence of the extra pseudoparticles reduces the central charge by precisely the right amount, to keep the total central charge the same (see below).

The action of $\mathcal{D}_{ij}$ on a symmetric matrix $\mathbb{K}_{\mathrm{qp}}$, in the case of two (physical) particles, can be described in the following way:

$$\mathbb{K}_{\mathrm{qp}} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \mathcal{D}_{12}\mathbb{K}_{\mathrm{qp}} = \frac{1}{\Delta}\begin{pmatrix} 1 & \Delta-1 & a-b-1 \\ \Delta-1 & 1 & c-b-1 \\ a-b-1 & c-b-1 & (1+b)^2-ac \end{pmatrix}, \tag{3.10}$$

where $\Delta = 2-(a-2b+c)$. In addition, in the transformed formulation, the particles 1 and 2 are pseudoparticles. When, in the original formulation, the particles i and j are physical, it is easily verified that the reduction of the central charge, in the transformed formulation, due to the particles i and j is in fact equal to one. This is precisely the value needed to give the transformed system the same central charge as the original formulation, as was to be expected.

The action of $\mathcal{D}_{12}$ on the fugacity and quantum number vectors $\mathbf{x}^{\mathrm{T}} = (x_1, x_2)$ and $\mathbf{q}_{\mathrm{qp}}^{\mathrm{T}} = (q_{\mathrm{qp},1}, q_{\mathrm{qp},2})$ is given by

$$\mathcal{D}_{12}\mathbf{x} = \begin{pmatrix} \left(\frac{x_1}{x_2}\right)^{1/\Delta} \\ \left(\frac{x_2}{x_1}\right)^{1/\Delta} \\ x_1^{(1+b-c)/\Delta} x_2^{(1+b-a)/\Delta} \end{pmatrix},$$

$$\mathcal{D}_{12}\mathbf{q}_{\mathrm{qp}} = \frac{1}{\Delta}\begin{pmatrix} q_{\mathrm{qp},1} - q_{\mathrm{qp},2} \\ q_{\mathrm{qp},2} - q_{\mathrm{qp},1} \\ (1+b-c)q_{\mathrm{qp},1} + (1+b-a)q_{\mathrm{qp},2} \end{pmatrix}. \tag{3.11}$$

If we have $x_1 = x_2 = x$ and hence, $q_{\mathrm{qp},1} = q_{\mathrm{qp},2} = \tilde{q}_{\mathrm{qp}}$, as will always be the case in this paper, we find

$$\mathcal{D}_{12}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ x \end{pmatrix}, \qquad \mathcal{D}_{12}\mathbf{q}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ 0 \\ \tilde{q}_{\mathrm{qp}} \end{pmatrix}. \tag{3.12}$$

From a character identity point of view, the transformation (3.10) is based on the q-binomial doubling formula

$$\begin{bmatrix} M+N \\ n \end{bmatrix} = \sum_{p-q=M-n} q^{(M-p)(N-q)} \begin{bmatrix} M \\ p \end{bmatrix}\begin{bmatrix} N \\ q \end{bmatrix}. \tag{3.13}$$

Indeed, considering the UCPF for two physical particles with $\mathbb{K}_{\mathrm{qp}}$ as in Eq. (3.10), i.e.,

$$Z = \sum \frac{q^{\frac{1}{2}(am_1^2+cm_2^2+2bm_1m_2)}}{(q)_{m_1}(q)_{m_2}} = \sum \frac{q^{\frac{1}{2}(am_1^2+cm_2^2+2bm_1m_2)}}{(q)_{m_1+m_2}} \begin{bmatrix} m_1+m_2 \\ m_1 \end{bmatrix}, \tag{3.14}$$

and then applying (3.13) with

$$\begin{aligned} M &= -(b-c)m_1 + (1+b-a)m_2, \\ N &= (1+b-c)m_1 - (b-a)m_2, \\ n &= m_1, \end{aligned} \tag{3.15}$$

to the q-binomial in (3.14), results in the UCPF based on $\mathcal{D}_{12}\mathbb{K}_{\rm qp}$ of (3.10), with the identifications

$$m'_1 = p, \qquad m'_2 = q, \qquad m'_3 = m_1 + m_2, \tag{3.16}$$

and where the first two particles in $\mathcal{D}_{12}\mathbb{K}_{\rm qp}$ are pseudo.

The general case can be derived from (3.13) as well, and is described in the following way. Again, we will focus on the case where we let $\mathcal{D}$ work on the first two particles. In addition, we will assume that both those particles are physical. For ease of presentation, we now define $\Delta = 2 - (b_{11} - 2b_{12} + b_{22})$, $\delta_1 = 1 + b_{12} - b_{11}$ and $\delta_2 = 1 + b_{12} - b_{22}$.

Using similar notation as in Eq. (3.3), we take (the symmetric) $\mathbb{K}_{\rm qp}$, the fugacities and quantum numbers

$$\mathbb{K}_{\rm qp} = \begin{pmatrix} b_{11} & b_{12} & \mathbf{b}_1^{\rm T} \\ b_{21} & b_{22} & \mathbf{b}_2^{\rm T} \\ \mathbf{b}_1 & \mathbf{b}_2 & \overline{\mathbb{K}}_{\rm qp} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \bar{\mathbf{x}} \end{pmatrix}, \qquad \mathbf{q}_{\rm qp} = \begin{pmatrix} q_{{\rm qp},1} \\ q_{{\rm qp},2} \\ \bar{\mathbf{q}}_{\rm qp} \end{pmatrix}. \tag{3.17}$$

The dual composite construction, applied on the first two particles is given by

$$\mathcal{D}_{12}\mathbb{K}_{\rm qp} = \frac{1}{\Delta} \left(\begin{array}{cc|c|c} 1 & \Delta - 1 & -\delta_1 & \mathbf{b}_1^{\rm T} - \mathbf{b}_2^{\rm T} \\ \Delta - 1 & 1 & -\delta_2 & \mathbf{b}_2^{\rm T} - \mathbf{b}_1^{\rm T} \\ \hline -\delta_1 & -\delta_2 & (1 + b_{12})^2 - b_{11}b_{22} & \delta_2\mathbf{b}_1^{\rm T} + \delta_1\mathbf{b}_2^{\rm T} \\ \hline \mathbf{b}_1 - \mathbf{b}_2 & \mathbf{b}_2 - \mathbf{b}_1 & \delta_2\mathbf{b}_1 + \delta_1\mathbf{b}_2 & \Delta\big(\overline{\mathbb{K}}_{\rm qp}\big)_{ij} + (\mathbf{b}_1 - \mathbf{b}_2)_i(\mathbf{b}_1 - \mathbf{b}_2)_j \end{array} \right). \tag{3.18}$$

The first two particles have become pseudoparticles, while the extra particle is a physical particle. Note that this construction based on the character identity Eq. (3.13) only works in the case that the particles on which it is applied are physical particles. We have not found a character identity for the case where the dual composite construction is applied to two pseudoparticles. However, we will show below that also in that case the central charge works out alright, so we suspect that there is indeed a character identity relating the two systems.

The action of the dual composite construction on the fugacities and quantum number vectors is given by

$$\mathcal{D}_{12}\mathbf{x} = \begin{pmatrix} \left(\frac{x_1}{x_2}\right)^{1/\Delta} \\ \left(\frac{x_2}{x_1}\right)^{1/\Delta} \\ x_1^{\delta_2/\Delta} x_2^{\delta_1/\Delta} \\ \bar{x}_i \left(\frac{x_1}{x_2}\right)^{(\vec{b}_1 - \vec{b}_2)_i/\Delta} \end{pmatrix},$$
$$\mathcal{D}_{12}\mathbf{q}_{\rm qp} = \frac{1}{\Delta} \begin{pmatrix} q_{{\rm qp},1} - q_{{\rm qp},2} \\ q_{{\rm qp},2} - q_{{\rm qp},1} \\ \delta_2 q_{{\rm qp},1} + \delta_1 q_{{\rm qp},2} \\ \Delta\bar{\mathbf{q}}_{\rm qp} + (\mathbf{b}_1 - \mathbf{b}_2)(q_{{\rm qp},1} - q_{{\rm qp},2}) \end{pmatrix}. \tag{3.19}$$

Again, specifying to the situation where $x_1 = x_2 = x$ and $q_{\mathrm{qp},1} = q_{\mathrm{qp},2} = \tilde{q}_{\mathrm{qp}}$, as holds in all the cases we consider, we find

$$\mathcal{D}_{12}\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ x \\ \bar{\mathbf{x}} \end{pmatrix}, \qquad \mathcal{D}_{12}\mathbf{q}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ 0 \\ \tilde{q}_{\mathrm{qp}} \\ \bar{\mathbf{q}}_{\mathrm{qp}} \end{pmatrix}. \tag{3.20}$$

The solutions $\{\lambda_i\}$ to the IOW-equations defined by $(\mathbb{K}_{\mathrm{qp}}, \mathbf{x})$ and $\{\lambda'_i\}$ defined by $(\mathbb{K}'_{\mathrm{qp}}, \mathbf{x}') = (\mathcal{D}_{ij}\mathbb{K}_{\mathrm{qp}}, \mathcal{D}_{ij}\mathbf{x})$ are, as was the case for the composite construction (compare (3.6)), related in a simple way

$$\begin{aligned} &\lambda'_i = \frac{\lambda_i\lambda_j - 1}{\lambda_j - 1}, \qquad \lambda'_j = \frac{\lambda_i\lambda_j - 1}{\lambda_i - 1}, \\ &\lambda'_{n+1} = \lambda_i\lambda_j, \qquad \lambda'_k = \lambda_k, \quad k \neq i, j, n+1. \end{aligned} \tag{3.21}$$

Using the relations (3.21) it is not hard to show that the IOW-equations based the two systems $(\mathbb{K}_{\mathrm{qp}}, \mathbf{x})$ and $(\mathcal{D}_{ij}\mathbb{K}_{\mathrm{qp}}, \mathcal{D}_{ij}\mathbf{x})$ are in fact equivalent. We also find that $\lambda_{\mathrm{tot}} = \lambda'_{\mathrm{tot}}$ by using the fact that the particles i and j are pseudoparticles after the dual composite construction has been applied. The composite particle which is created is a physical (pseudo) particle if particles i and j are physical (pseudo) in the original description.

From Eq. (3.21) it follows that the dual composite construction cannot be applied on a physical and pseudoparticle. In that case, λ'_{tot} cannot be made equal to λ_{tot}. Note that such a restriction does not apply to the composite construction of Section 3.2. Though we do not quite understand this difference, it will not affect any results in this paper.

Let us now focus on the central charge, and look at the case in which all the particles are physical particles first. Because the rank of the transformed matrices is increased by one, we need that the two created pseudoparticles reduce the central charge by one. This is easily verified. Also, because the central charge of the matrix $\mathcal{C}_{ij}\mathbb{K}_{\mathrm{e}}$ equals the central charge of $\mathbb{K}_{\mathrm{e}}$, we need to find the result that the central charge related to $\mathcal{D}_{ij}\mathbb{K}_{\mathrm{qp}}$ *without the pseudoparticle subtraction* equals the central charge related to $\mathbb{K}_{\mathrm{qp}}$ plus one. To show this, we need to relate the solutions to Eqs. (2.19), which we denote by ξ_i and ξ'_i for $\mathbb{K}_{\mathrm{qp}}$ and $\mathcal{D}_{ij}\mathbb{K}_{\mathrm{qp}}$, respectively. The relations are given by

$$\begin{aligned} &\xi'_i = \frac{\xi_i}{\xi_i + \xi_j - \xi_i\xi_j}, \qquad \xi'_j = \frac{\xi_j}{\xi_i + \xi_j - \xi_i\xi_j}, \\ &\xi'_{n+1} = \xi_i + \xi_j - \xi_i\xi_j, \qquad \xi'_k = \xi_k, \quad k \neq i, j, n+1. \end{aligned} \tag{3.22}$$

Because of the relation between the central charges, we require the following dilogarithm identity

$$L(x) + L(y) = L\left(\frac{x}{x + y - xy}\right) + L\left(\frac{y}{x + y - xy}\right) + L(x + y - xy) - L(1), \tag{3.23}$$

which is easily derived from Eq. (3.8) by applying Eq. (2.33) to each term, and making the change of variables $(x \mapsto 1 - x, y \mapsto 1 - y)$.

The argument above not only shows that the central charge works out correctly in the absence of pseudoparticles. It can also be used to show that the reduction of the central charge increases by one if we apply the dual composite construction on pseudoparticles.

What remains to be checked is the central charge if we apply the composite construction to physical particles, while pseudoparticles are present. For this, we need to compare the central charge equations for the original pseudoparticles with the ones where the additional two pseudoparticles are present. Though non-trivial, one can convince oneself that the solutions to the central charge equations of the original pseudoparticles do not change, while the solutions for the two pseudoparticles which are introduced add up to one and therefore increase the reduction by one, which gives the correct result.

3.4. P-transformations

In this section, we will discuss a transformation which is based on the (dual) composite construction. This construction is very useful in determining K-matrices for general affine Lie algebra CFTs. We will motivate this construction by using a simple example, which captures the essence of the method. In the end, this P-transformation is very similar to the W-transformations described in Section 3.1, with one important difference. After applying a P-transformation, some of the physical quasiparticles have transformed into pseudoparticles. One of the consequences of this is a reduction of the central charge.

As we will use the P-transformations mainly as a tool to obtain K-matrices for level-k affine Lie algebras from the direct sum of k level-1 algebras, we will explain the construction using the simplest case. Afterwards, we will present the general case. In the next section, we will use the results obtained here to find the K-matrices we are after.

3.4.1. The case $\mathfrak{sl}(2)_2$

The goal in this section is to obtain the K-matrices for the $\mathfrak{sl}(2)_2$ affine CFT, which describes the Moore–Read (or Pfaffian) quantum Hall state. The corresponding matrices are known, see, for instance, [2,3,45]. Let us recall the K-matrices for the (bosonic) $\nu = 1$ case, which corresponds to $\mathfrak{sl}(2)_2$

$$\mathbb{K}_{\mathrm{e}}^{\mathfrak{sl}(2)_2} = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}, \qquad \mathbf{t}_{\mathrm{e}} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{3.24}$$

The first particle can be identified with the (bosonic) electron, while the second is a composite of two electrons. In the quasiparticle sector

$$\mathbb{K}_{\mathrm{qp}}^{\mathfrak{sl}(2)_2} = \mathbb{K}_{\mathrm{e}}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad \mathbf{t}_{\mathrm{qp}} = -\mathbb{K}_{\mathrm{e}}^{-1} \cdot \mathbf{t}_{\mathrm{e}} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \tag{3.25}$$

where the first particle is a pseudoparticle. The K-matrices for the general Moore–Read state, at filling fraction $\nu = \frac{1}{M+1}$, are obtained by applying the so-called shift map, which is described in detail in [3]. Though the theory for general M has the same central charge, the theory does not have the underlying $\mathfrak{sl}(2)_2$ structure anymore, but rather a deformation (along the charge direction) of this. In this paper, we concentrate on the $M = 0$ case throughout; the K-matrices for general M are obtained by applying the shift map as indicated above. Note that the pseudoparticle matrices $\mathbb{K}_{\psi\psi}$ are unchanged under this shift map.

The main idea is now to obtain these $\mathfrak{sl}(2)_2$ matrices via an embedding of $\mathfrak{sl}(2)_2$ in $\mathfrak{sl}(2)_1 \oplus \mathfrak{sl}(2)_1$ (which we will call an Abelian covering, see also [13]). By introducing a composite, and projecting out some degrees of freedom, we obtain the K-matrices for $\mathfrak{sl}(2)_2$. In physical terms, we start from two, uncoupled, quantum Hall layers with filling $\nu = \frac{1}{2}$ (these are in fact bosonic Laughlin states). In a sense, this state is a covering state for the Moore–Read state at filling $\nu = 1$. By increasing the interactions between the two layers, one might encounter a phase transition to the Moore–Read state, as described in [30]. The bosons form pairs, and condense. In the terminology of an effective Landau–Ginzburg theory (see [22]), the difference of the gauge fields describing the bosons acquires a mass, and decouples from the spectrum. This is the Meissner effect.

On the level of the K-matrices, we can describe this in the following way. We first introduce the composite of the two bosonic particles, and afterwards simply delete (or 'project out') one of the original bosons. So we actually reduced the theory, as required. We start with the direct sum of two $\mathfrak{sl}(2)_1$ K-matrices

$$\mathbb{K}_{\mathrm{e}}^{\mathrm{cover}} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \mathbf{t}_{\mathrm{e}} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3.26}$$

$$\mathbb{K}_{\mathrm{qp}}^{\mathrm{cover}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad \mathbf{t}_{\mathrm{qp}} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}. \tag{3.27}$$

Now, applying the composite and dual composite constructions (Eqs. (3.4) and (3.18)) on these matrices gives the following, equivalent description

$$\widetilde{\mathbb{K}}_{\mathrm{e}} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{pmatrix}, \qquad \tilde{\mathbf{t}}_{\mathrm{e}} = -\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \tag{3.28}$$

$$\widetilde{\mathbb{K}}_{\mathrm{qp}} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}, \qquad \tilde{\mathbf{t}}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}. \tag{3.29}$$

Note that the first two particles of the quasiparticle matrix are pseudoparticles. To obtain the $\mathfrak{sl}(2)_2$ matrices, we have to project out one of these pseudoparticles, by putting it into the vacuum state. In addition, we discard one of the original bosons.

However, while projecting out one of the bosons in the electron sector simply corresponds to deleting the respective row and column in $\widetilde{\mathbb{K}}_{\mathrm{e}}$, projecting out one of the pseudoparticles is more subtle, due to the negative coupling between the pseudoparticles and the physical particle in $\widetilde{\mathbb{K}}_{\mathrm{qp}}$.

For explicitness, consider the UCPF corresponding to $\widetilde{\mathbb{K}}_{\mathrm{qp}}$ of (3.29)

$$\sum \frac{q^{\frac{1}{2}(m_1^2+m_2^2-(m_1+m_2)m_3+\frac{3}{4}m_3^2)}}{(q)_{m_3}} \begin{bmatrix} \frac{1}{2}m_3 \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}m_3 \\ m_2 \end{bmatrix}. \tag{3.30}$$

Due to the minus-sign in the coupling between particles 2 and 3 in $\widetilde{\mathbb{K}}_{\mathrm{qp}}$, the vacuum state for particle 2 is not achieved for $m_2 = 0$, but rather for $m_2 = \frac{1}{2}m_3$. Hence, rather than just omitting particle 2 from $\widetilde{\mathbb{K}}_{\mathrm{qp}}$, we need to set $m_2 = \frac{1}{2}m_3$ in the bilinear form. This results in

$$\mathbf{m}^{\mathrm{T}} \cdot \widetilde{\mathbb{K}}_{\mathrm{qp}} \cdot \mathbf{m} = m_1^2 + m_2^2 - (m_1 + m_2)m_3 + \frac{3}{4}m_3^2$$

$$\to m_1^2 + \left(\frac{1}{2}m_3\right)^2 - \left(m_1 + \frac{1}{2}m_3\right)m_3 + \frac{3}{4}m_3^2 = m_1^2 - m_1 m_3 + \frac{1}{2}m_3^2, \qquad (3.31)$$

which precisely corresponds to the matrix $\mathbb{K}_{\mathrm{qp}}$ of (3.25).

To summarize, the results of projecting out degrees of freedom in Eqs. (3.28) and (3.29), gives rise to the K-matrices of Eqs. (3.24) and (3.25). One of the key points of this section is that there is an elegant way of going from K-matrices for the (Abelian) coverings (Eqs. (3.26) and (3.27)) to the K-matrices of $\mathfrak{sl}(2)_2$, by what we call a 'P-transformation'. This also hold for the general case, as we will show below. We find

$$\mathbb{K}_{\mathrm{e}}^{\mathfrak{sl}(2)_2} = \mathbb{P} \cdot \mathbb{K}_{\mathrm{e}}^{\mathrm{cover}} \cdot \mathbb{P}^{\mathrm{T}}, \qquad \mathbb{K}_{\mathrm{qp}}^{\mathfrak{sl}(2)_2} = \left(\mathbb{P}^{-1}\right)^{\mathrm{T}} \cdot \mathbb{K}_{\mathrm{qp}}^{\mathrm{cover}} \cdot \mathbb{P}^{-1}. \qquad (3.32)$$

The vectors containing the quantum numbers (denoted by $\mathbf{q}_{\mathrm{e}}$ and $\mathbf{q}_{\mathrm{qp}}$) transform as

$$\tilde{\mathbf{q}}_{\mathrm{e}} = \mathbb{P} \cdot \mathbf{q}_{\mathrm{e}}, \qquad \tilde{\mathbf{q}}_{\mathrm{qp}} = \left(\mathbb{P}^{-1}\right)^{\mathrm{T}} \cdot \mathbf{q}_{\mathrm{qp}}. \qquad (3.33)$$

In the above, we have to take $\mathbb{P} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and hence $(\mathbb{P}^{-1})^{\mathrm{T}} = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. A few remarks need to be made here. First of all, the P-transformation described by Eqs. (3.32) and (3.33) closely resembles the W-transformation, as they act on the K-matrices in the same way (compare (3.1)). However, there are a few important differences. As we explained above, upon applying a P-transformation, we introduced a pseudoparticle in the quasiparticle sector. This is important, as the presence of a pseudoparticle changes the theory. For instance, the central charge is reduced, in the case at hand by $1/2$, which is precisely the difference in central charge between $\mathfrak{sl}(2)_1 \oplus \mathfrak{sl}(2)_1$ and $\mathfrak{sl}(2)_2$ (given by $c = 2$ and $c = 3/2$, respectively). So the P-transformation actually changes the theory, while the W-transformation is a basis transformation, which does not change the theory.

In the remainder of this section, we will show how a P-transformation works on a general K-matrix. These results are used in the next section to find the K-matrices for the general affine Lie algebra CFTs, in a similar way as we constructed the $\mathfrak{sl}(2)_2$ matrices above.

3.4.2. The general case

In this section, we will relate the introduction of a composite (in the electron sector), and the corresponding transformation in the quasiparticle sector to a general P-transformation. For notational simplicity, consider introducing a composite of particles 1 and 2 in a general symmetric K-matrix as given by Eq. (3.4). Now, suppose we delete particle 2 from the resulting matrix $\mathcal{C}_{12}\mathbb{K}_{\mathrm{e}}$, we then find a new K-matrix system $(\widetilde{\mathbb{K}}_{\mathrm{e}}, \tilde{\mathbf{q}}_{\mathrm{e}})$ given by

$$\widetilde{\mathbb{K}}_{\mathrm{e}} = \begin{pmatrix} a_{11} & a_{11} + a_{12} & \mathbf{a}_1^{\mathrm{T}} \\ a_{11} + a_{21} & a_{11} + 2a_{12} + a_{22} & \mathbf{a}_1^{\mathrm{T}} + \mathbf{a}_2^{\mathrm{T}} \\ \mathbf{a}_1 & \mathbf{a}_1 + \mathbf{a}_2 & \overline{\mathbb{K}} \end{pmatrix}, \qquad \tilde{\mathbf{q}}_{\mathrm{e}} = \begin{pmatrix} q_1 \\ q_1 + q_2 \\ \bar{\mathbf{q}} \end{pmatrix}. \qquad (3.34)$$

Notice that we can write the relation between $(\widetilde{\mathbb{K}}_{\mathrm{e}}, \tilde{\mathbf{q}}_{\mathrm{e}})$ and $(\mathbb{K}_{\mathrm{e}}, \mathbf{q}_{\mathrm{e}})$ as

$$\widetilde{\mathbb{K}}_{\mathrm{e}} = \mathbb{P} \cdot \mathbb{K}_{\mathrm{e}} \cdot \mathbb{P}^{\mathrm{T}}, \qquad \tilde{\mathbf{q}}_{\mathrm{e}} = \mathbb{P} \cdot \mathbf{q}_{\mathrm{e}}, \qquad (3.35)$$

with

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & \mathbf{0}^{\mathrm{T}} \\ 1 & 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{0} & \mathbb{I} \end{pmatrix}. \qquad (3.36)$$

Now consider the dual composite construction $\mathcal{D}_{12}\mathbb{K}_{\mathrm{qp}}$ (see Eq. (3.18)). In analogy with Eq. (3.30), putting the second pseudoparticle in its vacuum state amounts to setting

$$m_2 = -(\Delta - 1)m_1 + \delta_2 m_3 - (\mathbf{b}_2 - \mathbf{b}_1) \cdot \overline{\mathbf{m}}. \tag{3.37}$$

Substituting this in the quadratic form yields, after a lengthy calculation,

$$\mathbf{m}^{\mathrm{T}} \cdot (\mathcal{D}_{12}\mathbb{K}_{\mathrm{qp}}) \cdot \mathbf{m} \to \mathbf{m}^{\mathrm{T}} \cdot \widetilde{\mathbb{K}}_{\mathrm{qp}} \cdot \mathbf{m}, \tag{3.38}$$

where $\widetilde{\mathbb{K}}_{\mathrm{qp}}$ is given by

$$\widetilde{\mathbb{K}}_{\mathrm{qp}} = \begin{pmatrix} b_{11} - 2b_{12} + b_{22} & b_{12} - b_{22} & \mathbf{b}_1^{\mathrm{T}} - \mathbf{b}_2^{\mathrm{T}} \\ b_{12} - b_{22} & b_{22} & \mathbf{b}_2^{\mathrm{T}} \\ \mathbf{b}_1 - \mathbf{b}_2 & \mathbf{b}_2 & \overline{\mathbb{K}} \end{pmatrix}, \tag{3.39}$$

which is related to $\mathbb{K}_{\mathrm{qp}}$ by

$$\widetilde{\mathbb{K}}_{\mathrm{qp}} = \left(\mathbb{P}^{-1}\right)^{\mathrm{T}} \cdot \mathbb{K}_{\mathrm{qp}} \cdot \mathbb{P}^{-1}, \tag{3.40}$$

with

$$\left(\mathbb{P}^{-1}\right)^{\mathrm{T}} = \begin{pmatrix} 1 & -1 & \mathbf{0}^{\mathrm{T}} \\ 0 & 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{0} & \mathbb{I} \end{pmatrix}, \tag{3.41}$$

in accordance with Eq. (3.36). It is important to note that the first particle of $\widetilde{\mathbb{K}}_{\mathrm{qp}}$ in Eq. (3.39) is a pseudoparticle. The presence of this pseudoparticle causes the reduction of the central charge of the system $\widetilde{\mathbb{K}}_{\mathrm{e}} \oplus \widetilde{\mathbb{K}}_{\mathrm{qp}}$ with respect to $\mathbb{K}_{\mathrm{e}} \oplus \mathbb{K}_{\mathrm{qp}}$. Of course, this is to be expected when degrees of freedom are projected out.

Summarizing, a P-transformation acts on the K-matrices and quantum number vectors (denoted by $\mathbf{q}_{\mathrm{e}}$ and $\mathbf{q}_{\mathrm{qp}}$) as follows:

$$\widetilde{\mathbb{K}}_{\mathrm{e}} = \mathbb{P} \cdot \mathbb{K}_{\mathrm{e}} \cdot \mathbb{P}^{\mathrm{T}}, \qquad \widetilde{\mathbb{K}}_{\mathrm{qp}} = \left(\mathbb{P}^{-1}\right)^{\mathrm{T}} \cdot \mathbb{K}_{\mathrm{qp}} \cdot \mathbb{P}^{-1}, \tag{3.42}$$

and

$$\tilde{\mathbf{q}}_{\mathrm{e}} = \mathbb{P} \cdot \mathbf{q}_{\mathrm{e}}, \qquad \tilde{\mathbf{q}}_{\mathrm{qp}} = \left(\mathbb{P}^{-1}\right)^{\mathrm{T}} \cdot \mathbf{q}_{\mathrm{qp}}, \tag{3.43}$$

where in addition, some of the quasiparticles have been transformed into pseudoparticles.

In Section 4.1 we will repeatedly use the (dual) composite construction combined with the projecting out of degrees of freedom to determine K-matrices for a variety of CFTs. Rather than specifying the particles to which we consecutively apply this construction we will simply state the required resulting P-transformation, and specify which quasiparticles have become pseudoparticles.

Because the P-transformations take the form (3.42), properties such as the filling fraction (see (2.26)), are not changed upon performing the P-transformation. Of course, the statistics properties are changed in a profound way, because the induced pseudoparticles lead to non-trivial fusion rules as described in Section 2.2. In turn, this leads to the non-Abelian statistics of the physical quasiparticles (see, for instance, [38]).

One important remark needs to be made before closing this section. In the construction of the K-matrices, we will use the (dual) composite construction via the P-transformation.

We will always apply the dual composite construction to identical (quasi)particles. Hence, the quantum numbers of the quasiparticles (and also their electronic equivalents) are the same. Moreover, we will always have $a_{ii} = a_{jj}$ and $b_{ii} = b_{jj}$. As a result, it does not matter which of the electron-like particles is projected out. If $a_{ii} \neq a_{jj}$, the two different projections are related by $\mathbb{P}' = \mathbb{P}^{\mathrm{T}}$. The general form for $\mathbb{P}$ we use in this paper will be discussed in the next section (see, in particular, Eq. (4.13)).

4. K-matrices for affine Lie algebras

One of the main themes of this paper is the identification of the K-matrices for general affine Lie algebra CFTs. We will work in the so-called quantum Hall basis, as described above. In [3] (see also [2]), the K-matrices corresponding to the $\mathfrak{sl}(2)_k$ and $\mathfrak{sl}(3)_k$ CFTs were derived. Here, we will give an alternative construction of the $k > 1$ cases directly from the $k = 1$ cases, which can be found in [2]. This construction is based on the embedding of the level-k theory in the direct sum of k level-1 theories. By applying composite and dual composite constructions, we introduce pseudoparticles. After projecting out some of these, we have reduce the theory to the level-k theory. We will phrase all of this in terms of the P-transformations of the previous section. Apart from the $\mathfrak{sl}(2)_k$ and $\mathfrak{sl}(3)_k$ theories, we will also use this construction for the other (simply-laced) affine Lie algebra cases, and provide a few non-trivial checks to show that we indeed found the correct K-matrices. The non-simply-laced cases can be obtained by embedding the level-1 affine algebras into simply-laced algebras, and performing a similar construction as outlined above.

4.1. Constructing the matrices

We will use the techniques described in the previous section to construct the K-matrices for general affine Lie algebras.

In the this section, we will describe how this works in detail for the simplest examples, which have all the characteristics of the general case. Motivation of this construction can be found in the previous section. In Sections 4.2 and 4.3 we will present the results for the K-matrices for general affine Lie algebra CFTs.

4.1.1. Example: the case $\mathfrak{sl}(2)_k$

Let us illustrate the construction for the level $k > 2$ generalizations of the Moore–Read states, the so-called Read–Rezayi states [41]. The covering state in this case is the direct sum of k level-1 theories (instead of just 2 for the MR case). So we have

$$\mathbb{K}_{\mathrm{e}}^{\mathrm{cover}} = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix}, \qquad \mathbf{t}_{\mathrm{e}}^{\mathrm{cover}} = -\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{4.1}$$

(Here, and in the following we use the convention that ‘empty’ entries contain zeroes, if not implied otherwise by ‘dots’.) We also indicated the charge vector, containing the charge quantum numbers of the particles, as the transformation behavior of the quantum numbers under the P-transformation clearly shows that composites are introduced. To obtain the K-matrices for $\mathfrak{sl}(2)_k$, describing the Read–Rezayi states, we need to introduce all types of composites, from a pair up to a cluster made out of the k original particles. Thus $\mathbb{P}$ takes the following form:

$$\mathbb{P} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \ddots & \ddots & \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \tag{4.2}$$

This leads to following matrix $\mathbb{K}_\mathrm{e}$ and charge vector $\mathbf{t}_\mathrm{e}$ (by using Eqs. (3.42) and (3.43))

$$\mathbb{K}_\mathrm{e} = \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 \\ 2 & 4 & 4 & \cdots & 4 \\ 2 & 4 & 6 & \cdots & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 4 & 6 & \cdots & 2k \end{pmatrix}, \qquad \mathbf{t}_\mathrm{e} = -\begin{pmatrix} 1 \\ 2 \\ \vdots \\ k \end{pmatrix}, \tag{4.3}$$

which are indeed correct for the $\mathfrak{sl}(2)_k$ theory. The dual sector is simply obtained by using the duality relations (2.23), (2.25). Alternatively, we can apply the dual P-transformation on the dual (i.e., the inverse) of the covering matrix Eq. (4.1). The corresponding P-matrix is

$$\left(\mathbb{P}^{-1}\right)^\mathrm{T} = \begin{pmatrix} 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix}, \tag{4.4}$$

from which we find

$$\mathbb{K}_\mathrm{qp} = \left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & & & \\ -\frac{1}{2} & 1 & & & \\ & & \ddots & -\frac{1}{2} & \\ & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \hline & & & -\frac{1}{2} & \frac{1}{2} \end{array}\right), \qquad \mathbf{t}_\mathrm{qp} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{pmatrix}. \tag{4.5}$$

From the matrix equation (4.4) we read of that the first $k-1$ particles are pseudoparticles. These results are in perfect agreement with the results of [3,26].

4.1.2. Example: the case $\mathfrak{sl}(3)_k$

As an example of a case where the rank n of the affine Lie algebra is greater than 1, we show that a similar construction can be carried out to obtain the K-matrices related to the $\mathfrak{sl}(3)_k$ CFT. This is the underlying theory of the 'non-Abelian spin-singlet' quantum Hall states as defined in [5]. Finding the K-matrices when the rank $n>1$ is somewhat more complicated than for $n=1$. The K-matrices for the $\mathfrak{sl}(3)_k$ CFT were obtained in [3]. There, the basis was chosen in such a way that all the particles in the electron sector had the same sign for the charge. The reason for this choice was that the electron operators (for spin up and spin down) appearing in the construction of the quantum Hall state have the same sign of the charge. These electron operators are associated to the roots α_1 and $-\alpha_2$ of $\mathfrak{sl}(3)$. From mathematical point of view, it is more natural to work with α_1 and α_2, as the resulting K-matrices have a simpler structure. So here we will present the results using the (mathematically) more natural formulation, based on the positive roots. In Appendix D, we will explain the precise relationship between the two descriptions. Essentially, the relation is a W-transformation on the physical particles, which leaves the pseudoparticles unchanged. This is required, because the pseudoparticles are related to the fusion rules of the affine Lie algebra and they also determine the central charge. The K-matrix for the electron sector at level 1 takes the form in the representation chosen here

$$\mathbb{K}_{\mathrm{e}} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \mathbf{t}_{\mathrm{e}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \mathbf{s}_{\mathrm{e}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.6}$$

In the other formulation, used in [3], the off-diagonal elements of $\mathbb{K}_{\mathrm{e}}$ are 1, while the role of $\mathbf{t}_{\mathrm{e}}$ and $\mathbf{s}_{\mathrm{e}}$ is interchanged.

The K-matrix in Eq. (4.6) is the building block of the covering matrix, from which we construct the level-k K-matrices

$$\mathbb{K}_{\mathrm{e}}^{\mathrm{cover}} = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & & & & & \\ & & 2 & -1 & & & \\ & & -1 & 2 & & & \\ & & & & \ddots & & \\ & & & & & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix},$$

$$\mathbf{t}_{\mathrm{e}}^{\mathrm{cover}} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{s}_{\mathrm{e}}^{\mathrm{cover}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}. \tag{4.7}$$

At this point, we need to specify the matrix $\mathbb{P}$, which is used to project to the K-matrix for the $\mathfrak{sl}(3)_k$ theory. However, because we have $n = 2$ in this case, we can construct the composites (up to order k) in different ways. We will first state the form which gives the correct result, and comment on the other possibilities afterwards. The P-transformation which gives the correct central charge is given by

$$\mathbb{P} = \begin{pmatrix} \mathbb{I}_2 & & & \\ \mathbb{I}_2 & \mathbb{I}_2 & & \\ \vdots & \ddots & \ddots & \\ \mathbb{I}_2 & \cdots & \mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix}, \tag{4.8}$$

where $\mathbb{I}_2$ is the 2×2 identity matrix. The resulting K-matrix has the following form (explicit forms of the Cartan matrix $\mathbb{A}_2$ of A_2 and the symmetrized Cartan matrix $\mathbb{M}_k^{-1}$

of B_k can be found in Appendix A)

$$\mathbb{K}_e = \mathbb{A}_2 \otimes \mathbb{M}_k = \begin{pmatrix} 2 & -1 & 2 & -1 & \cdots & 2 & -1 & 2 & -1 \\ -1 & 2 & -1 & 2 & \cdots & -1 & 2 & -1 & 2 \\ 2 & -1 & 4 & -2 & \cdots & 4 & -2 & 4 & -2 \\ -1 & 2 & -2 & 4 & \cdots & -2 & 4 & -2 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & -1 & 4 & -2 & \cdots & 2(k-1) & -(k-1) & 2(k-1) & -(k-1) \\ -1 & 2 & -2 & 4 & \cdots & -(k-1) & 2(k-1) & -(k-1) & 2(k-1) \\ 2 & -1 & 4 & -2 & \cdots & 2(k-1) & -(k-1) & 2k & -k \\ -1 & 2 & -2 & 4 & \cdots & -(k-1) & 2(k-1) & -k & 2k \end{pmatrix}, \tag{4.9}$$

while the charge and spin quantum numbers are given by

$$\mathbf{t}_\mathrm{e} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \\ \vdots \\ k \\ -k \end{pmatrix}, \qquad \mathbf{s}_\mathrm{e} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ \vdots \\ k \\ k \end{pmatrix}. \tag{4.10}$$

It is not to hard to see that introducing the composites can be done in different ways. For instance, we could move some of the 1's in the lower-triangular part of the matrix $\mathbb{P}$ of Eq. (4.8) to the corresponding place in the upper-triangular part. If done systematically, we still would introduce all the composites, so the resulting quantum numbers would be the same. Luckily, all the essentially different possibilities result in different K-matrices, which have different central charge associated to them. So we can pick the, presumably, correct description by looking at the central charge and perform further checks to assure the validity of the chosen matrices. In all the cases we encountered, only one P-transformation gave rise to a rational central charge (as far as the numerical checks could tell), which indeed was the central charge corresponding to the affine Lie algebra CFT. We refer to Section 4.3 for more details on the checks of the central charge associated to the K-matrices. Whether or not the other possibilities correspond to (non-rational) CFTs is not clear at the moment.

The K-matrices and quantum numbers for the quasiparticle sector are obtained similarly as in the $\mathfrak{sl}(2)_k$ case, by applying the dual P-transformation to the dual of the covering. Now, the transformation matrix becomes the inverse transpose of Eq. (4.8)

$$\left(\mathbb{P}^{-1}\right)^{\mathrm{T}} = \begin{pmatrix} \mathbb{I}_2 & -\mathbb{I}_2 & & \\ & \mathbb{I}_2 & \ddots & \\ & & \ddots & -\mathbb{I}_2 \\ & & & \mathbb{I}_2 \end{pmatrix}, \tag{4.11}$$

with the results

$$\mathbb{K}_{\mathrm{qp}} = \mathbb{A}_2^{-1} \otimes \mathbb{M}_k^{-1} = \left(\begin{array}{c|c} \mathbb{A}_2^{-1} \otimes \mathbb{A}_{k-1} & \begin{array}{c} \\ -\mathbb{A}_2^{-1} \end{array} \\ \hline \begin{array}{cc} & \mathbb{A}_2^{-1} \end{array} & \mathbb{A}_2^{-1} \end{array}\right),$$

$$\mathbf{t}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \qquad \mathbf{s}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ -1 \end{pmatrix}. \tag{4.12}$$

The K-matrix is to be compared with the matrix (7.23) in [3]. Note that part of the K-matrix corresponding to the $2(k-1)$ pseudoparticles is the same in both cases. So, because we know the two descriptions are related (see Appendix D), we can say that by using the method of the P-transformations, we were able to obtain correct K-matrices for the $\mathfrak{sl}(3)_k$ theory. One important check is the central charge. Because the pseudoparticles are the same in both formulations, the central charge is also equal. In Section 4.3, the quasiparticle matrices for all simple affine Lie algebra CFTs will be given. The electron matrices are specified in Section 4.2. Before we come to that, we will first describe in detail how to construct the general K-matrices, using the P-transformations and suitable coverings.

4.1.3. The general case

Using the knowledge obtained in the previous section, we go on and propose a scheme to obtain the K-matrices for general affine Lie algebra CFTs. We will first concentrate on the simply-laced cases, and discuss the non-simply-laced cases afterwards. As we discussed the case of $\mathfrak{sl}(3)$, which has all the essential ingredients, in detail in the previous section, we will be brief here. We saw that in the case of $\mathfrak{sl}(3)_1$, we could use the particles related to the simple roots as the basis of the electron sector. Simple roots are roots which cannot be written as a sum of two positive roots. A Lie algebra of rank n has n simple roots, and their scalar products define the Cartan matrix. So we found that the K-matrix for the electron sector of $\mathfrak{sl}(3)_1$ was the Cartan matrix. In the following, we will assume that this is the case for all the simply-laced affine Lie algebras. What we need to do further to obtain the level-k K-matrices is construct the covering theory, which is just the direct sum of k level-1 theories, and apply the correct P-transformation. The form of the P-transformation is similar to the $\mathfrak{sl}(3)$ case, where the rank is the only thing which needs to be changed. So we find $\mathbb{P}$ for the simply-laced cases

$$\mathbb{P} = \begin{pmatrix} \mathbb{I}_n & & & \\ \mathbb{I}_n & \mathbb{I}_n & & \\ \vdots & \ddots & \ddots & \\ \mathbb{I}_n & \cdots & \mathbb{I}_n & \mathbb{I}_n \end{pmatrix}, \qquad \left(\mathbb{P}^{-1}\right)^{\mathrm{T}} = \begin{pmatrix} \mathbb{I}_n & -\mathbb{I}_n & & \\ & \mathbb{I}_n & \ddots & \\ & & \ddots & -\mathbb{I}_n \\ & & & \mathbb{I}_n \end{pmatrix}. \tag{4.13}$$

Applied to the covering matrix we find the result $\mathbb{K}_{\mathrm{e}} = \mathbb{P} \cdot (\mathbb{A}_n \otimes \mathbb{I}_k) \cdot \mathbb{P}^{\mathrm{T}} = \mathbb{A}_n \otimes \mathbb{M}_k$. See Section 4.2 for an explicit example. Of course, $\mathbb{A}_n$ can be replaced by the Cartan matrix of any other simply-laced algebra, $\mathbb{D}_n$ or $\mathbb{E}_n$. The K-matrix for the quasiparticle sector is obtained by applying $(\mathbb{P}^{-1})^{\mathrm{T}}$ to the dual covering $\mathbb{A}_n^{-1} \otimes \mathbb{I}_k$, resulting in $\mathbb{K}_{\mathrm{qp}} = \mathbb{A}_n^{-1} \otimes \mathbb{M}_k^{-1}$. From the form of $(\mathbb{P}^{-1})^{\mathrm{T}}$ we find that the first $n(k-1)$ particles are in fact pseudoparticles. These matrices will be given explicitly in Section 4.3. For now, we note that the central charge associated to these systems does indeed have the correct value. More on this can be found in Section 4.3.

Let us now focus our attention to the non-simply-laced case. The idea is to apply the same construction as for the simply-laced cases. However, we need to find the correct starting point, that is, the level $k = 1$ formulation. The non-simply-laced affine algebras have non-trivial fusion rules already at level-1, so we already need pseudoparticles at level-1. This is also reflected in the central charge, which is non-integer. To find the K-matrices, we embed the non-simply-laced algebra in a simply-laced one, and basically do the same construction as before: project out some degrees of freedom by introducing pseudoparticles. As an example, we quote the case for $\mathfrak{so}(5)_1$, which is related to the spin-charge separated quantum Hall states of [4] (see also [9,11]). There, the K-matrices for the $\mathfrak{so}(5)_1$ were obtained from the $\mathfrak{so}(6)_1$ K-matrices using the construction outlined above. It turns out that in general, the matrices for the non-simply-laced affine Lie algebras are equal to the (simply-laced) affine Lie algebra in which they are embedded. The difference is the presence of pseudoparticles in the non-simply-laced cases, as described above. Alternative descriptions are possible, e.g., for $G_{2,k}$ we have an alternative description (which is used in connection with the corresponding parafermion CFT), where the $k = 1$ K-matrix has a couple of sign changes in comparison to the Cartan matrix of the algebra used for the embedding, see Appendix C.

To check that we indeed found the correct matrices, we will provide another way to obtain the K-matrix for non-simply-laced CFTs at level one. This time, we directly use the exclusion statistics parameters of the electron-like operators, corresponding to the root lattice of the algebra. It is important to know the exclusion statistics of the corresponding parafermions (which are part of the electron operators, see Section 5.3 and also [25]), but we can borrow results from the literature here. We will show how this works for the case $\mathfrak{so}(5)_1$ in Appendix B, while G_2 at level-1 can be found in Appendix C. The other non-simply-laced cases can be obtained in a similar way.

Having identified the $k = 1$ K-matrices for the non-simply-laced algebras, we can go on, and take the direct sum of k of the level-1 matrices, and do exactly the same P-transformations as in the simply-laced case. Because the covering matrices for the non-simply-laced cases are identical to the ones used for the corresponding simply-laced cases, the resulting K-matrices will be identical as well. The only difference is the number of pseudoparticles, as there will be more pseudoparticles in the non-simply-laced case. So, specifying the nature of the particles is the only way to tell the difference between the two. It is important to note that in the P-transformation, (dual) composites are made only out of identical particles. We never have the situation where a physical particle is paired with a pseudoparticle, in accordance with the results of Section 3.3.

4.2. The matrices $\mathbb{K}_{\mathrm{e}}$

The building blocks of all the K-matrices are the Cartan matrices $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_n$ and their inverses. In addition, we need the symmetrized Cartan matrix of B_n, which we denote by $\mathbb{M}_k$, and its inverse. All these matrices can be found explicitly in Appendix A.

From Section 4.1.3, we have the results that for the simply-laced cases $A_{n,k}$, $D_{n,k}$ and $E_{n,k}$ the matrices $\mathbb{K}_{\mathrm{e}}$ take the form $\mathbb{A}_n \otimes \mathbb{M}_k$, $\mathbb{D}_n \otimes \mathbb{M}_k$ and $\mathbb{E}_n \otimes \mathbb{M}_k$, respectively. As an example, we will give the result for $D_{4,2}$ explicitly

$$\mathbb{K}_{\mathrm{e}} = \mathbb{D}_4 \otimes \mathbb{M}_2 = \left(\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 & 0 & 2 \\ \hline 2 & -1 & 0 & 0 & 4 & -2 & 0 & 0 \\ -1 & 2 & -1 & -1 & -2 & 4 & -2 & -2 \\ 0 & -1 & 2 & 0 & 0 & -2 & 4 & 0 \\ 0 & -1 & 0 & 2 & 0 & -2 & 0 & 4 \end{array}\right). \tag{4.14}$$

For the non-simply-laced cases, we have to take the Cartan matrix corresponding to affine Lie algebra which we used for the embedding. We find that the matrices $\mathbb{K}_{\mathrm{e}}$ are $\mathbb{D}_{n+1} \otimes \mathbb{M}_k$, $\mathbb{A}_{2n-1} \otimes \mathbb{M}_k$, $\mathbb{E}_6 \otimes \mathbb{M}_k$ and $\mathbb{D}_4 \otimes \mathbb{M}_k$ for $B_{n,k}$, $C_{n,k}$, $F_{4,k}$ and $G_{2,k}$, respectively.

4.3. The matrices $\mathbb{K}_{\mathrm{qp}}$

The matrices $\mathbb{K}_{\mathrm{qp}}$ can be obtained from $\mathbb{K}_{\mathrm{e}}$ by a simple inversion (see (2.23)). In the following, we will explicitly give these matrices, and indicate which particles are in fact the pseudoparticles. With this knowledge, one can calculate the central charge corresponding to $\mathbb{K}_{\mathrm{e}} \oplus \mathbb{K}_{\mathrm{qp}}$ by using Eq. (2.34). As this is hard to do analytically in general, we determined the central charge numerically for some low values of (n, k). All the cases up to rank $n = 10$ have been checked up to level $k = 20$. We found that the central charge corresponding to the matrices was equal to the central charge of the CFTs up to 10^{-20} or better. The central charge of an affine Lie algebra CFT is given by (cf. (2.13))

$$c_{\mathrm{ALA}} = \frac{k \dim X_n}{k + \mathrm{h}^{\vee}}, \tag{4.15}$$

where $\dim X_n$ is the dimension and $\mathrm{h}^{\vee}$ the dual Coxeter number of the Lie algebra X_n. Both can be found in Appendix A for every simple Lie algebra.

In the following, we will denote the ith column of the matrix M by $(M)_i$. Recall that the quasiparticle matrices are of the form (see Eq. (2.29))

$$\mathbb{K}_{\mathrm{qp}} = \left(\begin{array}{c|c} \mathbb{K}_{\psi\psi} & \mathbb{K}_{\psi\phi} \\ \hline \mathbb{K}_{\phi\psi} & \mathbb{K}_{\phi\phi} \end{array}\right),$$
$$\mathbb{K}_{\psi\psi}^{\mathrm{T}} = \mathbb{K}_{\psi\psi}, \qquad \mathbb{K}_{\phi\phi}^{\mathrm{T}} = \mathbb{K}_{\phi\phi}, \qquad \mathbb{K}_{\psi\phi}^{\mathrm{T}} = \mathbb{K}_{\phi\psi}, \tag{4.16}$$

where ψ denotes the pseudoparticles, and ϕ the physical quasiparticles.

4.3.1. The case $A_{n,k}$

The quasiparticle matrix $\mathbb{K}_{\mathrm{qp}}$ for $\mathfrak{sl}(n+1)_k$ is given by

$$\mathbb{K}_{\mathrm{qp}} = \mathbb{A}_n^{-1} \otimes \mathbb{M}_k^{-1} = \left(\begin{array}{c|c} \mathbb{A}_n^{-1} \otimes \mathbb{A}_{k-1} & \\ & -\mathbb{A}_n^{-1} \\ \hline -\mathbb{A}_n^{-1} & \mathbb{A}_n^{-1} \end{array}\right). \tag{4.17}$$

In particular, the pseudoparticle matrix is given by $\mathbb{K}_{\psi\psi} = \mathbb{A}_n^{-1} \otimes \mathbb{A}_{k-1}$.

4.3.2. The case $B_{n,k}$

As already pointed out, we need to an embedding to obtain the $B_{n,1}$ description first. This is done for $\mathfrak{so}(5)$ in Appendix B, where we used $D_{3,1}$ for the embedding. In general, we need $D_{n+1,1}$. We find that we need one extra pseudoparticle, which corresponds to the first node of the Dynkin diagram of D_{n+1}. This extra particle has exclusion statistics parameter 1, which gives a reduction of the central charge by $\frac{1}{2}$, which is indeed the difference of the central charge of the theories $D_{n+1,1}$ and $B_{n,1}$. At general level we find that $\mathbb{K}_{\mathrm{qp}} = \mathbb{D}_{n+1}^{-1} \otimes \mathbb{M}_k^{-1}$, which is characterized by

$$\mathbb{K}_{\psi\psi} = \begin{pmatrix} 2\mathbb{D}_{n+1}^{-1} & -\mathbb{D}_{n+1}^{-1} & & & \\ -\mathbb{D}_{n+1}^{-1} & 2\mathbb{D}_{n+1}^{-1} & \ddots & & \\ & \ddots & \ddots & -\mathbb{D}_{n+1}^{-1} & \\ & & -\mathbb{D}_{n+1}^{-1} & 2\mathbb{D}_{n+1}^{-1} & -(\mathbb{D}_{n+1}^{-1})_1 \\ & & & -(\mathbb{D}_{n+1}^{-1})_1^{\mathrm{T}} & 1 \end{pmatrix}, \tag{4.18}$$

where we see explicitly that there is an extra pseudoparticle next to the $\mathbb{D}_{n+1}^{-1} \otimes \mathbb{A}_{k-1}$ part.

Accordingly, the matrix $\mathbb{K}_{\phi\phi}$ is the inverse Cartan matrix of D_{n+1}, with the first row and column omitted (denoted by $\mathbb{D}_{n+1}^{-1}|_{\not{1}}$)

$$\mathbb{K}_{\phi\phi} = \mathbb{D}_{n+1}^{-1}|_{\not{1}} = \begin{pmatrix} 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 3 & \cdots & 3 & \frac{3}{2} & \frac{3}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 3 & \cdots & n-1 & \frac{n-1}{2} & \frac{n-1}{2} \\ 1 & \frac{3}{2} & \cdots & \frac{n-1}{2} & \frac{n+1}{4} & \frac{n-1}{4} \\ 1 & \frac{3}{2} & \cdots & \frac{n-1}{2} & \frac{n-1}{4} & \frac{n+1}{4} \end{pmatrix}. \tag{4.19}$$

Finally, we have

$$\mathbb{K}_{\psi\phi} = \begin{pmatrix} \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ -(\mathbb{D}_{n+1}^{-1})_2 & \cdots & \cdots & \cdots & -(\mathbb{D}_{n+1}^{-1})_{n+1} \\ -1 & \cdots & -1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \tag{4.20}$$

where $\mathbf{0}$ stands for the (column) vector with all zeroes (of 'length' $(n+1)(k-2)$ in this case). Putting the parts together, we find

$$\mathbb{K}_{\mathrm{qp}} = \begin{pmatrix} \mathbb{K}_{\psi\psi} & \mathbb{K}_{\psi\phi} \\ \mathbb{K}^{\mathrm{T}}_{\psi\phi} & \mathbb{K}_{\phi\phi} \end{pmatrix} = \mathbb{D}^{-1}_{n+1} \otimes \mathbb{M}^{-1}_{k}. \tag{4.21}$$

To put emphasis on the fact that the pseudoparticle matrix is bigger that the one for the D_{n+1} CFT, we gave the matrices $\mathbb{K}_{\psi\psi}$ etc. explicitly, as we will do for all non-simply-laced cases.

4.3.3. *The case $C_{n,k}$*

In this case, we need $A_{2n-1,k}$ as the theory for the embedding. For level $k=1$ we need the particles corresponding to the *even* nodes to be pseudoparticles. These will be the extra pseudoparticles for $k>1$, giving $n-1$ extra pseudoparticles. We again will specify the matrix $\mathbb{K}_{\mathrm{qp}}$ by its parts $\mathbb{K}_{\psi\psi}$, etc.

$$\mathbb{K}_{\psi\psi} = \begin{pmatrix}
2\mathbb{A}^{-1}_{2n-1} & -\mathbb{A}^{-1}_{2n-1} & & & & & \\
-\mathbb{A}^{-1}_{2n-1} & 2\mathbb{A}^{-1}_{2n-1} & \ddots & & & & \\
& \ddots & \ddots & -\mathbb{A}^{-1}_{2n-1} & & & \\
& & -\mathbb{A}^{-1}_{2n-1} & 2\mathbb{A}^{-1}_{2n-1} & -(\mathbb{A}^{-1}_{2n-1})_2 & \cdots & -(\mathbb{A}^{-1}_{2n-1})_{2n-2} \\
& & & -(\mathbb{A}^{-1}_{2n-1})^{\mathrm{T}}_2 & & & \\
& & & \vdots & & 2\mathbb{A}^{-1}_{n-1} & \\
& & & -(\mathbb{A}^{-1}_{2n-1})^{\mathrm{T}}_{2n-2} & & &
\end{pmatrix}, \tag{4.22}$$

$$\mathbb{K}_{\psi\phi} = \begin{pmatrix}
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
-(\mathbb{A}^{-1}_{2n-1})_1 & -(\mathbb{A}^{-1}_{2n-1})_3 & \cdots & -(\mathbb{A}^{-1}_{2n-1})_{2n-1} \\
& \mathbb{K}_{\psi_e\phi} & &
\end{pmatrix}. \tag{4.23}$$

The matrix $\mathbb{K}_{\psi_e\phi}$, which contains the coupling between the physical particles and the *extra* pseudoparticles, is described most easily by specifying its entries explicitly. Let us first recall the elements of the inverse Cartan matrix of A_{2n-1} (compare with Appendix A)

$$\begin{aligned}
&(\mathbb{A}^{-1}_{2n-1})_{2i-1,2j-1} = \min(2i-1, 2j-1) - \frac{(2i-1)(2j-1)}{2n}, \\
&\quad i, j = 1, \ldots, 2n-1.
\end{aligned} \tag{4.24}$$

Then we have

$$\begin{aligned}
&(\mathbb{K}_{\psi_e\phi})_{i,j} = \min(2i-1, 2j) - \frac{(2i-1)2j}{2n}, \\
&\quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n-1.
\end{aligned} \tag{4.25}$$

For the matrix $\mathbb{K}_{\phi\phi}$ we have

$$(\mathbb{K}_{\phi\phi})_{i,j} = \min(2i-1, 2j-1) - \frac{(2i-1)(2j-1)}{2n}, \quad i, j = 1, 2, \ldots, n. \tag{4.26}$$

Note that the elements of the matrix describing the extra pseudoparticles $(\mathbb{K}_{\psi_e\psi_e})_{i,j} = \min(2i, 2j) - \frac{(2i)(2j)}{2n}$, where $i, j = 1, \ldots, n-1$ is indeed equal to $2\mathbb{A}^{-1}_{n-1}$.

4.3.4. The case $D_{n,k}$

As we already used the matrix corresponding to $D_{n+1,k}$ in the case of $B_{n,k}$, we will be brief here:

$$\mathbb{K}_{\rm qp} = \mathbb{D}_n^{-1} \otimes \mathbb{M}_k^{-1} = \left(\begin{array}{c|c} \mathbb{D}_n^{-1} \otimes \mathbb{A}_{k-1} & -\mathbb{D}_n^{-1} \\ \hline -\mathbb{D}_n^{-1} & \mathbb{D}_n^{-1} \end{array}\right). \tag{4.27}$$

So we have $n(k-1)$ pseudoparticles, and n physical ones.

4.3.5. The cases $E_{n,k}$ with $n = 6, 7, 8$

For $E_{n,k}$, we simply have a similar result as for the other simply-laced cases:

$$\mathbb{K}_{\rm qp} = \mathbb{E}_n^{-1} \otimes \mathbb{M}_k^{-1} = \left(\begin{array}{c|c} \mathbb{E}_n^{-1} \otimes \mathbb{A}_{k-1} & -\mathbb{E}_n^{-1} \\ \hline -\mathbb{E}_n^{-1} & \mathbb{E}_n^{-1} \end{array}\right). \tag{4.28}$$

so the $n(k-1)$ pseudoparticles couple via $\mathbb{E}_n \otimes \mathbb{A}_{k-1}$.

4.3.6. $F_{4,k}$

The embedding used this time is based upon $E_{6,k}$. Now we expect to have two extra pseudoparticles, based on the level-1 case (cf. (2.58), Section 2.3), which turns out to be true. The couplings of these extra pseudoparticles are related to the nodes 1 and 5 (see Appendix A). For general k, we have the pseudoparticle matrix

$$\mathbb{K}_{\psi\psi} = \begin{pmatrix} 2\mathbb{E}_6^{-1} & -\mathbb{E}_6^{-1} & & & & \\ -\mathbb{E}_6^{-1} & 2\mathbb{E}_6^{-1} & \ddots & & & \\ & \ddots & \ddots & -\mathbb{E}_6^{-1} & & \\ & & -\mathbb{E}_6^{-1} & 2\mathbb{E}_6^{-1} & -(\mathbb{E}_6^{-1})_1 & -(\mathbb{E}_6^{-1})_5 \\ & & & -(\mathbb{E}_6^{-1})_1^{\rm T} & \frac{4}{3} & \frac{2}{3} \\ & & & -(\mathbb{E}_6^{-1})_5^{\rm T} & \frac{2}{3} & \frac{4}{3} \end{pmatrix}, \tag{4.29}$$

while the physical particles have

$$\mathbb{K}_{\phi\phi} = \begin{pmatrix} \frac{10}{3} & 4 & \frac{8}{3} & 2 \\ 4 & 6 & 4 & 3 \\ \frac{8}{3} & 4 & \frac{10}{3} & 2 \\ 2 & 3 & 2 & 2 \end{pmatrix}. \tag{4.30}$$

Physical and pseudoparticles are coupled via

$$\mathbb{K}_{\psi\phi} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbb{E}_6^{-1})_2 & -(\mathbb{E}_6^{-1})_3 & -(\mathbb{E}_6^{-1})_4 & -(\mathbb{E}_6^{-1})_6 \\ \frac{5}{3} & 2 & \frac{4}{3} & 1 \\ \frac{4}{3} & 2 & \frac{5}{3} & 1 \end{pmatrix}. \tag{4.31}$$

Again, if we combine the physical and extra pseudoparticles in the right way, we find the matrix $\mathbb{E}_6^{-1}$.

4.3.7. $G_{2,k}$

Finally we come to the last case, which is $G_{2,k}$. This case is special in the sense that if we use a similar procedure as we used in all the other cases, we find a description in which the number of physical particles does not equal the rank of the algebra, as was the situation in the other cases. This will have consequences as we consider the related parafermions in Section 5.3. In Appendix C we will provide a different description of $G_{2,k}$, which does have two physical particles. For now, we will just use the description based on the K-matrices for $D_{4,k}$, in which we embed $G_{2,k}$. It turns out that we need three extra pseudoparticles, leaving only one physical particle. Note that the coupling of the extra pseudoparticles is given by Eq. (2.59) in Section 2.3.

$$\mathbb{K}_{\psi\psi} = \begin{pmatrix} 2\mathbb{D}_4^{-1} & -\mathbb{D}_4^{-1} & & & & & \\ -\mathbb{D}_4^{-1} & 2\mathbb{D}_4^{-1} & \ddots & & & & \\ & \ddots & \ddots & -\mathbb{D}_4^{-1} & & & \\ & & -\mathbb{D}_4^{-1} & 2\mathbb{D}_4^{-1} & -(\mathbb{D}_4^{-1})_1 & -(\mathbb{D}_4^{-1})_3 & -(\mathbb{D}_4^{-1})_4 \\ & & & -(\mathbb{D}_4^{-1})_1^{\mathrm{T}} & 1 & \frac{1}{2} & \frac{1}{2} \\ & & & -(\mathbb{D}_4^{-1})_3^{\mathrm{T}} & \frac{1}{2} & 1 & \frac{1}{2} \\ & & & -(\mathbb{D}_4^{-1})_4^{\mathrm{T}} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \tag{4.32}$$

$$\mathbb{K}_{\phi\phi} = (2), \tag{4.33}$$

$$\mathbb{K}_{\psi\phi} = \begin{pmatrix} \mathbf{0} \\ -(\mathbb{D}_4^{-1})_2 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{4.34}$$

5. K-matrices for coset conformal field theories

Having identified the K-matrices for the affine Lie algebra CFTs, one might hope to find K-matrices for more general CFTs. An obvious class to look at are the coset conformal field theories, as most CFTs can be written in a coset form. In this section, we will provide K-matrices for a class of coset CFTs. In our search for the K-matrices for coset CFTs, we will be mainly guided by the central charge. We can test our results by comparing to known coset K-matrices. For diagonal cosets of simply-laced affine Lie algebras, the results of the K-matrices are due to McCoy and co-workers. See, for instance, [6].

Having obtained a scheme, we will apply it to the cosets $\mathfrak{so}(2n)_k/\mathfrak{so}(2n-1)_k$ with $k = 1, 2$, where the latter is the non-trivial one. The parafermionic cosets are dealt with in

Section 5.3, as they require a different approach. This already shows that the scheme we found is by no means unique, but useful anyway.

5.1. Diagonal cosets

As said, the central charge is an important quantity to keep in mind in determining the K-matrices for the cosets. Let us take a look at the general coset G/H, where $H \subset G$ is maximal. Let us assume that both G and H are of the form $\mathbb{K}_{\mathrm{e}} \oplus \mathbb{K}_{\mathrm{qp}}$, with equal rank n. Also, both quasiparticle matrices can contain pseudoparticles. So the central charge of these theories (denoted by $c(G)$ and $c(H)$) is given by

$$c(G) = n - c\big(\mathbb{K}_{\psi\psi}(G)\big), \qquad c(H) = n - c\big(\mathbb{K}_{\psi\psi}(H)\big), \tag{5.1}$$

where $c(\mathbb{K}_{\psi\psi}(G))$ denotes the central charge corresponding to the pseudoparticle matrix of G. Let us further assume that all the pseudoparticles which appear in $\mathbb{K}_{\psi\psi}(G)$ also appear in $\mathbb{K}_{\psi\psi}(H)$. This restricts the applicability of the construction, but still covers a large class of cosets. Now the argument of the central charge suggests to take the pseudoparticle K-matrix of H, and change the pseudoparticles which do *not* appear in the pseudoparticle matrix of G into physical particles. The central charge corresponding to this matrix is $c(\mathbb{K}_{\psi\psi}(H)) - c(\mathbb{K}_{\psi\psi}(G))$. This indeed equals the central charge of the coset theory, which is given by $c(G) - c(H)$. Note that the matrix we propose for the coset theory is not of the form $\mathbb{K} \oplus \mathbb{K}^{-1}$. This is in fact consistent with known results for K-matrices of coset conformal field theories, as we will discuss below. This construction does work for the cosets of the type $X_{n,k} \oplus X_{n,l}/X_{n,k+l}$, where X_n is a simply-laced Lie algebra. Indeed, using this, we reproduce the results of McCoy for these diagonal cosets, see, for instance, [6].

The construction above is in fact more generally applicable as we will show in the next subsection, where we will show a non-trivial example based on the coset of $\mathfrak{so}(2n)_k/\mathfrak{so}(2n-1)_k$.

5.2. $\mathfrak{so}(2n)_k/\mathfrak{so}(2n-1)_k$

Applying the construction above to the coset $\mathfrak{so}(2n)_k/\mathfrak{so}(2n-1)_k$ at level $k=1$, we find the K-matrix $\mathbb{K} = (1)$, which is obviously the correct result for this $c = \frac{1}{2}$ CFT. Another coset with $c \leqslant 1$ is the case $k=2$, which has $c=1$. We find the following K-matrix

$$\mathbb{K} = \left(\begin{array}{c|ccccc} 1 & -1 & \cdots & -1 & -\frac{1}{2} & -\frac{1}{2} \\ \hline -1 & & & & & \\ \vdots & & & & & \\ -1 & & & 2\mathbb{D}_n^{-1} & & \\ -\frac{1}{2} & & & & & \\ -\frac{1}{2} & & & & & \end{array}\right), \tag{5.2}$$

where only the first particle is physical. As mentioned, this matrix yields the correct central charge $c=1$ by construction. That it indeed describes the correct $c=1$ CFT can be seen as follows. Applying the dual composite construction to

$$\mathbb{K} = \begin{pmatrix} 1-\frac{1}{2n} & \frac{1}{2n} \\ \frac{1}{2n} & 1-\frac{1}{2n} \end{pmatrix}, \tag{5.3}$$

where both particles are physical, we find

$$\mathcal{D}_{12}\mathbb{K} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{n}{2} & 1-\frac{n}{2} \\ -\frac{1}{2} & 1-\frac{n}{2} & \frac{n}{2} \end{pmatrix}. \tag{5.4}$$

Now applying the composite construction to the two pseudoparticles in (5.4) $(n-2)$ times we find (5.2). On the other hand, the UCPF based on (5.3), summed over $m_1 + m_2 \equiv 0 \mod 2n$, equals the $c = 1$ $\mathfrak{u}(1)$-character

$$\frac{1}{(q)_\infty} \sum_{k\in\mathbb{Z}} q^{n(2n-1)k^2}, \tag{5.5}$$

by using the Durfee square identity (see, e.g., [1])

$$\frac{1}{(q)_\infty} = \sum_{m\geqslant 0} \frac{q^{m^2}}{(q)_m (q)_m}. \tag{5.6}$$

So we indeed find that the matrix (5.2) describes a $c = 1$ conformal field theory, namely, the free boson compactified on a circle.

In addition to this non-trivial example, also the equivalence used in the theory of G_2-holonomy—namely, between $\mathfrak{so}(7)_1/G_{2,1}$ and the tricritical Ising model—works, if we take the G_2 (level $k = 1$) description of Appendix C. We find the K-matrix

$$\mathbb{K} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \tag{5.7}$$

with one physical and one pseudoparticle. This is indeed the K-matrix corresponding to the minimal model with $c = 7/10$.

5.3. *Parafermions*

Generalized parafermionic conformal field theories were defined by Gepner [25] as a generalization of the $\mathbb{Z}_k$ parafermions of [49]. The generalized parafermion theories can be viewed as cosets based on general affine Lie algebras (ALAs) and $\mathfrak{u}(1)$ theories

$$X^{\text{pf}}_{n,k} = \frac{X_{n,k}}{\mathfrak{u}(1)^n}, \tag{5.8}$$

where n is the rank of the Lie algebra X_n, and k the level. The central charge of the parafermion CFT (5.8) is given by

$$c_{\text{pf}} = c_{\text{ALA}} - n, \tag{5.9}$$

where c_{ALA} is the central charge of the corresponding affine Lie algebra theory (see Eq. (2.13)). The parafermion cosets (5.8) are somewhat different in comparison to the diagonal cosets of Section 5.1, and need to be treated differently. Before we come to the discussion of the K-matrices, we first fix some notations concerning the parafermion fields, following [25].

The primary fields of the theory Φ_λ^Λ are labeled by a (highest) weight Λ and a charge λ, which is also an element of the weight lattice, and is defined modulo $k\mathcal{M}_L$, i.e., k times the long root lattice. To obtain a complete, independent set of parafermion fields, one has to impose the following restrictions. The charge λ must be 'accessible' from Λ by subtracting roots (including α_0) from Λ. Furthermore, the (proper) external automorphisms σ (see [23]) of the affine Lie algebra give rise to field identifications

$$\Phi_\lambda^\Lambda \equiv \Phi_{\lambda+\sigma(0)}^{\sigma(\Lambda)}, \tag{5.10}$$

where $\sigma(0)$ denotes the image of the affine weight $k\Lambda_0$ under σ.

An important check on the K-matrices for the parafermionic CFTs is based on the relation between the parafermionic partition functions and the string functions c_λ^Λ of the corresponding affine Lie algebras [25]

$$Z_{\mathrm{pf}}^{\Lambda,\lambda} = (\eta)^n c_\lambda^\Lambda, \tag{5.11}$$

where $\eta = q^{1/24}\prod_{k=1}^\infty (1-q^k)$ is the Dedekind function. As an example, we will express the partition function $Z_{\mathrm{pf}}^{\Lambda,\lambda}$ with $\Lambda = (0,\ldots,0) \equiv \mathbf{1}$ in terms of UCPFs based on the K-matrices for the parafermion CFTs. Using Eq. (5.11), we can check our results against the known (tabulated) string functions.

We will use the matrices $\mathbb{K}_e$ of the corresponding affine Lie algebras as a starting point for obtaining the parafermionic matrices $\mathbb{K}^{\mathrm{pf}}$. The matrices $\mathbb{K}_e$ correspond to the (elementary) electron-like particles and composites (up to order k) of these elementary particles. The operators corresponding to these (elementary) particles have the form

$$\Phi_\lambda^{\mathbf{1}} {:}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\varphi}}{:}, \tag{5.12}$$

where $\boldsymbol{\varphi} = (\varphi_1,\ldots,\varphi_n)$ is a set of bosonic fields, which correspond to the $\mathfrak{u}(1)$ degrees of freedom and determine the quantum numbers of the particles via the constants α_i. For the order k composites, the parafermion fields are trivial, i.e., $\Phi_{k\mu}^{\mathbf{1}} = \mathbf{1}$, for $\mu \in \mathcal{M}_L$ (μ a long root), in which case only the vertex operator part remains.

In this section, we are interested in the K-matrices for the parafermionic CFTs. These can be obtained from the matrices $\mathbb{K}_e$ of the corresponding affine Lie algebra theories by subtracting from the particles which have a non-trivial parafermion field $\Phi_\lambda^{\mathbf{1}}$ the part of the exclusion statistics which corresponds to the vertex operator ${:}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\varphi}}{:}$. This can be done 'by hand' by calculating the exclusion statistics of the vertex operators. Actually, because there are always particles which do not have a parafermion field (or equivalently, a trivial parafermion field), this can be done by applying what we will call an X-transformation. Such a transformation is like a W-transformation. However, the matrices associated to an X-transformation are not $SL(r,\mathbb{Z})$ matrices, but rather $SL(r,\mathbb{Q})$. This is because the quantum numbers of the largest composites (which are the particles with trivial parafermion fields) are k times the quantum numbers of the particles in the $k=1$ formulation. In general, the non-zero non-diagonal entries take the form l/k, with $l = 1,\ldots,k-1$. Explicitly, in the case of the $\mathbb{Z}_k = \mathfrak{sl}(2)_k/\mathfrak{u}(1)$ parafermions we find the following

$$\mathbb{X} = \begin{pmatrix} 1 & & & & -\frac{1}{k} \\ & 1 & & & -\frac{2}{k} \\ & & \ddots & & \vdots \\ & & & 1 & -\frac{k-1}{k} \\ & & & & 1 \end{pmatrix}. \tag{5.13}$$

For more general parafermions, the matrices are (a little) more complicated. In fact, each entry of the matrix (5.13) becomes an $n \times n$ matrix. Although fractions appear in $\mathbb{X}$, the quantum numbers of the particles after the transformation are still integers, because the largest composite is of order k. More precisely, the X-transformation is such that all the quantum numbers of the transformed particles are in fact zero; in a sense all the vertex operators containing the chiral boson fields are stripped of from the parafermionic fields. The transformed matrix $\mathbb{K}_{\mathrm{e}}$ splits in two pieces, namely, a part containing the order k composites and the part corresponding to the parafermions $\Phi_{\lambda}^{\mathbf{1}}$, which is the matrix we are looking for. We will denote this matrix by $\mathbb{K}^{\mathrm{pf}}$. In the quasiparticle sector, the pseudoparticles will completely decouple from the physical quasiparticles and hence the transformed matrix is of the form $\mathbb{K}_{\psi\psi} \oplus \widetilde{\mathbb{K}}_{\phi\phi}$, where $\widetilde{\mathbb{K}}_{\phi\phi}$ is a deformed quasiparticle matrix. So we conjecture that the K-matrices for parafermionic CFTs are given by the inverse of the pseudoparticle matrix $\mathbb{K}_{\psi\psi}$, of the corresponding affine Lie algebra CFT

$$\mathbb{K}^{\mathrm{pf}} = \mathbb{K}_{\psi\psi}^{-1}. \tag{5.14}$$

A first check on the proposed matrices is the corresponding central charge. The central charge corresponding to the matrices $\mathbb{K}_{\psi\psi}$ is given by

$$c_{\psi\psi} = (n+p)k - c_{\mathrm{ALA}}, \tag{5.15}$$

where p is the difference in rank between the affine algebra under consideration and the one used to 'build' the K-matrices (thus for simply-laced algebras, $p = 0$). The rank of the matrix $\mathbb{K}_{\psi\psi}$ is $(k-1)(n+p)+e$, where e is the number of 'extra' pseudoparticles needed for the non-simply-laced algebras. Thus we have the following result for the central charge of matrices $\mathbb{K}^{\mathrm{pf}}$

$$c_{\mathrm{pf}} = c_{\mathrm{ALA}} - n - (p-e). \tag{5.16}$$

For all the affine algebras, except $G_{2,k}$, the K-matrices of Section 4.3 have $p = e$, so we obtain the correct result of Eq. (5.9). However, we also find that the construction above does not work for the description of $G_{2,k}$ as given in Section 4.3.7, because there the number of physical quasiparticles is 1 instead of 2, which is the rank of G_2. Luckily, there exists another way to represent the $G_{2,k}$ affine Lie algebra, which does have two physical quasiparticles. The inverse of the pseudoparticle matrix therefore has the correct central charge. The corresponding K-matrices can be found in Appendix C. It has been checked that this $G_{2,k}$ parafermion K-matrix does give rise to the corresponding string functions (for $k = 2, 3$).

5.3.1. The case $\mathfrak{so}(5)_2$ as an example

As an example, we will discuss the characters of the parafermionic theory associated to $\mathfrak{so}(5)_2$.

The conjectured pseudoparticle $\mathbb{K}_{\psi\psi}$ for $\mathfrak{so}(5)_2$ is given by Eq. (4.18) with $n = 2, k = 2$

$$\mathbb{K}_{\psi\psi} = \begin{pmatrix} (\mathbb{D}_3^{-1})_{11} & -(\mathbb{D}_3^{-1})_1^{\mathrm{T}} \\ -(\mathbb{D}_3^{-1})_1 & 2\mathbb{D}_3^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 2 & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}. \tag{5.17}$$

The K-matrix which is supposed to describe the $\mathfrak{so}(5)_2$ parafermions is simply the inverse of the pseudoparticle matrix, where it is assumed that all particles are physical

$$\mathbb{K}^{\mathrm{pf}} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix}. \tag{5.18}$$

The UCPF based on this K-matrix, namely,

$$Z_{\mathrm{pf}}^{\Lambda=\mathbf{1}} = \sum \frac{q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}^{\mathrm{pf}}\cdot\mathbf{m}}}{\prod_i (q)_{m_i}}, \tag{5.19}$$

with $\mathbf{m}$ a 4-dimensional vector, is the sum over string-functions

$$Z_{\mathrm{pf}}^{\Lambda=\mathbf{1}} = \sum_{\lambda} (\eta)^l c_{\lambda}^{\Lambda}. \tag{5.20}$$

The sum over λ runs over the independent parafermion fields $\Phi_{(\lambda_1,\lambda_2)}^{(0,0)}$ (where we assume that the first root is the short root). The various string-functions $c_{(\lambda_1,\lambda_2)}^{(0,0)}$ are obtained by restricting the sum in Eq. (5.19). Explicitly, we have

$$c_{\lambda}^{(0,0)} = \frac{q^{-1/12}}{(q)_\infty^2} \sum_{res(\lambda)} \frac{q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}^{\mathrm{pf}}\cdot\mathbf{m}}}{\prod_i (q)_{m_i}}, \tag{5.21}$$

where

$$res(\lambda) = \begin{cases} 2m_1 + m_2 + 2m_3 = 0 \bmod 4, \\ m_3 + m_4 = 0 \bmod 2, \ \text{for } \lambda = (0,0); \\ 2m_1 + m_2 + 2m_3 = 0 \bmod 4, \\ m_3 + m_4 = 1 \bmod 2, \ \text{for } \lambda = (2,0); \\ 2m_1 + m_2 + 2m_3 = 2 \bmod 4, \\ m_3 + m_4 = 0 \bmod 2, \ \text{for } \lambda = (0,2); \\ 2m_1 + m_2 + 2m_3 = 1 \bmod 4, \\ m_3 + m_4 = 0 \bmod 2, \ \text{for } \lambda = (0,1). \end{cases} \tag{5.22}$$

The string functions $c_{(\lambda_1,\lambda_2)}^{(2,0)}$ can be obtained by using a shift vector; more explicitly, by changing the power of q in Eq. (5.19) to $\frac{1}{2}\mathbf{m}\cdot\mathbb{K}^{\mathrm{pf}}\cdot\widetilde{\mathbf{m}}$, where $\widetilde{\mathbf{m}} = (m_1 - 1, m_2, m_3, m_4)$.

We have not yet found similar expressions for the other (independent) string functions, such as $c^{(0,1)}_{(\lambda_1,\lambda_2)}$ and $c^{(1,0)}_{(\lambda_1,\lambda_2)}$.

5.3.2. Cases checked

The cases for which we checked that the conjectured matrices do give the string functions $c^{\mathbf{1}}_{\boldsymbol{\lambda}}$ include all the affine Lie algebras up to rank $n=3$ and level $k=2$. In addition, we also checked $\mathfrak{so}(5)_3$, $\mathfrak{so}(8)_2$, $E_{6,2}$, $E_{7,2}$, $E_{8,2}$ and $F_{4,2}$. The checks were performed by numerically calculating the partition functions up to a certain order in q, depending on the dimension of the K-matrix. These results were compared to the weight-multiplicity tables of Kass et al. [33]. Note that despite the fact that for the higher rank algebras the checks were performed to rather low order in q, we believe that the formulas hold to all orders in q.

As an example, we give the K-matrix associated to the F_4 parafermions at level $k=2$:

$$\mathbb{K}^{\mathrm{pf}}(F_{4,2}) = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{4} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 2 & -1 \\ -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & -1 & 2 \end{pmatrix}. \tag{5.23}$$

Explicitly, the relation between the parafermionic character based on the matrix in Eq. (5.23), namely,

$$\sum_{\lambda} Z^{\mathbf{1}}_{\lambda} = \sum_{\{m_i\}} \frac{q^{(\frac{1}{2}\mathbf{m}\cdot\mathbb{K}^{\mathrm{pf}}\cdot\mathbf{m})}}{\prod_i (q)_{m_i}}, \tag{5.24}$$

and the string functions is as follows. Upon splitting the character in pieces containing powers of q which differ by integers, one finds

$$\sum_{\{m_i\}}{}' \frac{q^{(\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m})}}{\prod_i (q)_{m_i}} = q^{\frac{1}{6}}(q)^4_\infty\big(c^{\mathbf{1}}_{(0,0,0,0)} + 3c^{\mathbf{1}}_{(0,0,0,2)}\big) \quad \big(q^n;\ n\in\mathbb{N}\big), \tag{5.25}$$

$$\sum_{\{m_i\}}{}' \frac{q^{(\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m})}}{\prod_i (q)_{m_i}} = 12q^{\frac{1}{6}}(q)^4_\infty c^{\mathbf{1}}_{(1,0,0,0)} \quad \big(q^{n+\frac{1}{2}};\ n\in\mathbb{N}\big), \tag{5.26}$$

$$\sum_{\{m_i\}}{}' \frac{q^{(\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m})}}{\prod_i (q)_{m_i}} = 24q^{\frac{1}{6}}(q)^4_\infty c^{\mathbf{1}}_{(0,0,1,0)} \quad \big(q^{n+\frac{1}{4}};\ n\in\mathbb{N}\big), \tag{5.27}$$

$$\sum_{\{m_i\}}{}' \frac{q^{(\frac{1}{2}\mathbf{m}\cdot\mathbb{K}\cdot\mathbf{m})}}{\prod_i (q)_{m_i}} = 24q^{\frac{1}{6}}(q)^4_\infty c^{\mathbf{1}}_{(0,0,0,1)} \quad \big(q^{n+\frac{3}{4}};\ n\in\mathbb{N}\big). \tag{5.28}$$

The primes on the sums denote the restriction to the powers of q as indicated. The various numerical constants for the string functions $c_\lambda^{\mathbf{1}}$ are the number of independent fields of the form $\Phi_{\lambda'}^{\mathbf{1}}$ which have the same conformal dimension as the field $\Phi_\lambda^{\mathbf{1}}$.

6. Application to level restricted Kostka polynomials

In Section 2.2 we have argued that there exists an intimate relation between the fusion rules of a CFT and the pseudoparticle K-matrix as both count paths on the fusion diagram. In fact, there exists a natural q-deformation of the number of fusion paths giving rise to the so-called level truncated Kostka polynomial. This deformation shows up as part of the UCPF expression for the characters of WZW models, as conjectured in Section 2.1.2. One would thus expect that the level truncated Kostka polynomials can be expressed as UCPFs with the K-matrices found in this paper.

Concretely, if $\phi_i = \phi_{\Lambda_i}$, $i = 1, \ldots, r$, denotes the field corresponding to the ith fundamental weight of $\mathfrak{g}$, the multiplicity of the field ϕ_λ in the fusion rule

$$\phi_1^{n_1} \times \cdots \times \phi_r^{n_r} \tag{6.1}$$

is given by $\left(N_1^{n_1} \ldots N_r^{n_r}\right)_0{}^\lambda$. By associating a power of q to each path, determined through the crystal graph of $\mathfrak{g}$, we obtain a q-deformation of this number. This is referred to as the (dual) level-k truncated Kostka polynomial (or truncated q Clebsch–Gordan coefficient) of $\mathfrak{g}$ and we will denote it by $M_{\lambda\mu}^{(k)}(q)$ where $\mu = \sum_i n_i \Lambda_i$. An explicit expression of $M_{\lambda\mu}^{(k)}(q)$ for $k \to \infty$ is known (see, e.g., [12] and references therein) and originates in Bethe-ansatz techniques [36]. Explicit UCPF type expressions for finite k are known for $\mathfrak{g} = \mathfrak{sl}(n)$ (see [42] for the most general result and also [7,15,28,34]) and $\mathfrak{so}(5)_1$ [12]. In [29], UCPF type expressions for Kostka polynomials for general (non-twisted) affine Lie algebras were conjectured. Proofs for some of these conjectures and expressions for some twisted cases can be found in, for instance, [42] and [40]. The relation between the K-matrices used in these expressions and the ones brought forward in this paper is not clear at the moment. We are gratefull to Ole Warnaar for bringing these references to our attention.

According to the UCPF conjecture, $M_{\lambda\mu}^{(k)}(q)$ should be closely related to

$$q^{\frac{1}{2}\mathbf{n}\cdot\mathbb{K}_{\phi\phi}\cdot\mathbf{n}-\frac{1}{2}\mathbf{n}'\cdot\mathbb{K}_{\phi\phi}\cdot\mathbf{n}'} \sum_{\mathbf{m}}{}' q^{\frac{1}{2}\mathbf{m}\cdot\mathbb{K}_{\psi\psi}\cdot\mathbf{m}+\mathbf{n}\cdot\mathbb{K}_{\phi\psi}\cdot\mathbf{m}} \times \prod_i \begin{bmatrix} ((\mathbb{I}-\mathbb{K}_{\psi\psi})\cdot\mathbf{m})_i - (\mathbb{K}_{\psi\phi}\cdot\mathbf{n})_i + \mathbf{u}_i \\ m_i \end{bmatrix}, \tag{6.2}$$

where $\lambda = \sum n_i' \Lambda_i$ and $\mu = \sum_i n_i \Lambda_i$. (We have set $\mathbf{Q} = 0$ as we are only discussing paths starting at the identity representation.)

In the simply-laced case, it has been conjectured before (see, e.g., [10] and references therein) that $M_{\lambda\mu}^{(k)}(q)$ can indeed be written in terms of the UCPF based on $\mathbb{K}_{\mathrm{qp}} = \mathbb{X}_n^{-1} \otimes \mathbb{M}_k^{-1}$. Here we will focus on a specific non-simply-laced example, namely $\mathfrak{so}(5)$ at levels $k = 1, 2$. We defer a general investigation to future work. An explicit recipe for computing $M_{\lambda\mu}^{(k)}(q)$ for $\mathfrak{g} = \mathfrak{so}(5)$, at level 1, was given in [48]. Explicit formulae for the level $k = 1$ case were given in [12]. Concretely,

$$
\begin{aligned}
&M^{(1)}_{(0,0),(n_1,n_2)}(q) = q^{\frac{1}{2}(n_1^2+n_1n_2)+\frac{3}{8}n_2^2} \sum_{m_1} q^{\frac{1}{2}(m_1^2-m_1n_2)} \begin{bmatrix} \frac{1}{2}n_2 \\ m_1 \end{bmatrix}, \\
&\quad n_1 + \frac{1}{2}n_2 + m_1 \text{ even},\ n_2 \text{ even}, \\
&M^{(1)}_{(1,0),(n_1,n_2)}(q) = q^{\frac{1}{2}(n_1^2+n_1n_2)+\frac{3}{8}n_2^2-\frac{1}{2}} \sum_{m_1} q^{\frac{1}{2}(m_1^2-m_1n_2)} \begin{bmatrix} \frac{1}{2}n_2 \\ m_1 \end{bmatrix}, \\
&\quad n_1 + \frac{1}{2}n_2 + m_1 \text{ odd},\ n_2 \text{ even}, \\
&M^{(1)}_{(0,1),(n_1,n_2)}(q) = q^{\frac{1}{2}(n_1^2+n_1n_2)+\frac{3}{8}n_2^2-\frac{3}{8}} \sum_{m_1} q^{\frac{1}{2}(m_1^2-m_1n_2)} \begin{bmatrix} \frac{1}{2}(n_2+1) \\ m_1 \end{bmatrix}, \\
&\quad \frac{1}{2}(n_2+1) + m_1 \text{ even},\ n_2 \text{ odd}.
\end{aligned} \tag{6.3}
$$

The above formulae are of the UCPF form with

$$
\mathbb{K} = \left(\begin{array}{c|cc} 1 & 0 & -\frac{1}{2} \\ \hline 0 & 1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{4} \end{array}\right), \tag{6.4}
$$

which is to be compared to the $B_{2,1}$ quasiparticle K-matrix of Section 4.3.2, given by

$$
\mathbb{K} = \left(\begin{array}{c|cc} 1 & \frac{1}{2} & \frac{1}{2} \\ \hline \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \end{array}\right). \tag{6.5}
$$

While the pseudoparticle part of Eqs. (6.4) and (6.5) agree, the K-matrices obviously differ in the physical particle part. Both K-matrices are reminiscent of $\mathfrak{so}(6)$, but while (6.5) has physical particles inherited from the **4** and $\bar{\mathbf{4}}$ of $\mathfrak{so}(6)$, Eq. (6.4) contains physical particles inherited from the **4** and the **6** of $\mathfrak{so}(6)$. Since $\mathbf{6} = \mathbf{5} \oplus \mathbf{1}$ under $\mathfrak{so}(5)$, the matrix (6.4) does indeed seem to be better suited to describe general (truncated) Kostka polynomials for $\mathfrak{so}(5)$, although we expect that the $\mathfrak{so}(5)$ Kostka polynomials can also be expressed in terms of a UCPF based on (6.5). Unfortunately, it seems that Eq. (6.4) does not have a straightforward higher level generalization.

Therefore, motivated by the decomposition of finite-dimensional irreducible representations $W_{(n_1,n_2,n_3)}$ of $\mathfrak{so}(6)$ into those of $\mathfrak{so}(5)$ under the regular embedding $\mathfrak{so}(5) \to \mathfrak{so}(6)$, i.e.,

$$
W_{(n_1,n_2,0)} \cong \bigoplus_{l=0}^{n_1} W_{(n_1-l,n_2)}, \tag{6.6}
$$

we introduce

$$
\widetilde{M}^{(1)}_{(0,0),(n_1,n_2)}(q) = \sum_{k=0}^{n_1} \begin{bmatrix} n_1 \\ k \end{bmatrix} M^{(1)}_{(0,0),(n_1-k,n_2)}(q). \tag{6.7}
$$

Inserting the expression for $M^{(1)}_{(0,0),(n_1-k,n_2)}(q)$, and changing $k \to n_1 - k$ in the summation, we find

$$\widetilde{M}^{(1)}_{(0,0),(n_1,n_2)}(q) = \sum_{k,l;k+l+n_2/2 \text{ even}} q^{\frac{1}{2}(k^2+kn_2)+\frac{3}{8}n_2^2+\frac{1}{2}l^2-\frac{1}{2}ln_2} \begin{bmatrix} \frac{1}{2}n_2 \\ l \end{bmatrix} \begin{bmatrix} n_1 \\ k \end{bmatrix}. \tag{6.8}$$

Now, let $p = k - l$, then

$$\begin{aligned}\widetilde{M}^{(1)}_{(0,0),(n_1,n_2)}(q) &= \sum_p \sum_{k,l;k-l=p} q^{\frac{1}{2}p^2+\frac{1}{2}pn_2+\frac{3}{8}n_2^2} q^{kl} \begin{bmatrix} \frac{1}{2}n_2 \\ l \end{bmatrix} \begin{bmatrix} n_1 \\ k \end{bmatrix} \\ &= \sum_p q^{\frac{1}{2}p^2+\frac{1}{2}pn_2+\frac{3}{8}n_2^2} \begin{bmatrix} n_1 + \frac{1}{2}n_2 \\ n_1 - p \end{bmatrix},\end{aligned} \tag{6.9}$$

where, in the last step, we have used a finite version of the Durfee square formula (see [8]). Finally, letting $p \to n_1 - p$, we find

$$\widetilde{M}^{(1)}_{(0,0),(n_1,n_2)}(q) = q^{\frac{1}{2}(n_1^2+n_1n_2)+\frac{3}{8}n_2^2} \sum_p q^{\frac{1}{2}p^2-pn_1-\frac{1}{2}pn_2} \begin{bmatrix} n_1 + \frac{1}{2}n_2 \\ p \end{bmatrix}. \tag{6.10}$$

A similar computation can be given for the other sectors $M^{(1)}_{(n_1',n_2'),(n_1,n_2)}(q)$ of (6.3). Now, Eq. (6.10) is of the UCPF form with

$$\mathbb{K} = \left(\begin{array}{c|cc} 1 & -1 & -\frac{1}{2} \\ \hline -1 & 1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{4} \end{array}\right), \tag{6.11}$$

which has the same $\mathbb{K}_{\psi\psi}$ and $\mathbb{K}_{\phi\phi}$ parts as (6.4), but differs in the coupling $\mathbb{K}_{\psi\phi}$.

Now consider the $\mathfrak{so}(5)$, level $k = 2$ case. As an ansatz we take the pseudoparticle matrix of Section 4.3.2 (see also Eq. (5.17)), and the physical particles of Eq. (6.11), and adjust the coupling between them. Specifically, let

$$\mathbb{K} = \left(\begin{array}{c|ccc|cc} 1 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \hline -1 & 2 & 1 & 1 & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & 1 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{4} \\ \hline 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right). \tag{6.12}$$

Note that this matrix is not invertible, as is the case for the matrix in Eq. (6.11). Thus, Eq. (6.2) reads explicitly

$$\begin{aligned}\widetilde{M}^{(2)}_{(n_1',n_2'),(n_1,n_2)}(q) &= q^{\frac{1}{2}(n_1^2+n_1n_2)+\frac{3}{8}n_2^2-\frac{1}{2}(n_1'^2+n_1'n_2')-\frac{3}{8}n_2'^2} \\ &\times \sum_{\mathbf{m}}{}' q^{\frac{1}{4}(2m_1^2+4m_2^2+3m_3^2+3m_4^2)-\frac{1}{2}m_1(2m_2+m_3+m_4)+m_2(m_3+m_4)+\frac{1}{2}m_3m_4}\end{aligned}$$

$$\begin{aligned}
&\times q^{-\frac{1}{2}n_1(2m_2+m_3+m_4)-\frac{1}{4}n_2(2m_2+3m_3+m_4)} \\
&\times \begin{bmatrix} \frac{1}{2}(2m_2+m_3+m_4)+u_1 \\ m_1 \end{bmatrix} \\
&\times \begin{bmatrix} m_1-(m_2+m_3+m_4)+n_1+\frac{1}{2}n_2+u_2 \\ m_2 \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{2}\bigl(m_1-(2m_2+m_3+m_4)\bigr)+\frac{1}{2}n_1+\frac{3}{4}n_2+u_3 \\ m_3 \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{2}\bigl(m_1-(2m_2+m_3+m_4)\bigr)+\frac{1}{2}n_1+\frac{1}{4}n_2+u_4 \\ m_4 \end{bmatrix},
\end{aligned} \tag{6.13}$$

with some appropriate restriction on the summation over $(m_1,\dots,m_4)$.

Numerical evidence suggests the following conjecture (cf. (6.7))

$$\widetilde{M}^{(2)}_{(n_1',n_2'),(n_1,n_2)}(q)=\sum_{k=0}^{n_1}\begin{bmatrix} n_1 \\ k \end{bmatrix}\sum_{l=0}^{n_1'} M^{(2)}_{(n_1'-l,n_2'),(n_1-k,n_2)}(q), \tag{6.14}$$

or equivalently,

$$\begin{aligned}
M^{(2)}_{(n_1',n_2'),(n_1,n_2)}(q)=&\sum_{k=0}^{n_1}(-1)^k q^{\frac{1}{2}k(k-1)}\begin{bmatrix} n_1 \\ k \end{bmatrix} \\
&\times\sum_{l=0}^{n_1'}(-1)^l \widetilde{M}^{(2)}_{(n_1'-l,n_2'),(n_1-k,n_2)}(q),
\end{aligned} \tag{6.15}$$

where the vectors $\mathbf{u}$ in (6.13), for given (n_1',n_2'), are given in Table 1.[4] The summation restrictions are such that $2m_2+m_3+3m_4\equiv 2((n_1-n_1')+(n_2-n_2')) \bmod 4$, and $n_1+\frac{n_2}{2}+m_1\equiv\bigl(n_1'+\frac{n_2'}{2}\bigr) \bmod 2$.

Again, the conjectured formula (6.14) is strongly reminiscent of the decomposition of finite-dimensional irreducible representations (6.6). This suggests that while the procedure of Section 4.3.2 does produce a pseudoparticle K-matrix leading to the correct central charge, it still overcounts the number of fusion paths. This overcounting can also be seen by applying the analysis of Section 2.2, as the pseudoparticle K-matrix does not give rise to

Table 1
The vectors $\mathbf{n}'$ and $\mathbf{u}$ for the $\mathfrak{so}(5)_2$ Kostka polynomials

(n_1', n_2')	$(u_1; u_2, u_3, u_4)$
$(0,0)$	$(0;0,0,0)$
$(1,0)$	$\bigl(0;1,\frac{1}{2},\frac{1}{2}\bigr)$
$(0,1)$	$\bigl(0;\frac{1}{2},\frac{1}{4},\frac{3}{4}\bigr)$

[4] We have not been able to find the $\mathbf{u}$-vectors corresponding to the remaining integrable highest weight modules at level 2, i.e., $(n_1',n_2')=(2,0),(1,1)$ and $(0,2)$.

the same recursion relations as the $\mathfrak{so}(5)_2$ fusion rules. For this reason we also expect that the $\mathfrak{so}(5)_2$ characters, when written in UCPF form using the K-matrices of Section 4.3.2, will need alternating sign corrections.

7. Discussion

In this paper, we proposed a scheme to obtain the K-matrices for the CFTs with affine Lie algebra symmetry. This construction was based on character identities, which were applied to certain Abelian covering states. After projecting out some degrees of freedom, the K-matrices were obtained. Subsequently, these K-matrices were used to obtain the K-matrices of coset CFTs. Also, they appeared in some expressions for the level-k restricted Kostka polynomials.

It would be interesting to investigate if the K-matrices obtained here indeed are the central objects in the Kostka polynomials related to a general affine Lie algebra. An interesting open question is whether similar K-matrices can be used for more general CFTs, such as the twisted affine Lie algebras (and their parafermions), which were studied in [16] and [17]. Another interesting class of theories which might be addressed in a similar fashion are the affine Lie superalgebras and the related parafermions (see, for instance, [14] and [32] for the case $\mathfrak{osp}(1|2)$).

Most of our consistency checks on whether we obtained the correct K-matrices were based on the fact that the central charge worked out correctly. Even though this proved to be an extremely restrictive 'guide', the ultimate verification of course relies on the construction of the CFT characters in the UCPF form using these K-matrices. While we have proved this in special cases, and did numerical checks in others, a complete verification requires tools beyond the scope of this paper, and will require proving a host of new q-identities. A systematic approach towards a full proof will undoubtedly benefit from a better algebra-geometric understanding of the role of K-matrices (see, e.g., [10,19–21] for some initial studies).

Note added

In an earlier version of this paper we referred to the Kostka polynomials of Section 6 as "generalized Kostka polynomials" to indicate the generalization of the standard A_n Kostka polynomials to general simple Lie algebras. In order to avoid confusion with the "generalized Kostka polynomials", introduced independently by Schilling and Warnaar [43] and by Kirillov and Shimozono [37] (cf. [40] for a discussion), which are more general than the Kostka polynomials which are the subject of this paper, we will simply refer to the polynomials in this paper as (level restricted) Kostka polynomials. We thank Ole Warnaar for communication on these points.

Acknowledgements

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Appendix A. Cartan matrices and their inverses

In this appendix, we will list of the Cartan matrices of the simple Lie algebras, to clarify the conventions used in this paper. In addition, we will give some other properties, namely the dimension and the dual Coxeter number. Other properties can be found, for instance, in [23].

In the Cartan matrices, the empty entries correspond to zeros, unless otherwise implied by the dots. Even though we only use matrices corresponding to simply laced Lie algebras, we will give the Cartan matrices of all the simple Lie algebras, for completeness. We will denote the Cartan matrix corresponding to the Lie algebra X_n by $\mathbb{X}_n$.

$\mathbf{A_n}$: The Cartan matrix for A_n is given by

$$\mathbb{A}_n = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \tag{A.1}$$

$$\mathbb{A}_n^{-1} = \frac{1}{n+1} \begin{pmatrix} n & n-1 & n-2 & \dots & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \dots & 4 & 2 \\ n-2 & 2(n-2) & 3(n-3) & \dots & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \dots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{pmatrix}. \tag{A.2}$$

$\mathbf{B_n}$:

$$\mathbb{B}_n = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix},$$

$$\mathbb{B}_n^{-1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ \frac{1}{2} & 1 & \frac{3}{2} & \cdots & \frac{n-1}{2} & \frac{n}{2} \end{pmatrix} \tag{A.3}$$

$\mathbf{C_n}$:

$$\mathbb{C}_n = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix},$$

$$\mathbb{C}_n^{-1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} \\ 1 & 2 & 2 & \cdots & 2 & 1 \\ 1 & 2 & 3 & \cdots & 3 & \frac{3}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & & n-1 & \frac{n-1}{2} \\ 1 & 2 & 3 & \cdots & n-1 & \frac{n}{2} \end{pmatrix}. \tag{A.4}$$

$\mathbf{D_n}$:

$$\mathbb{D}_n = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & -1 & & \\ & & & -1 & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix},$$

$$\mathbb{D}_n^{-1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 1 & 2 & 3 & \cdots & 3 & \frac{3}{2} & \frac{3}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-2 & \frac{n-2}{2} & \frac{n-2}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & \cdots & \frac{n-2}{2} & \frac{n}{4} & \frac{n-2}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & \cdots & \frac{n-2}{2} & \frac{n-2}{4} & \frac{n}{4} \end{pmatrix}. \tag{A.5}$$

$\mathbf{E_6}$:

$$\mathbb{E}_6 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix},$$

$$\mathbb{E}_6^{-1} = \frac{1}{3} \begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}. \tag{A.6}$$

$\mathbf{E_7}$:

$$\mathbb{E}_7 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix},$$

$$\mathbb{E}_7^{-1} = \frac{1}{2} \begin{pmatrix} 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 4 & 8 & 10 & 12 & 8 & 4 & 6 \\ 5 & 10 & 15 & 18 & 12 & 6 & 9 \\ 6 & 12 & 18 & 24 & 16 & 8 & 12 \\ 4 & 8 & 12 & 16 & 12 & 6 & 8 \\ 2 & 4 & 6 & 8 & 6 & 4 & 4 \\ 3 & 6 & 9 & 12 & 8 & 4 & 7 \end{pmatrix}. \tag{A.7}$$

$\mathbf{E_8}$:

$$\mathbb{E}_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix},$$

$$\mathbb{E}_8^{-1} = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix}. \tag{A.8}$$

$\mathbf{F_4}$:

$$\mathbb{F}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \qquad \mathbb{F}_4^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}. \tag{A.9}$$

$\mathbf{G_2}$:

$$\mathbb{G}_2 = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \qquad \mathbb{G}_2^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}. \tag{A.10}$$

In Table 2 we list some of the properties of the simple Lie algebras. The black nodes in the Dynkin diagrams correspond to the short roots.

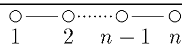
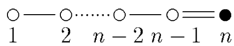
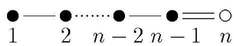
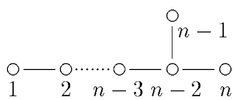
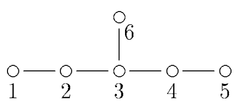
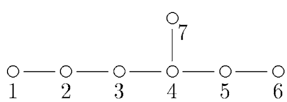
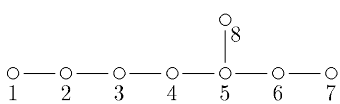
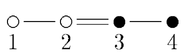
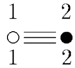
In addition to the Cartan matrices given above, we will frequently use the symmetrized Cartan matrix of B_k, which we denote by $\mathbb{M}_k^{-1}$. Explicitly, we have

$$\mathbb{M}_k = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & k \end{pmatrix}, \qquad \mathbb{M}_k^{-1} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}. \tag{A.11}$$

The simple Lie algebras are labeled by X_n, where n is the rank, and X can be $A, B, \ldots, G$. As we will only be dealing with the untwisted affine Lie algebras, we will use the notation $X_{n,k}$, rather than $(X_n^{(1)})_k$, which is more common in the literature. Sometimes, we will use the notation $\mathfrak{sl}(n)_k$, $\mathfrak{so}(2n-1)_k$, $\mathfrak{sp}(2n)_k$ and $\mathfrak{so}(2n)_k$ for the infinite series of untwisted affine Lie algebras. Here, and in the rest of the paper, the level is denoted by k.

Blackboard bold, such as $\mathbb{A}$ is used for matrices, while vectors are in boldface, such as $\mathbf{Q}$. If we want to specify a column of a matrix, say $\mathbb{A}$, we use the notation $(\mathbb{A})_c$, where the integer c denotes the column we want to specify. In bilinear forms such as $\mathbf{m}^{\mathrm{T}} \cdot \mathbb{K} \cdot \mathbf{m}$, we will frequently omit the transposition symbol $^{\mathrm{T}}$.

Table 2
Some properties of the finite-dimensional simple Lie algebras

X_n	Dynkin diagram	dim X_n	$h^\vee$
A_n		$n(n+2)$	$n+1$
B_n		$n(2n+1)$	$2n-1$
C_n		$n(2n+1)$	$n+1$
D_n		$2n(n-1)$	$2n-2$
E_6		78	12
E_7		133	18
E_8		248	30
F_4		52	9
G_2		14	4

Appendix B. Obtaining the $\mathfrak{so}(5)_1$ matrices

The electron matrix for $\mathfrak{so}(5)_1$ can be obtained by using knowledge about the root diagram and the associated parafermions (see [25] for general parafermion theories). We will anticipate that it is in fact possible to use a quantum Hall type of basis for this theory. So we define a set of electron operators, where the vertex operator part is chosen in such a way that the spin and charge are such that we actually have electron-like operators. The matrix $\mathbb{K}_e$ is obtained via the connection with the exclusion statistics, i.e., we calculate the associated exclusion statistics parameters of these electron operators. From [25] we obtain that at level $k = 1$, the short roots of $\mathfrak{so}(5)$ come with a parafermion operator, which is in fact the Majorana fermion ψ, which has the same exclusion statistics parameter as a fermion, namely, 1. The root diagram of $\mathfrak{so}(5)$ is given in Fig. 1. The electron operators we take to be part of the quantum Hall basis correspond to $\overline{\Psi}^{\downarrow}$, $\Delta_s^{\uparrow\uparrow}$ and Δ_c. These operators take the form (at level $k = 1$)

$$\overline{\Psi}^{\downarrow} = \psi :e^{\frac{i}{\sqrt{2}}(\varphi_c+\varphi_s)}:, \qquad \Delta_s^{\uparrow\uparrow} = :e^{i\sqrt{2}\varphi_s}:, \qquad \Delta_c = :e^{i\sqrt{2}\varphi_c}:, \tag{B.1}$$

where φ_s and φ_c are spin and charge bosons, respectively, chosen according to the spin and charge direction indicated in Fig. 1. From these operators, we infer the following exclusion

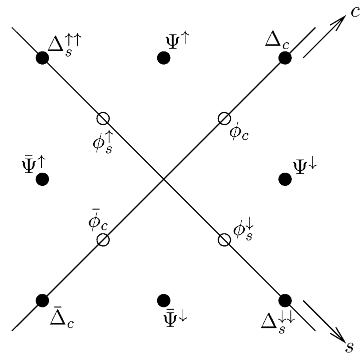


Fig. 1. The roots (•) and weights (○) of $\mathfrak{so}(5)$.

statistics matrix

$$\mathbb{K}_{\mathrm{e}} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \qquad \mathbf{t}_{\mathrm{e}} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \mathbf{s}_{\mathrm{e}} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}. \tag{B.2}$$

We should comment on a few things here. First of all, the matrix we found is equal to the Cartan matrix of $\mathfrak{so}(6)$, which relates to the so-called covering state of the state related to $\mathfrak{so}(5)$. This is analogous to the situation of the Moore–Read state, which is related to a two-layer state. So we could have started from this K-matrix, and performed a similar construction as was done in Section 3.4.1 to find the K-matrices for the Moore–Read state. This would lead to the same matrix (B.2). In addition, in the quasiparticle sector, there is a pseudoparticle, just as in the Moore–Read case. The matrix for the quasiparticle sector can simply be obtained by inverting the matrix (B.2). As said, it is important to notice that the particle in the quasiparticle sector which has trivial quantum numbers, is to be considered as a pseudoparticle. Otherwise, we would not obtain the correct central charge, and hence, not the correct description. We find

$$\mathbb{K}_{\mathrm{qp}} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \qquad \mathbf{t}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{s}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}. \tag{B.3}$$

To obtain the K-matrices for $\mathfrak{so}(5)$ at general level, we take k copies of the level-1 formulation, and do a similar construction as described in Section 3.2. This gives the result of Section 4.3.

Appendix C. The case $G_{2,k}$

In Section 4.3 we found that the K-matrices for the affine Lie algebra $G_{2,k}$ are special in the sense that the number of physical quasiparticles is not equal to the rank of this algebra (which is 2), if we use the standard construction of Section 4.1. Here, we will find another way of describing this theory, which does have two physical quasiparticles.

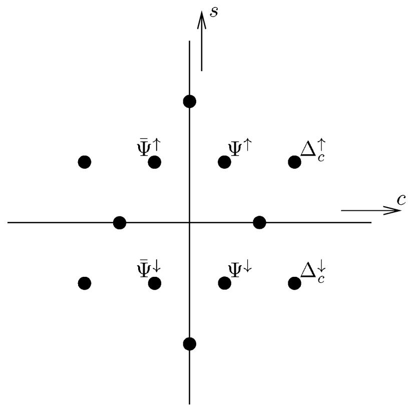


Fig. 2. The roots of G_2.

We will start by deriving the K-matrices for level $k=1$, in a similar way as we did for $\mathfrak{so}(5)_1$ in Appendix B. We continue by explaining how to obtain the K-matrices for general level k. This is a little different from Section 4.1, as the P-transformation which is needed is different.

The root lattice for the Lie algebra G_2 is given in Fig. 2. In fact, it is not possible to pick four electron-like operators, such that the K-matrix is the Cartan matrix of the enveloping algebra $\mathfrak{so}(8)$, but we will stay as close as possible.

The short roots come with two types of parafermions, ψ_1 and ψ_2, which belong to the $\mathbb{Z}_3$ parafermion theory. The operators needed to form the quantum Hall basis are

$$\Psi^{\uparrow} = \psi_1 {:}\exp\left\{\frac{i}{\sqrt{6}}\phi_c + \frac{i}{\sqrt{2}}\phi_s\right\}{:}, \qquad \overline{\Psi}^{\downarrow} = \psi_2 {:}\exp\left\{-\frac{i}{\sqrt{6}}\phi_c - \frac{i}{\sqrt{2}}\phi_s\right\}{:}, \tag{C.1}$$

$$\Delta_c^{\uparrow} = {:}\exp\left\{i\frac{3}{\sqrt{6}}\phi_c + \frac{i}{\sqrt{2}}\phi_s\right\}{:}, \qquad \Delta_c^{\downarrow} = {:}\exp\left\{i\frac{3}{\sqrt{6}}\phi_c - \frac{i}{\sqrt{2}}\phi_s\right\}{:}, \tag{C.2}$$

where $\phi_{c,s}$ are the charge and spin boson. As the K-matrix for the $\mathbb{Z}_3$ parafermions is given by

$$\mathbb{K}_{\mathbb{Z}_3}^{\mathrm{pf}} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}, \tag{C.3}$$

and the statistics parameters due to the vertex operators of the spin and charge bosons are easily calculated, we find the following data for the 'electron' sector of the $G_{2,k=1}$ theory

$$\mathbb{K}_{\mathrm{e}} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \qquad \mathbf{t}_e = -\begin{pmatrix} 1 \\ -1 \\ 3 \\ 3 \end{pmatrix}, \qquad \mathbf{s}_e = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}. \tag{C.4}$$

By the duality construction, we find the dual data

$$\mathbb{K}_{\mathrm{qp}} = \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & 1 & -\frac{1}{2} \\ \hline -1 & 1 & 2 & -1 \\ \frac{1}{2} & -\frac{1}{2} & -1 & 1 \end{array}\right), \qquad \mathbf{t}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{s}_{\mathrm{qp}} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \tag{C.5}$$

where the first two particles are pseudoparticles, which reduce the central charge, and take care of the non-Abelian statistics. Note that we do not use the usual ordering of the Cartan matrix (compare Appendix A), because in the quasiparticle sector, we want the first to particles to be the pseudoparticles.

Picking the operators associated to the right roots is crucial in finding a basis for the G_2 affine Lie algebra. The way we have chosen them here gives a description which does give the right central charge, and has two physical quasiparticles.

We would like to comment on the difference between the pseudoparticle matrices for the two descriptions of $G_{2,1}$. If we apply the composite construction on the 2×2 pseudoparticle matrix of this appendix, we indeed find the pseudoparticle matrix (at level 1) of Section 4.3. This matrix also appeared in Section 2.2, Eq. (2.59). So the pseudoparticles are equivalent in both cases.

We now proceed by constructing the matrices for level k. As usual, the covering is of the form $\mathbb{K}_{\mathrm{e}} \otimes \mathbb{I}_k$. The required P-transformation turns out to be of the form (compare with Appendix D)

$$\mathbb{P}' = \begin{pmatrix} \mathbb{I}_4 & \mathbb{J}_4^u & \cdots & \mathbb{J}_4^u \\ \mathbb{J}_4^l & \mathbb{I}_4 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{J}_4^u \\ \mathbb{J}_4^l & \cdots & \mathbb{J}_4^l & \mathbb{I}_4 \end{pmatrix}, \tag{C.6}$$

where $\mathbb{J}_4^u$ and $\mathbb{J}_4^l$ are given by

$$\mathbb{J}_4^u = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \qquad \mathbb{J}_4^l = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}. \tag{C.7}$$

Because $\mathbb{J}_4^u + \mathbb{J}_4^l = \mathbb{I}_4$, all composites up to order k are formed. To display the resulting matrix, it is most convenient to reorder the particles in the order of increasing quantum numbers (this is not done automatically, because of the form of the P-transformation). To conveniently display the 'permuted' K-matrix for the electron sector, we define a modified Cartan matrix of $\mathbb{D}_4$

$$\mathcal{M}(a,b,c) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & c & 0 \\ b & c & a & b \\ 0 & 0 & b & a \end{pmatrix}. \tag{C.8}$$

Then, the electron K-matrix for $G_{2,k}$ can be described by

$$\mathbb{K}_{\mathrm{e}}^{G_{2,k}} = \begin{pmatrix} \mathcal{M}(2,0,-1) & \mathcal{M}(2,0,-1) & \cdots & & \cdots & \mathcal{M}(2,1,-1) \\ \mathcal{M}(2,0,-1) & \mathcal{M}(4,0,-2) & & & & \mathcal{M}(4,2,-2) \\ \vdots & & \ddots & & & \vdots \\ & & & \mathcal{M}\big(2\min(i,j), \max(i+j-k,0), -\min(i,j)\big) & & \\ \vdots & & & & \ddots & \vdots \\ \mathcal{M}(2,1,-1) & \mathcal{M}(4,2,-2) & \cdots & & \cdots & \mathcal{M}(2k,k,-k) \end{pmatrix}. \tag{C.9}$$

To make this a little more clear, we give the result for $k = 2$ explicitly

$$\mathbb{K}_e = \left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 2 \\ \hline 2 & 0 & 1 & 0 & 4 & 0 & 2 & 0 \\ 0 & 2 & -1 & 0 & 0 & 4 & -2 & 0 \\ 1 & -1 & 2 & 1 & 2 & -2 & 4 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 & 4 \end{array}\right). \tag{C.10}$$

The quasiparticle sector for $k > 2$ is characterized by the following matrices (compare with Section 4)

$$\mathbb{K}_{\psi\psi} = \begin{pmatrix} 2\mathbb{D}_4^{-1} & -\mathbb{D}_4^{-1} & & & -\big(\mathbb{D}_4^{-1}\big)_1 & \\ -\mathbb{D}_4^{-1} & 2\mathbb{D}_4^{-1} & \ddots & & & \\ & \ddots & \ddots & -\mathbb{D}_4^{-1} & & \\ & & -\mathbb{D}_4^{-1} & 2\mathbb{D}_4^{-1} & & -\big(\mathbb{D}_4^{-1}\big)_3 \\ -\big(\mathbb{D}_4^{-1}\big)_1^{\mathrm{T}} & & & & 1 & 0 \\ & & & -\big(\mathbb{D}_4^{-1}\big)_3^{\mathrm{T}} & 0 & 1 \end{pmatrix}, \tag{C.11}$$

$$\mathbb{K}_{\phi\psi} = \begin{pmatrix} & -\big(\mathbb{D}_4^{-1}\big)_2^{\mathrm{T}} & 0 & 1 \\ -\big(\mathbb{D}_4^{-1}\big)_4^{\mathrm{T}} & & \frac{1}{2} & 0 \end{pmatrix}, \tag{C.12}$$

$$\mathbb{K}_{\phi\phi} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{C.13}$$

$$\mathbf{t}_{\mathrm{qp}} = (0, \ldots, 0; 1, 1), \tag{C.14}$$

$$\mathbf{s}_{\mathrm{qp}} = (0, \ldots, 0; -1, 1). \tag{C.15}$$

So, although the form of the K-matrix differs from the general description, we still find that all the elements are related to the (inverse) Cartan matrix of the Lie algebra D_4.

Now that we have a description of G_2 which does have two quasiparticles (for every k), we can use the same conjecture (5.14) to find the K-matrices for the parafermions, namely, the parafermion theory $G_{2,k}/[\mathfrak{u}(1)]^2$. So, without giving the explicit form, it is found that the parafermion K-matrix $\mathbb{K}^{\mathrm{pf}}_{G_2} = \mathbb{K}^{-1}_{\psi\psi}$ does have the right properties. It gives the correct central charge, and reproduces the string functions as described in Section 5.3.

For the case $k = 1$, we indeed find that the parafermions associated to G_2 are the $\mathbb{Z}_3$ parafermions. At level $k \geqslant 2$ we find the K-matrices of the G_2 parafermions, which for $k = 2$ is given by

$$\mathbb{K}^{\mathrm{pf}}_{G_2,k=2} = \begin{pmatrix} \frac{5}{3} & \frac{1}{3} & -\frac{1}{2} & 0 & \frac{4}{3} & \frac{3}{2} \\ \frac{1}{3} & \frac{5}{3} & -\frac{1}{2} & 0 & \frac{2}{3} & \frac{4}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ \frac{4}{3} & \frac{2}{3} & 0 & 0 & \frac{8}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{4}{3} & 0 & 0 & \frac{4}{3} & \frac{8}{3} \end{pmatrix}. \tag{C.16}$$

Note the ‘asymmetry’ between the parafermions 3 and 4.

Appendix D. Relating different bases

In Section 4.1 we pointed out that the K-matrices for $\mathfrak{sl}(3)_k$ found in [3] differ from the ones we presented here. The reason for this was also given. In [3], all the particles in the electron sector were chosen such that their charge all had the same sign. Consequently, the K-matrix for level-1 was based on the roots α_1 and $-\alpha_2$. This resulted in the following K-matrix and quantum number vectors

$$\mathbb{K}'^{(k=1)}_{\mathrm{e}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{t}'_{\mathrm{e}} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{s}'_{\mathrm{e}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{D.1}$$

In this appendix, we will explain in detail the relation between this approach and the one followed in this paper. The matrix (D.1) can also be used to obtain K-matrices for $\mathfrak{sl}(3)_k$. This formulation is different, but can be related to the one obtained in Section 4.1. We will first show that we can construct the $\mathfrak{sl}(3)_k$ K-matrices found in [3] using the P-transformations. We then explicitly relate the two constructions.

So, let us begin with the covering matrix based on Eq. (D.1), which is constructed in the usual way, by taking a direct sum of k copies: $\mathbb{K}'^{\mathrm{cover}}_{\mathrm{e}} = \mathbb{K}'^{(k=1)}_{\mathrm{e}} \otimes \mathbb{I}_k$. Now the P-transformation is different than the one used in Section 4.1. It will be such that all composites up to order k are formed (for both spin up and spin down particles). However, $\mathbb{P}$ is not lower triangular, but instead we have

$$\mathbb{P}' = \begin{pmatrix} \mathbb{I}_2 & \mathbb{J}^u_2 & \cdots & \mathbb{J}^u_2 \\ \mathbb{J}^l_2 & \mathbb{I}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{J}^u_2 \\ \mathbb{J}^l_2 & \cdots & \mathbb{J}^l_2 & \mathbb{I}_2 \end{pmatrix}. \tag{D.2}$$

Here, $\mathbb{J}^u_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\mathbb{J}^l_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The transformed K-matrix $\mathbb{P}' \cdot \mathbb{K}'^{\mathrm{cover}}_{\mathrm{e}} \cdot \mathbb{P}'^{T}$ is most easily described after a suitable permutation of the particles, which orders the particles according to their quantum numbers; as indicated before, all composites (up to order k) are formed, because $\mathbb{J}_2^u + \mathbb{J}_2^l = \mathbb{I}_2$. The quantum numbers after applying the P-transformation to the covering and the permutation to order them, are given by $\mathbf{t}'_\mathrm{e} = -(1,1,2,2,\ldots,k,k)$ and $\mathbf{s}'_\mathrm{e} = (1,-1,2,-2,\ldots,k,-k)$. The K-matrix becomes

$$\mathbb{K}'_\mathrm{e} = \begin{pmatrix} 2 & 0 & 2 & 0 & \cdots & 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 & \cdots & 0 & 2 & 1 & 2 \\ 2 & 0 & 4 & 0 & \cdots & 4 & 1 & 4 & 2 \\ 0 & 2 & 0 & 4 & \cdots & 1 & 4 & 2 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & 4 & 1 & \cdots & 2(k-1) & k-2 & 2(k-1) & k-1 \\ 0 & 2 & 1 & 4 & \cdots & k-2 & 2(k-1) & k-1 & 2(k-1) \\ 2 & 1 & 4 & 2 & \cdots & 2(k-1) & k-1 & 2k & k \\ 1 & 2 & 2 & 4 & \cdots & k-1 & 2(k-1) & k & 2k \end{pmatrix}. \tag{D.3}$$

This matrix is to be compared with $\mathbb{K}_\mathrm{e}$ of Eq. (4.9). The diagonal part of the 2×2 blocks is the same, namely, $2\min(i,j)$, where i,j label the blocks. The off-diagonal parts are given by $\max(k-i-j,0)$. The inverse is found to be (again, after a suitable permutation of the particles)

$$\mathbb{K}'_\mathrm{qp} = \left(\begin{array}{cccc|cc} & & & & -\left(\mathbb{A}_2^{-1}\right)_1 & 0 \\ & & & & 0 & 0 \\ & \mathbb{A}_2^{-1}\otimes\mathbb{A}_{k-1} & & & \cdots & \cdots \\ & & & & 0 & 0 \\ & & & & 0 & -\left(\mathbb{A}_2^{-1}\right)_2 \\ \hline -\left(\mathbb{A}_2^{-1}\right)_1^\mathrm{T} & 0 \;\cdots\; 0 & & 0 & \frac{2}{3} & 0 \\ 0 & 0 \;\cdots\; 0 & & -\left(\mathbb{A}_2^{-1}\right)_2^\mathrm{T} & 0 & \frac{2}{3} \end{array}\right), \tag{D.4}$$

which is to be compared with $\mathbb{K}_\mathrm{qp}$ of Eq. (4.12). To relate the two descriptions, we make use of the fact that we know how to relate the matrices for $k=1$. The difference is the use of α_2 in the description detailed in Section 4.1 and $-\alpha_2$ in the description of this appendix and [3]. Recall that the K-matrix for level $k=1$ from Section 4.1 is given by $\mathbb{K}_\mathrm{e}^{(k=1)} = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. So we find that we can relate the two K-matrices for level-1 by a W-transformation, which is given by $\mathbb{K}_\mathrm{e}'^{(k=1)} = \mathbb{W}\cdot\mathbb{K}_\mathrm{e}^{(k=1)}\cdot\mathbb{W}^\mathrm{T}$, where $\mathbb{W} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Because we also know how to transform the coverings into the corresponding K-matrices for $\mathfrak{sl}(3)_k$, we can relate the two descriptions in terms of a W-transformation. Apart from the extra permutations which are involved, the calculation is straightforward, and we find the relation $\mathbb{K}'_\mathrm{e} = \mathbb{W}_\mathrm{e}\cdot\mathbb{K}_\mathrm{e}\cdot\mathbb{W}_\mathrm{e}^\mathrm{T}$, with (dropping the subscript 2)

$$\mathbb{W}_{\mathrm{e}} = \begin{pmatrix} -\mathbb{J}^l & & & -\mathbb{J}^u & \mathbb{J}^u \\ & \ddots & \iddots & & \mathbb{J}^u \\ & & \vdots\!\cdot\!\vdots & & \vdots \\ & \iddots & \ddots & & \mathbb{J}^u \\ -\mathbb{J}^u & & & -\mathbb{J}^l & \mathbb{J}^u \\ & & & & \mathbb{J}^u - \mathbb{J}^l \end{pmatrix}, \tag{D.5}$$

where $\vdots\!\cdot\!\vdots$ stands for

$$\begin{pmatrix} -\mathbb{J}^l & -\mathbb{J}^u \\ -\mathbb{J}^u & -\mathbb{J}^l \end{pmatrix}$$

if k is odd and for

$$\begin{pmatrix} -\mathbb{J}^l & & -\mathbb{J}^u \\ & -\mathbb{I} & \\ -\mathbb{J}^u & & -\mathbb{J}^l \end{pmatrix}$$

if k is even. Note that $\mathbb{W}_{\mathrm{e}}^{-1} = \mathbb{W}_{\mathrm{e}}$. For the quasiparticle sector we have a similar relation, $\mathbb{K}'_{\mathrm{qp}} = \mathbb{W}_{\mathrm{qp}} \cdot \mathbb{K}_{\mathrm{qp}} \cdot \mathbb{W}_{\mathrm{qp}}^{\mathrm{T}}$. But because we needed the extra permutations, we do not have the relation $\mathbb{W}_{\mathrm{qp}} = (\mathbb{W}_{\mathrm{e}}^{-1})^{\mathrm{T}}$. This only holds for the case at hand if we undo this permutation. Instead, we have

$$\mathbb{W}_{\mathrm{qp}} = \begin{pmatrix} -\mathbb{I} & & & & \\ & -\mathbb{I} & & & \\ & & \ddots & & \\ & & & -\mathbb{I} & \\ \mathbb{J}^u & \mathbb{J}^u & \cdots & \mathbb{J}^u & \mathbb{J}^u - \mathbb{J}^l \end{pmatrix}. \tag{D.6}$$

Note that in going from the one formulation to the other, we are only transforming the physical quasiparticles, the pseudoparticles are not changed. This should be the case, as the pseudoparticles govern the fusion rules and the central charge.

Let us end this discussion by mentioning that the formulation for $\mathfrak{sl}(3)_k$ of the type of Eq. (D.3) can be generalized to arbitrary affine Lie algebra CFTs. The relations between the description in this paper is precisely analogous to the relation for $\mathfrak{sl}(3)$ as described in this appendix. The only difference would be in the form of the matrices $\mathbb{J}^u$ and $\mathbb{J}^l$. However, they still would only have non-zero elements on the diagonal, subject to the constraint $\mathbb{J}^u + \mathbb{J}^l = \mathbb{I}$.

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