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Non-abelian spin-singlet quantum Hall states: wave functions and quasihole state counting

E. Ardonne [a], N. Read [b], E. Rezayi [c], K. Schoutens [a]

[a] *Institute for Theoretical Physics, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*
[b] *Department of Physics, Yale University, P.O. Box 208120, New Haven, CT 06520-8120, USA*
[c] *Department of Physics, CSU-Los Angeles, Los Angeles, CA 90032, USA*

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Abstract

We investigate a class of non-abelian spin-singlet (NASS) quantum Hall phases, proposed previously. The trial ground and quasihole excited states are exact eigenstates of certain $(k+1)$-body interaction Hamiltonians. The $k=1$ cases are the familiar Halperin abelian spin-singlet states. We present closed-form expressions for the many-body wave functions of the ground states, which for $k>1$ were previously defined only in terms of correlators in specific conformal field theories. The states contain clusters of k electrons, each cluster having either all spins up, or all spins down. The ground states are non-degenerate, while the quasihole excitations over these states show characteristic degeneracies, which give rise to non-abelian braid statistics. Using conformal field theory methods, we derive counting rules that determine the degeneracies in a spherical geometry. The results are checked against explicit numerical diagonalization studies for small numbers of particles on the sphere.

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1. Introduction and summary

The observation [1–3] of a quantum Hall (QH) state at an even-denominator filling factor, $\nu = 5/2$, stimulated the development of trial wave functions outside the usual hierarchy (or later, composite fermion) approach, which generates only odd-denominator fractions. The $5/2$ state is interpreted as half-filling of the first excited Landau level (LL), the lowest one being filled with electrons of both spins, and can be mapped to half-filling of the lowest LL, with a suitable Hamiltonian. There are now strong indications that this state is spin-polarized [4,5], and described [4] by the paired "pfaffian" state of Moore and Read (MR) [6], which has filling factor $1/2$. This state was originally proposed as

E-mail address: ardonne@science.uva.nl (E. Ardonne).

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an example that manifests non-abelian braid statistics of its quasiparticle excitations [6]. Generalizations exist in which the particles are "clustered" in k-plets ($k = 1, 2, 3, \dots$), but still spin-polarized [7]. In these states, the non-abelian statistics are associated with parafermion conformal field theories (CFTs).

It is well known that, despite the presence of a strong magnetic field, spin-singlet QH states are sometimes favored over their spin-polarized counterparts. The possibility to manipulate the Zeeman splitting by the application hydrostatic pressure and by tilting the magnetic field has opened the possibility of systematic studies of transitions between spin-polarized and non-polarized QH states (see, e.g., [8]). In this light, states analogous to the spin-polarized clustered QH states of [7], but with a spin-singlet structure, have been constructed [9]. In [9], trial wave functions for these non-abelian spin-singlet (NASS) states have been written in terms of correlators in a CFT describing parafermions associated to the Lie algebra SU(3). We remark that one may consider an alternative series of NASS states, whose algebraic structure is related to SO(5) rather than SU(3). The simplest state of this type is a paired spin-singlet state that exhibits a separation of spin and charge in the quasihole excitation spectrum [10]. The SO(5)-based NASS states will not be discussed in this paper.

In the present paper, we study in detail some of the properties of the NASS states, paying special attention to the case $k = 2$. We give explicit closed form expressions for the ground state wave functions, and study the degeneracies of their quasihole excitations. The degeneracies of states with fixed spins and fixed well-separated positions of the quasiparticles are the origin of the non-abelian braid statistics.

We first strengthen the case for the existence of the incompressible phases of matter with the universal properties of the states of Ardonne and Schoutens (AS) [9], by showing that their trial wave functions for the ground state and for states with quasiholes are exact zero-energy eigenstates of certain $(k + 1)$-body interaction Hamiltonians for particles in a single LL, in a similar way as the spin-polarized cases [7]. The explicit closed-form wave functions for the ground states are obtained. In the study of the quasihole degeneracies, we then follow two complementary approaches. The first is an analytical path, which relies heavily on the formal structure of the associated parafermion CFT, and on the analogy with earlier studies for spin-polarized non-abelian QH states [11–13]. While at present we lack explicit expressions for the many-body wave functions describing the quasiholes, we have enough control to derive explicit counting formulas for the degeneracies, for $k = 2$, for particles on a sphere. The second approach is a numerical study of the $(k + 1)$-body Hamiltonian for the case $k = 2$, on the sphere. The numbers of zero-energy states for each number of electrons N and of quasiholes n considered are in exact agreement with the analytical derivation. In addition, we study the excitation spectrum of the same Hamiltonian, and compare the ground state with that of electrons interacting via the lowest LL Coulomb interaction.

A highlight of the analytical approach in this paper comes in the derivation of the total degeneracies of quasihole states. In the CFT set up (which will be described in more detail in Section 2) the QH states are described as conformal blocks of "particle" and "quasihole" operators. The particle operator factorizes as a product of a vertex operator and a parafermion field, and the quasihole operator is the product of a vertex operator and a so-called spin field of the parafermion CFT. The nontrivial fusion rules of these spin fields cause a degeneracy of the ground states in the presence of quasiholes at fixed positions and spins. There is also further degeneracy associated with the positions and spins of the quasiholes, which is finite on the sphere. The two contributions need to be combined in the right way. It turns out, as in earlier cases, that the various states which stem from the non-trivial fusion rules have a different spatial degeneracy. Therefore, we can not just multiply the two degeneracy factors, but we need to break up the degeneracies due to the fusion rules. To accomplish this task, and arrive at the final counting formula, we analyze "truncated chiral spectra" in the SU(3) parafermion CFT, using the methods of [14].

This paper is organized as follows. In Section 2, we explain in which way CFT is used to describe QH states, and review the NASS states as correlators. In Section 3, we introduce the $(k+1)$-body interaction, and show that the AS correlators give zero-energy ground states. In Section 4 we give the explicit ground state wave functions for the NASS states, and discuss their spin-singlet properties. Section 5 describes the correlators which give the states with quasiholes present. The derivation of the counting formula is done in Sections 6 to 9, using the method which is outlined above, with Eq. (49) as the final outcome. Explicit results of this formula for the degeneracy of the ground states in the presence of quasiholes are given in the same section for several N (the number of electrons) and n (the number of quasiholes). In Section 10 we present numerical diagonalization studies on a sphere, finding full agreement with the analytical expression obtained in Section 9, and compare the states with the ground state of the Coulomb interaction.

2. QHE–CFT correspondence

In the QHE, following Laughlin [15], trial wave functions have long been used as paradigms that represent an entire phase of incompressible behavior. This notion was reviewed in Ref. [7], so in this paper we will concentrate on the properties of trial states and their position-space wave functions. As explained by MR [6], many QH trial wave functions of the $(2+1)$-dimensional system can be obtained as conformal blocks (i.e., chiral parts of correlation functions or "correlators") in a suitable chiral CFT in 2 euclidean dimensions, as we will briefly explain.

For particles with complex coordinates $z_j = x_j + iy_j$, $j = 1, \ldots, N$, for N particles, we will use reduced wave functions $\widetilde{\Psi}(z)$, and neglect spin temporarily. For the lowest Landau level (LLL), the reduced wave function must be holomorphic in the z_i's. For particles in the plane, the full wave function $\Psi(z)$ is recovered by multiplying $\widetilde{\Psi}(z)$ by $\exp(-\sum_i |z_i|^2/4)$; we have set the magnetic length to 1. For particles on the sphere [16], the coordinate z_i refers to stereographic projection, and the full wave function is recovered by multiplying by $\prod_i (1+|z_i|^2/4R^2)^{-(1+N_\phi/2)}$, where N_ϕ is the total number of magnetic flux quanta through the sphere [7,12]. In the latter case, the reduced wave function $\widetilde{\Psi}(z)$ must be a polynomial of degree no higher than N_ϕ in each z_i, so that the z component of angular momentum of each particle lies between $N_\phi/2$ and $-N_\phi/2$. Note that we use the term "particles" for the underlying particles, which are either charged bosons or charged fermions (electrons), because it is convenient to consider cases of either statistics together.

The simplest example of a state with a uniform density (a state of zero total angular momentum on the sphere [16]) is the Laughlin wave function [15]:

$$\widetilde{\Psi}_{\mathrm{L}}^{M}(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^M, \tag{1}$$

for a fixed integer M. The filling factor can be defined for a sequence of states as $\nu = \lim_{N\to\infty} N/N_\phi$, where N_ϕ is identified with the largest power of any z_i in the state. For the Laughlin state it is $\nu = 1/M$. Note that this function is antisymmetric (describes fermions) for M odd, and symmetric (describes bosons) for M even.

The Laughlin wave function can be obtained as

$$\widetilde{\Psi}_{\mathrm{L}}^{M}(z_1,\ldots,z_N)=\lim_{z_\infty\to\infty}(z_\infty)^a\big\langle V_e(z_1)\cdots V_e(z_N)e^{-i\sqrt{M}N\varphi}(z_\infty)\big\rangle, \tag{2}$$

with $V_e(z)=\exp(i\sqrt{M}\,\varphi)$ a chiral vertex operator in the $c=1$ chiral CFT describing a single scalar field φ compactified on a radius $R^2=M$. The operator $e^{-i\sqrt{M}N\varphi}(z_\infty)$ brings in a positive background charge, which guarantees the overall neutrality of the system. The constant a must be chosen in such a way that the effect of the background charge does not go to zero in the limit $z_\infty\to\infty$; in Eq. (2) we need $a=MN^2$. This procedure is simpler for our purposes than the uniform background charge used in MR [6], though the latter has the additional feature of reproducing the factors in the unreduced wave function.

Other, more complicated, QH states can be constructed by invoking more complicated CFTs. The CFT framework guarantees that a number of consistency requirements for such states are met [6]. The trial wave functions become more meaningful, and the corresponding phase really exists, when there is a (local) Hamiltonian for which the trial state is the nondegenerate ground state, and the excitation spectrum (for the same N_ϕ as the ground state) has a gap in the thermodynamic limit. Short range 2-body interaction Hamiltonians with these properties for the Laughlin state were found by Haldane [16], and 3-body interactions for which the MR state is an exact zero energy eigenstate were found beginning from the work of Ref. [17]. Read and Rezayi (RR) discovered [7] that these constructions are the first two cases in an infinite sequence, and found the parafermion states as the exact zero energy eigenstates of $(k+1)$-body interactions for all k. Here we will show similarly that, in the case of particles with spin, the NASS states of Ref. [9] are exact eigenstates of zero energy for $(k+1)$-body spin-independent interactions, with the Halperin state and the corresponding 2-body interaction [16] as the only case known previously. First, we recall the construction in Ref. [9], then in the following section we establish that the wave function defined by a correlator here is a zero-energy eigenstate of a $(k+1)$-body interaction Hamiltonian.

The NASS states proposed in [9] can be viewed as non-abelian generalizations of the abelian spin-singlet Halperin states labeled as $(m+1,m+1,m)$ (see Eq. (6)), or alternatively, as spin-singlet analogs of the spin-polarized “clustered” or “parafermion” states of RR [7]. The filling fraction of the NASS states is $\nu=\frac{2k}{2kM+3}$, with M an integer (M is odd when the particles are fermions, even when they are bosons). The wave functions of these states are constructed as conformal blocks in basically the same way as was done above for the Laughlin state. In the basic case $M = 0$, the component of the ground state with any set of $N/2$ of the particles having spin up, and the remainder spin down, is defined (up to phases that may be needed when reconstructing the full state from these components) as the correlator [9]

$$\widetilde{\Psi}^{k,M=0}_{\mathrm{NASS}} = \lim_{z_\infty\to\infty} (z_\infty)^{3N^2/(2k)} \left\langle \exp\left(i\frac{N}{2\sqrt{k}}(\alpha_2-\alpha_1)\cdot\vec{\varphi}\right)(z_\infty) \times B_{\alpha_1}(z_1^\uparrow)\cdots B_{\alpha_1}(z_{N/2}^\uparrow)B_{-\alpha_2}(z_1^\downarrow)\cdots B_{-\alpha_2}(z_{N/2}^\downarrow)\right\rangle. \tag{3}$$

In this equation the "particle operators" (to avoid confusion, we emphasize that this means the operators that correspond to the particles in the CFT, not the operators that create actual particles in the $(2+1)$-dimensional system) are current operators $B_\alpha(z)$ in an $\mathrm{SU}(3)_k$ (i.e., level k) Wess–Zumino–Witten CFT. These currents can be written in terms of two bosons $\vec{\varphi} = (\varphi_1, \varphi_2)$ and a Gepner parafermion ψ_α associated to $\mathrm{SU}(3)_k/[\mathrm{U}(1)]^2$ [18]. The currents are labeled by the corresponding roots α of SU(3)

$$B_\alpha(z) = \psi_\alpha \exp\left(i\alpha\cdot\vec{\varphi}/\sqrt{k}\right)(z). \tag{4}$$

The roots are given by $\alpha_1 = (\sqrt{2}, 0)$, $\alpha_2 = (-\sqrt{2}/2, \sqrt{6}/2)$. We note that these two roots form an SU(2) doublet under an SU(2) subalgebra of SU(3); the embedding of the subalgebra is isomorphic to that given in terms of 3×3 Hermitian matrices [generators of SU(3)] by the 2×2 Hermitian blocks at the upper left corner. For such an embedding, there is also a U(1) subalgebra [generated by "hypercharge" diag$(1, 1, -2)$; the particles carry hypercharge 1] that commutes with the SU(2) and corresponds to the particle number. Note that a (hyper-)charge at infinity is again needed for neutrality.

Working out the vertex-operator part of this correlator, we arrive at the following factorized form of the NASS state (after multiplication by an additional Laughlin factor to obtain general M [1])

$$\begin{aligned}&\widetilde{\Psi}^{k,M}_{\mathrm{NASS}}\left(z_1^\uparrow,\ldots,z_{N/2}^\uparrow; z_1^\downarrow,\ldots,z_{N/2}^\downarrow\right)\\ &\quad= \left\langle \psi_{\alpha_1}(z_1^\uparrow)\cdots\psi_{\alpha_1}(z_{N/2}^\uparrow)\psi_{-\alpha_2}(z_1^\downarrow)\cdots\psi_{-\alpha_2}(z_{N/2}^\downarrow)\right\rangle\\ &\qquad\times \left[\widetilde{\Psi}^{(2,2,1)}_{\mathrm{H}}(z_i^\uparrow; z_j^\downarrow)\right]^{1/k}\widetilde{\Psi}^{M}_{\mathrm{L}}(z_i^\uparrow; z_j^\downarrow).\end{aligned} \tag{5}$$

The wave function $\widetilde{\Psi}^{(2,2,1)}_{\mathrm{H}}$ is one of the Halperin wave functions [19]

$$\begin{aligned}&\widetilde{\Psi}^{(m',m',m)}_{\mathrm{H}}\left(z_1^\uparrow,\ldots,z_{N/2}^\uparrow; z_1^\downarrow,\ldots,z_{N/2}^\downarrow\right)\\ &\quad= \prod_{i<j}\left(z_i^\uparrow - z_j^\uparrow\right)^{m'}\prod_{i<j}\left(z_i^\downarrow - z_j^\downarrow\right)^{m'}\prod_{i,j}\left(z_i^\uparrow - z_j^\downarrow\right)^{m}.\end{aligned} \tag{6}$$

The latter give rise to spin-singlet states whenever $m' = m+1$ [16]. The wave function Eq. (5) contains a term which is a correlator of parafermions, the explicit form of which will be found below.

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[1] We can also obtain the Laughlin factor by using the particle operators (19) and (20) in the correlator, together with a suitably adjusted background charge.

We also mention here that the CFT construction implies that the number of sectors of edge states (representations of the chiral algebra), and the number of ground states (conformal blocks with N particle operators inserted) on the torus for N divisible by $2k$, are both given by $(k+1)(k+2)(2kM+3)/6$, which is an integer. For $M=0$, this coincides with the numbers for $\mathrm{SU}(3)_k$ current algebra. The filling factor is $P/Q = 2k/(2kM+3)$, so if P and Q are defined as being coprime, then the denominator $Q = 2kM+3$ unless $2k$ and 3 have a common factor, that is unless k is divisible by 3, in which case we have $Q = (2kM+3)/3$. The number of ground states on the torus is then always divisible by the denominator Q of the filling factor, as it should be. We also expect that for some k values there are ground states on the torus for other N values, as for the MR states [20] and RR states.

3. Solution of $(k+1)$-body Hamiltonian

It is known [16] that the abelian Halperin spin-singlet state is the unique (on the sphere) exact zero-energy eigenstate of a two-body interaction Hamiltonian. Other than the trivial case $(1,1,0)$, which is two filled Landau levels, where the Hamiltonian in question is zero, the simplest case is $m=1$ in Eq. (6), the $(2,2,1)$ state, which corresponds to $M=0$ in Eq. (3) or (5). In the latter case this Hamiltonian is simply a repulsive δ-function interaction between any two particles. As in Refs. [7,12], it is simplest to start by generalizing this $M=0$ case. Because higher M values are obtained by multiplying by additional Laughlin factors, the Hamiltonians for $M=0$ can be straightforwardly extended to $M>0$ by extending the range of the $(k+1)$-body part, and adding 2-body interactions as needed, which have the effect of requiring zero-energy states to contain the Laughlin factors. We will not detail this here, however, see Section 10.

The natural choice of Hamiltonian for $M=0$ is to consider the $(k+1)$-body δ-function as in Ref. [7], but here for particles with spin. The Hamiltonian (acting within the LLL) is

$$H = V \sum_{i_1<i_2<\cdots<i_{k+1}} \delta^2(z_{i_1}-z_{i_2})\delta^2(z_{i_2}-z_{i_3})\cdots\delta^2(z_{i_k}-z_{i_{k+1}}), \tag{7}$$

with $V>0$. Note that here we have reverted to labeling the particles independently of their spin. For this Hamiltonian, a state is a zero-energy eigenstate if it vanishes whenever any $k+1$ particles coincide; for this to be satisfied for some nontrivial states, the particles must be bosons.

We will now show that the correlator as in Eq. (3) is such a zero-energy eigenstate. The argument we give here makes direct use of the current algebra satisfied by the currents, and also sheds new light on the previous spinless case of RR, where a slightly different argument was used. Without loss of generality, we can consider letting the first $k+1$ particles, of either spin, come to the same point, say $z=0$, that is $z_1^{\sigma_1}, z_2^{\sigma_2}, \ldots, z_{k+1}^{\sigma_{k+1}}$ all $\to 0$. In the standard radial quantization scheme for CFT, we can consider the current operators as acting in a Hilbert space that is built starting from a highest weight state that in the present case is simply the vacuum $|0\rangle$ of radial quantization about the origin

$z = 0$. As $z_i^{\sigma_i}$ tend to 0 one by one, the resulting operator product expansion (ope) contains no singular terms. This follows from the standard current-algebra ope's of the currents, together with the fact that the roots α_1 and $-\alpha_2$ do not sum to either 0 or another root (for simplicity, let us replace these two roots by the natural notation $\sigma = \uparrow$ and $\downarrow$, respectively). Indeed the only nonvanishing term as the $z_i^{\sigma_i}$ tend to zero sequentially is the operator at 0 that corresponds to the state

$$B_{\sigma_1,-1} B_{\sigma_2,-1} \cdots B_{\sigma_{k+1},-1} |0\rangle \tag{8}$$

in the highest weight representation, and we need to show that this vanishes for all choices of $\sigma_1, \ldots, \sigma_{k+1}$. Here we have used the modes of the currents,

$$B_a(z) = \sum_n z^{-n-1} B_{a,n}, \tag{9}$$

which holds for all generators a of SU(3), not only roots. In fact, the commutation relations of the affine Lie algebra for these modes also imply that $B_{\uparrow,-1}$ and $B_{\downarrow,-1}$ commute, so we need only to prove

$$(B_{\downarrow,-1})^m (B_{\uparrow,-1})^{k+1-m} |0\rangle = 0, \tag{10}$$

for all m in the range $0 \leqslant m \leqslant k+1$.

Let us begin by choosing $m = 0$. Then we need to show that

$$(B_{\uparrow,-1})^{k+1} |0\rangle = 0. \tag{11}$$

But this is simply the pure-current-algebra null-vector equation, which first entered the physics literature in Refs. [21,22]. Thus this is satisfied in the irreducible, unitary vacuum highest weight module of the SU(3) affine Lie algebra at level k, and there are similar equations for all current algebras, and for each "integrable" highest weight representation. This already suffices to rederive the RR states, which are related to SU(2) current algebra [7]. The RR states are in fact the same as those in Eq. (3) above, but with all spins $\uparrow$, so the only root that appears lies in an SU(2) subalgebra (the spin singlet property of the above states is of course lost when this is done, and the charge at infinity should be adjusted). Hence, we have very quickly rederived the fact that the RR states for $M = 0$ are the zero-energy states of the $(k+1)$-body δ-function Hamiltonian.

It is straightforward to complete the proof for the NASS states. Essentially, we use the SU(2) symmetry under which $B_{\uparrow,-1}$ and $B_{\downarrow,-1}$ form a doublet, and notice that the set of states labeled by m forms a highest weight multiplet under this algebra, of SU(2) spin $k/2$. (This is clear from the $2+1$ point of view, where we are looking at states of $k+1$ spin $1/2$ bosons all at the same point.) Then since the highest weight vanishes, all the others do also, which completes the proof.

As in RR, a similar argument also establishes that quasihole states, written as similar correlators but with spin fields (primary fields of the SU(3) current algebra) inserted at the quasihole positions [9], are exact zero-energy eigenstates. These are considered explicitly in Section 5. Similar arguments also imply that the zero-energy ground states of H on the torus are given by correlators for some number N of the above fields inserted, with the number of such ground states (for N divisible by $2k$) already given in Section 2.

4. Ground state wave function

Based on the results of the previous section, we expect the structure of the trial wave functions — that is, of the chiral correlators (5) — of the NASS states to be similar to that of the RR states, and also to generalize the Halperin $(2, 2, 1)$ state. The RR wave functions were constructed by dividing the (same-spin) particles into clusters of k, writing down a product of factors for each pair of clusters, and finally symmetrizing over all ways of dividing the particles into clusters. Hence in the case with spin, we guess that we should divide the up particles into groups of k, the downs into groups of k and then multiply together factors that connect up with up, down with down, or up with down clusters, and finally ensure that the function is of the correct permutational symmetry type to yield a spin-singlet state (in particular, it should be symmetric in the coordinates of the up particles, and also in those of the downs). We expect that the up–up and down–down parts of this should closely resemble the RR wave functions, before the symmetrization; it was shown in Ref. [7] that the functions found there vanish when $k+1$ particles come to the same point, even inside the sum over permutations that symmetrizes the final function. These considerations guided the following construction.

Due to the spin-singlet nature of the state, the wave function will be nonzero only if the number of spin-up and spin-down particles is the same. Furthermore, there must be an integer number of clusters, so the total number of particles N must be divisible by $2k$, and will be written as $N = 2kp$, where $p \in \mathbb{N}$. One example was already given in [9], namely the wave function for the case $k = 2$, $M = 0$ with the number of particles equal to 4 (i.e., $p = 1$),

$$\widetilde{\Psi}^{k=2,M=0}_{\mathrm{NASS}}\left(z_1^{\uparrow}, z_2^{\uparrow}; z_1^{\downarrow}, z_2^{\downarrow}\right) = \left(z_1^{\uparrow} - z_1^{\downarrow}\right)\left(z_2^{\uparrow} - z_2^{\downarrow}\right) + \left(z_1^{\uparrow} - z_2^{\downarrow}\right)\left(z_2^{\uparrow} - z_1^{\downarrow}\right). \tag{12}$$

This is part of the two-dimensional irreducible representation of the permutation group on 4 objects, S_4, as can easily be seen. This is the correct symmetry type to obtain a spin-singlet state, as we discuss further below.

We will now describe the different factors that enter the NASS wave functions. Because the only effect of M being nonzero is to give an overall Laughlin factor, we will assume at first that $M = 0$. First we give the factors that involve particles of the same spin, say spin up. They are the same as in RR [7]. We will divide the particles into clusters of k in the simplest way,

$$\left(z_1^{\uparrow}, \ldots, z_k^{\uparrow}\right), \left(z_{k+1}^{\uparrow}, \ldots, z_{2k}^{\uparrow}\right), \ldots, \left(z_{(p-1)k+1}^{\uparrow}, \ldots, z_{pk}^{\uparrow}\right), \tag{13}$$

and the same for the $z^{\downarrow}$'s. (In a more precise treatment, we would say that the first $N/2$ particles are spin up, the remainder spin down.) We write down factors that connect the ath with the bth cluster:

$$\begin{aligned}\chi^{z^{\uparrow}}_{a,b} = {} & \left(z^{\uparrow}_{(a-1)k+1} - z^{\uparrow}_{(b-1)k+1}\right)\left(z^{\uparrow}_{(a-1)k+1} - z^{\uparrow}_{(b-1)k+2}\right)\left(z^{\uparrow}_{(a-1)k+2} - z^{\uparrow}_{(b-1)k+2}\right) \\ & \times \left(z^{\uparrow}_{(a-1)k+2} - z^{\uparrow}_{(b-1)k+3}\right) \cdots \left(z^{\uparrow}_{ak} - z^{\uparrow}_{bk}\right)\left(z^{\uparrow}_{ak} - z^{\uparrow}_{(b-1)k+1}\right).\end{aligned} \tag{14}$$

For $k = 1$, we would write $\chi^{z^{\uparrow}}_{a,b} = (z^{\uparrow}_a - z^{\uparrow}_b)^2$. The factors that connect up with down spins are simpler:

$$\chi_{a,b}^{z^{\uparrow},z^{\downarrow}} = \left(z^{\uparrow}_{(a-1)k+1} - z^{\downarrow}_{(b-1)k+1}\right)\left(z^{\uparrow}_{(a-1)k+2} - z^{\downarrow}_{(b-1)k+2}\right)\cdots\left(z^{\uparrow}_{ak} - z^{\downarrow}_{bk}\right). \tag{15}$$

For $k = 1$, the factor would be $\chi_{a,b}^{z^{\uparrow},z^{\downarrow}} = (z^{\uparrow}_a - z^{\downarrow}_b)$. We multiply all these factors for all pairs of clusters, up–up, down–down, or up–down:

$$\prod_{a<b}^{p} \chi_{a,b}^{z^{\uparrow}} \prod_{c,d}^{p} \chi_{c,d}^{z^{\uparrow},z^{\downarrow}} \prod_{e<f}^{p} \chi_{e,f}^{z^{\downarrow}}. \tag{16}$$

Notice that for $k = 1$, we do obtain the Halperin $(2, 2, 1)$ wave function.

To obtain a spin-singlet state when the spatial function is combined with the spin state (which lies in the tensor product of N spins $1/2$), some symmetry properties must be satisfied. For the $M = 0$ case, the particles are bosons, hence the full wave function must be invariant under permutations of spins and coordinates of any two particles. This can be used to obtain the correct form of the function from that component in which, say the first $N/2$ are spin up, the rest spin down, as above, so knowledge of that component is sufficient. The requirement that the full wave function be a spin-singlet can be shown to reduce to the Fock conditions: the component just defined must be symmetric under permutations of the coordinates of the up particles, and also of the down particles, and must also obey the Fock cyclic condition, as given in Ref. [23] (modified in an obvious way for the boson case). These three conditions can be shown to imply that the spatial wave function is of a definite permutational symmetry type (belongs to a certain irreducible representation of the permutation group), that corresponds to the Young diagram with two rows of $N/2$ boxes each. In general, given a function of arbitrary symmetry, a Young operator can be constructed that projects it onto a member of the correct representation (though the result may vanish); this construction generalizes the familiar symmetrization and antisymmetrization operations. For the present case, the Young operator is the following operation, equivalent to summing over the function with various permutations of its arguments, and some sign changes: first, antisymmetize in z_1, $z_{N/2+1}$; then in z_2, $z_{N/2+2}$; $\ldots$, $z_{N/2}$, z_N; then symmetrize in $z_1, \ldots, z_{N/2}$; then finally symmetrize in $z_{N/2+1}, \ldots, z_N$. This clearly satisfies the first two requirements of Fock, and can be proved to satisfy also the cyclic condition. It remains to check that it is nonzero, we believe it is. Incidentally, the application of the Young operator is the analog of symmetrizing over the down spins in the spatial wave function of the permanent state (see, e.g., Ref. [12]), to which it reduces for the case of BCS paired wave functions of spin $1/2$ bosons (there are similar statements in the more familiar case of spin-singlet pairing of spin-$1/2$ fermions). However, based on the example of the Halperin ($k = 1$) case, we also considered the function defined as in Eq. (16), and then simply symmetrized over all the ups and over all the downs. For the Halperin function [which in fact is already symmetric in Eq. (16)], this satisfies the cyclic condition, as can be seen using the fact that the $(1, 1, 0)$ state is a Landau level filled with both spins, plus the Pauli exclusion principle for fermions. For $k = 2$, 3, we verified the cyclic condition numerically for several moderate sizes. Hence, we expect that this simpler form actually works for all k (as well as for all N divisible by k). Apparently, this procedure and the application of the Young operator give the same function in the end (up to a normalization).

For $M=0$, our wave function is then:

$$\widetilde{\Psi}^{k,0}_{\text{NASS}} = \text{Sym} \prod_{a<b}^{p} \chi^{z^\uparrow}_{a,b} \prod_{c,d}^{p} \chi^{z^\uparrow,z^\downarrow}_{c,d} \prod_{e<f}^{p} \chi^{z^\downarrow}_{e,f}, \tag{17}$$

where Sym stands for the symmetrization over the ups and also over the downs. This function is nonzero, as may be seen by letting the up coordinates coincide in clusters of k each, and also the downs, all clusters at different locations, and making use of the result in RR [7] that only one term in the symmetrization is nonzero in the limit. This term is the Halperin $(2k, 2k, k)$ function for $2p$ particles. To obtain the wave function for general M, we multiply by an overall Laughlin factor, $\widetilde{\Psi}^M_{\text{L}}$.

We can give a simple proof that our wave function (for $M=0$) vanishes if any $k+1$ particles, each of either spin, come to the same point. This works also for the RR wave functions, and is simpler, though less informative, than the proof in RR [7]. It works term by term, inside the sum over permutations in the symmetrizer. Thus, without loss of generality, we may use the simple clustering considered above. We note that on the clock face formed by the labels $1, \ldots, k$ within each cluster, there is always a factor connecting any two particles at the same position, regardless of their spin. This factor vanishes when the particles coincide. Since there are only k distinct positions, when $k+1$ particles come to the same point, the clock positions must coincide in at least two cases, so that the wave function vanishes, which completes the proof.

We do not have a direct general proof of the equality of these explicit wave functions and the formal expressions Eq. (5), but we have performed a number of consistency checks. First, the wave functions are polynomials of the correct degree. From Eq. (5), we can infer what the total degree should be. The parafermions of the correlator contribute with (see [18]) $-1 \cdot 2kp \cdot (1-\frac{1}{k})$. The factors of the $(2,2,1)$ part are $2 \cdot \frac{2}{k} \cdot \frac{1}{2}kp(kp-1)$ and $1 \cdot \frac{1}{k} \cdot (kp)^2$. Adding these gives, for $M=0$, $pk(3p-2)$. We need to check whether Eq. (16) gives the same degree. For the ith up particle, the degree of $z_i^\uparrow$ in the product of up–up factors is $2(p-1)$, and in the up–down factors is p. Thus the net degree in $z_i^\uparrow$ is $N_\phi = 3p-2 = 3N/2k-2$, or for general M, $N_\phi = 3p+M(N-1)-2 = (M+3/2k)N-2-M$. This gives the filling factor $\nu = 2k/(2kM+3)$ [9], which reduces to that for the Halperin states for $k=1$, and also the shift, defined as $N_\phi = N/\nu - \mathcal{S}$, which here is $\mathcal{S} = M+2$ on the sphere (for more on the shift, see Ref. [24]). Finally, the total degree is $N/2$ times that in $z_i^\uparrow$, namely $kp(3p-2)$ for $M=0$, the same as for the correlator. Also, the numerical work described in Section 10 confirms that the ground state of the appropriate Hamiltonian on the sphere for $k=2$, $M=1$ at the given number of flux does have a unique spin-zero ground state at zero energy, so that the correlator and the wave function constructed above must coincide. This also implies that the wave functions above must be spin-singlet.

5. Correlators corresponding to states with quasiholes

To obtain wave functions for NASS states with quasiholes, one inserts corresponding operators into the correlator that corresponds to the ground state wave function. Here we will give the form of the correlators, using standard CFT techniques, but not the wave functions. The operators we insert have the form of a spin field times a vertex operator, similar to the RR states [7] (note that the term "spin field" is traditional, and has no relation to the SU(2) "spin" symmetry). In the exponent of the vertex operators, the fundamental bosons are multiplied by (fundamental) *weights* of the Lie algebra SU(3): $\varpi_1 = (\sqrt{2}/2, \sqrt{6}/6)$, $\varpi_2 = (0, \sqrt{6}/3)$. We take the quasiholes from the triplet **3** of SU(3). The corresponding operators are

$$C_{\varpi_1}(w^{\uparrow}) = \sigma_{\varpi_1} \exp(i\varpi_1 \cdot \vec{\varphi}/\sqrt{k})(w^{\uparrow}),$$
$$C_{-\varpi_2}(w^{\downarrow}) = \sigma_{-\varpi_2} \exp(-i\varpi_2 \cdot \vec{\varphi}/\sqrt{k})(w^{\downarrow}). \tag{18}$$

In order to find the wave function for general M, we note the following. The two bosons $\vec{\varphi} = (\varphi_1, \varphi_2)$ can be written in terms of a (hyper-)charge and spin boson (φ_c and φ_s, respectively) by means of a simple rotation: $\varphi_1 = \frac{\sqrt{3}}{2}\varphi_c + \frac{1}{2}\varphi_s$ and $\varphi_2 = -\frac{1}{2}\varphi_c + \frac{\sqrt{3}}{2}\varphi_s$. The M-dependence is then brought in via a rescaling of the scale associated with the charge boson φ_c. The particle and quasihole operators for general M become in terms of these bosons

$$B'_{\alpha_1} = \psi_1 \exp\left(\frac{i}{\sqrt{2k}}\left(\sqrt{2kM+3}\,\varphi_c + \varphi_s\right)\right)(z^{\uparrow}), \tag{19}$$

$$B'_{-\alpha_2} = \psi_2 \exp\left(\frac{i}{\sqrt{2k}}\left(\sqrt{2kM+3}\,\varphi_c - \varphi_s\right)\right)(z^{\downarrow}), \tag{20}$$

$$C'_{\varpi_1} = \sigma_{\uparrow} \exp\left(\frac{i}{\sqrt{2k}}\left(\frac{1}{\sqrt{2kM+3}}\varphi_c + \varphi_s\right)\right)(w^{\uparrow}), \tag{21}$$

$$C'_{-\varpi_2} = \sigma_{\downarrow} \exp\left(\frac{i}{\sqrt{2k}}\left(\frac{1}{\sqrt{2kM+3}}\varphi_c - \varphi_s\right)\right)(w^{\downarrow}), \tag{22}$$

where we have written $\psi_{\alpha_1} = \psi_1$, $\psi_{-\alpha_2} = \psi_2$, $\sigma_{\varpi_1} = \sigma_{\uparrow}$ and $\sigma_{-\varpi_2} = \sigma_{\downarrow}$ for simplicity. The most basic spin fields $\sigma_{\uparrow,\downarrow}$ transform as a doublet of the SU(2) subalgebra we identify with the spin of the particles. Also, the hypercharge of the quasihole operators has the same sign as that of the particle operators. This implies that these are indeed quasiholes, as in earlier cases [6,7]; the wave functions are nonsingular as any particle coordinate z_i approaches any quasihole coordinate w_j.

Note that when these operators are used in the CFT correlator (together with a suitably chosen background charge), the extra Laughlin factor is automatically generated. The correlator for the component of the wave function with $N_{\uparrow,\downarrow}$ spin-up and -down particles and $n_{\uparrow,\downarrow}$ spin-up and -down quasiholes is given by

$$\begin{aligned}\widetilde{\Psi}^{k,M}_{\mathrm{NASS,qh}} = \lim_{z_\infty \to \infty} (z_\infty)^a \Big\langle &\exp\left(\frac{-i}{\sqrt{2k}}\left\{\left[\sqrt{2kM+3}\,(N_\uparrow + N_\downarrow) + \frac{n_\uparrow + n_\downarrow}{\sqrt{2kM+3}}\right]\varphi_c\right.\right.\\ &\left.\left.+ [N_\uparrow - N_\downarrow + n_\uparrow - n_\downarrow]\varphi_s\right\}\right)(z_\infty)\\ &\times C'_{\varpi_1}(w_1^\uparrow)\cdots C'_{\varpi_1}(w_{n_\uparrow}^\uparrow) C'_{-\varpi_2}(w_1^\downarrow)\cdots C'_{-\varpi_2}(w_{n_\downarrow}^\downarrow)\\ &\times B'_{\alpha_1}(z_1^\uparrow)\cdots B'_{\alpha_1}(z_{N_\uparrow}^\uparrow) B'_{-\alpha_2}(z_1^\downarrow)\cdots B'_{-\alpha_2}(z_{N_\downarrow}^\downarrow)\Big\rangle.\end{aligned} \tag{23}$$

The value of a will be given momentarily. In the wave function (23) we inserted the most general background charge required for neutrality in the Cartan subalgebra of SU(3). However, the background charge can involve only the charge boson φ_c, which corresponds to the spin-independent background magnetic field in the QH problem. Thus we find the condition

$$N_\uparrow + n_\uparrow = N_\downarrow + n_\downarrow, \tag{24}$$

which is part of the requirement of SU(2) symmetry for the correlator. The correlator is a spin-singlet, which means that the wave function *for the particles* is a nonzero spin state of the particles, with spin determined by the quasiholes. Effectively, the quasiholes carry spin 1/2 which is added to the spin-singlet ground state. Note that a quasihole labeled up carries a spin down from the latter point of view, by Eq. (24), just as it carries negative charge (hence the term quasihole), since there is a deficiency of particles at its location. For $N = N_\uparrow + N_\downarrow$ sufficiently large, $N \geqslant n$ in fact, the spin (up or down) for each quasihole can be chosen freely, as we will see in some examples. Using condition (24), we can calculate that a must be

$$a = \frac{2kM+3}{2k}\left(N + \frac{n}{2kM+3}\right)^2, \tag{25}$$

where $n = n_\uparrow + n_\downarrow$, in order that the limit $z_\infty \to \infty$ exists and is nonzero.

By working out the vertex-operator part, we arrive at the following form

$$\begin{aligned}
&\widetilde{\Psi}^{k,M}_{\mathrm{NASS,qh}}\big(z^\uparrow_1,\ldots,z^\uparrow_{N_\uparrow};z^\downarrow_1,\ldots,z^\downarrow_{N_\downarrow};w^\uparrow_1,\ldots,w^\uparrow_{n_\uparrow};w^\downarrow_1,\ldots,w^\downarrow_{n_\downarrow}\big)\\
&\quad=\Big\langle\sigma_\uparrow\big(w^\uparrow_1\big)\cdots\sigma_\uparrow\big(w^\uparrow_{n_\uparrow}\big)\sigma_\downarrow\big(w^\downarrow_1\big)\cdots\sigma_\downarrow\big(w^\downarrow_{n_\downarrow}\big)\\
&\qquad\times\psi_1\big(z^\uparrow_1\big)\cdots\psi_1\big(z^\uparrow_{N_\uparrow}\big)\psi_2\big(z^\downarrow_1\big)\cdots\psi_2\big(z^\downarrow_{N_\downarrow}\big)\Big\rangle\\
&\qquad\times\big[\widetilde{\Psi}^{(2,2,1)}_{\mathrm{H}}\big(z^\uparrow_i;z^\downarrow_j\big)\big]^{1/k}\widetilde{\Psi}^{M}_{\mathrm{L}}\big(z^\uparrow_i;z^\downarrow_j\big)\prod_{i,j}\big(z^\uparrow_i-w^\uparrow_j\big)^{1/k}\prod_{i,j}\big(z^\downarrow_i-w^\downarrow_j\big)^{1/k}\\
&\qquad\times\prod_{i<j}\big(w^\uparrow_i-w^\uparrow_j\big)^{\frac{1}{2kM+3}(\frac{2}{k}+M)}\prod_{i,j}\big(w^\uparrow_i-w^\downarrow_j\big)^{\frac{-1}{2kM+3}(\frac{1}{k}+M)}\\
&\qquad\times\prod_{i<j}\big(w^\downarrow_i-w^\downarrow_j\big)^{\frac{1}{2kM+3}(\frac{2}{k}+M)}.
\end{aligned}\tag{26}$$

Note that the correlator is nonzero only if the parafermion and spin fields can be fused to yield the identity operator.

The number of magnetic flux N_ϕ seen by any particle is found to be

$$N_\phi = \frac{2kM+3}{2k}N + \frac{1}{2k}n - (M+2), \tag{27}$$

where we used Eq. (24). Since N_ϕ must be an integer (so that the wave function is a polynomial in the z_i's), this gives another condition, that $(3N+n)/2$ [which is an integer by Eq. (24)] must be divisible by k. [For the RR states, there is an analogous condition, $2N+n$ must be divisible by k. For k even, this means that n is even, as in the $k=2$ case (the

MR state) [6]. In Ref. [7], only the case n and N both divisible by k was considered.] From Eq. (27), we can deduce that the quasiparticle charge is $1/(2kM+3)$. This corresponds to a fractional flux, $1/2k$ of the usual flux quantum. In effect, the flux quantum has been reduced by $1/k$ by the formation of clusters, as in paired states and in spin-polarized RR states [7]. The factor of $1/2$ is present already in the Halperin $k=1$ case. So if k is not divisible by 3, the quasihole charge is $1/Q$ (Q is the denominator of the filling factor, defined in Section 2), as in many other cases, but if k is divisible by 3, the quasihole charge is $1/3Q$. This is similar to what happens in the MR and RR states, where the quasihole charge is further fractionalized (smaller than $1/Q$) when k is divisible by 2 [7].

The conditions (24) and that N_ϕ be integer are clearly necessary, but in fact are also sufficient, to ensure that the quasihole wave functions are nonzero polynomials in the z_i's, except in the special case $n=1$ where the function vanishes. To see this, one must examine the fusion rules for the parafermion system, and check that the fields can be fused to the identity operator under the stated conditions. This will be considered in the next section.

For completeness, we give the conformal dimensions of the particle and quasihole operators $B'_\alpha(z)$ and $C'_\varpi(w)$. To obtain these, we need the dimensions of the parafermionic and spin fields, which are $\Delta_\psi = 1-1/k$ and $\Delta_\sigma = \frac{k-1}{k(k+3)}$, respectively [18]. Using these, we find (see also [9]) $\Delta_{\text{part}} = (M+2)/2$ and $\Delta_{\text{qh}} = \frac{(5k-1)M+8}{2(k+3)(2kM+3)}$.

We can show that the quasihole states we have obtained are zero-energy eigenstates for the $(k+1)$-body Hamiltonian above, as follows. We again concentrate on the case $M=0$. The argument using the ope's of the currents $B_{\uparrow,\downarrow}(z)$ again applies [7], as long as the $k+1$ z_i's are brought to a point that does not coincide with a quasihole coordinate w_j. To complete the argument, we must also consider the case where the latter condition does not hold. There are two ways to do this. One is to note first that, as a function of the z_i's for fixed w_j's, the correlator is a polynomial, as it must be to be a valid QHE wave function. (It is *not* a polynomial in the quasihole coordinates w_j.) This is because we chose to examine quasiholes rather than the opposite charge objects. Then the fact that it vanishes when $k+1$ z_i's coincide away from a quasihole coordinate w_j also implies that it vanishes when they are at a w_j, by continuity, which holds because the function is a polynomial in the z_i's.

A second argument is also instructive. We may generalize the argument using the current algebra null vectors to directly address the limit of B's approaching a w_j. There is a generalization of the central equation for this case,

$$(B_{\uparrow,-1})^k|\uparrow\rangle = 0. \tag{28}$$

Here the state $|\uparrow\rangle = C_{\varpi_1}(0)|0\rangle$ is the state in radial quantization corresponding to the quasihole operator at 0. There are similar equations, with successively lower exponents, for the higher-order quasiholes (with multiples of the charge of the basic one) obtained by successively fusing quasiholes together. These are the null vector equations for the highest weights in distinct representations of the affine Lie algebra (or for the distinct primary fields) [21,22]. We want to emphasize that the equation says that for certain choices of the spins, the wave function vanishes when only k particles come to the same point (or fewer for the higher-order quasiholes). This is a generalization of the fact that the Laughlin quasihole is defined as the factor $\prod_i(z_i - w)$ which vanishes when any one particle approaches w. (This generalization applies already to the spinless RR states.) It is also a generalization of the Halperin case, where a quasihole is a factor $\prod_i (z_i^\uparrow - w^\uparrow)$ which vanishes when a single up particle goes to $w^\uparrow$, but not when a single down particle does. When k basic quasiholes are fused at w (taking the leading term at each fusion), the null vector equations state that the function does vanish when a single particle of appropriate spin approaches w, so we have a Laughlin- (or rather Halperin-) type quasihole in that case, as was already known for the MR ($k = 2$ RR) state, for example. Note that the above spin $k/2$ state is the highest weight in a multiplet, so there is a set of $k + 1$ such null vectors in total. This does not include all possible spin choices, as we have pointed out. To complete the proof that the function vanishes when any $k + 1$ z_i's (i.e., $k + 1$ particles of any spin) come to w_j, we must show that, for all m,

$$\begin{aligned}&(B_{\downarrow,-1})^m (B_{\uparrow,-1})^{k+1-m} |\uparrow\rangle = 0,\\ &(B_{\downarrow,-1})^m (B_{\uparrow,-1})^{k+1-m} |\downarrow\rangle = 0.\end{aligned} \tag{29}$$

This can be done by an elementary argument, applying the SU(2) lowering operator to Eq. (28), then another $B_{\uparrow,-1}$, using the same equation, and then lowering further, and so on.

6. Fusion rules

From now on, we focus mainly on the case $k = 2$, which is a spin-singlet analogue of the MR state. The numerical studies reported in Section 10 were all performed for this special case. Analytical results for $k \geqslant 2$ will be presented elsewhere [27].

As pointed out in the introduction, the non-trivial fusion rules play a crucial role in the ground state degeneracies. In fact, the correlator in Eq. (26) does not represent a single wave function, because in general there is more than one way in which the spin fields and parafermion fields can be fused to the identity. To show how this works, we will give the fusion rules, and explain that they can be written in terms of a Bratteli diagram. By using the correspondence between the fields of the parafermion theory and fields of the corresponding Wess–Zumino–Witten models (see [18]), one finds the fusion rules listed in Table 1.

Table 1
Fusion rules of the parafermion and spin fields associated to the parafermion theory $\mathrm{SU}(3)_2/[\mathrm{U}(1)]^2$ introduced by Gepner [18]

$\times$	$\sigma_\uparrow$	$\sigma_\downarrow$	σ_3	ρ	ψ_1	ψ_2	ψ_{12}
$\sigma_\uparrow$	$\mathbb{1} + \rho$						
$\sigma_\downarrow$	$\psi_{12} + \sigma_3$	$\mathbb{1} + \rho$					
σ_3	$\psi_1 + \sigma_\downarrow$	$\psi_2 + \sigma_\uparrow$	$\mathbb{1} + \rho$				
ρ	$\psi_2 + \sigma_\uparrow$	$\psi_1 + \sigma_\downarrow$	$\psi_{12} + \sigma_3$	$\mathbb{1} + \rho$			
ψ_1	σ_3	ρ	$\sigma_\uparrow$	$\sigma_\downarrow$	$\mathbb{1}$		
ψ_2	ρ	σ_3	$\sigma_\downarrow$	$\sigma_\uparrow$	ψ_{12}	$\mathbb{1}$	
ψ_{12}	$\sigma_\downarrow$	$\sigma_\uparrow$	ρ	σ_3	ψ_2	ψ_1	$\mathbb{1}$

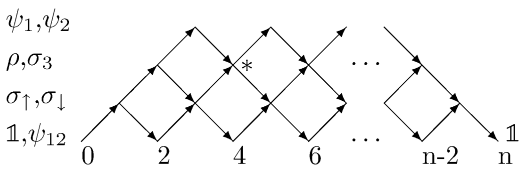

Fig. 1. Bratteli diagram for the spin fields of $SU(3)_2/[U(1)]^2$.

An examination of the fusion rules shows that there are different cases, according to the parity, even or odd, of $n_\uparrow$, $n_\downarrow$, $N_\uparrow$, $N_\downarrow$. In the case where all four numbers are even, the spin fields and the parafermions can be fused to the identity separately. In the case where all four are odd, the spin fields and the parafermions can be fused to ψ_{12} separately, and these two ψ_{12}'s can then be fused to the identity. Because the quasiholes only involve the $\sigma_{\uparrow,\downarrow}$ fields, we in fact only need the first two columns of Table 1. With this restriction, the fusion rules can be written in terms of a Bratteli diagram, see Fig. 1.

Each arrow stands for either a $\sigma_\uparrow$ or $\sigma_\downarrow$ field. The arrow starts at a certain field which can only be one of the fields on the left of the diagram at the same height. This last field is fused with the one corresponding to the arrow, while the arrow points at a field present in this fusion. As an example, the arrows starting at the $*$ are encoding the fusion rules $\rho \times \sigma_{\uparrow(\downarrow)} = \psi_{2(1)} + \sigma_{\uparrow(\downarrow)}$ and $\sigma_3 \times \sigma_{\uparrow(\downarrow)} = \psi_{1(2)} + \sigma_{\downarrow(\uparrow)}$. One checks that the diagram is in accordance with the first two columns of Table 1. The symbol $\mathbb{1}$ at the right-hand side of Fig. 1 indicates that in the end we have fused the fields to the identity. This is possible only when $n_\uparrow$ and $n_\downarrow$ are both even; in the case where both numbers are odd, ψ_{12} is obtained at that position in the diagram. In the remaining two cases, where $n = n_\uparrow + n_\downarrow$ is odd, we can draw a similar diagram with the last point at the top, representing ψ_1 or ψ_2 (except when $n = 1$, to which we return in a moment). In these cases, $N = N_\uparrow + N_\downarrow$ must also be odd, in order that the fusion of the parafermions ψ_1 and ψ_2 for the particles can produce the appropriate field which can fuse with the result of the spin field fusions to finally give the identity. In the case $n = 1$, it is not possible to fuse the spin fields to obtain a parafermion, and the correlator vanishes.

For the counting formula we need to know the number of ways in which a given number of spin fields can be fused to give a field in the parafermion sector, that is 1, ψ_1, ψ_2, or ψ_{12}. This number equals the number of paths (of length the number of spin fields) on the Bratteli diagram leading to the corresponding point on the diagram. One finds that when the numbers of spin-up and -down quasiholes are $n_\uparrow$ and $n_\downarrow$, respectively, this number, the number of fusion channels, is $d_{n_\uparrow,n_\downarrow} = f_{n-2}$, where the Fibonacci number is defined by $f_m = f_{m-1} + f_{m-2}$, with the initial conditions $f_0 = 1$ and $f_1 = 1$. This is valid for $n \geqslant 2$ in all four cases of $n_\uparrow$, $n_\downarrow$ even or odd, while for $n = 1$, $d_{n_\uparrow,n_\downarrow} = 0$, and for $n = 0$, $d_{0,0} = 1$. The result is obtained by examining the Bratelli diagram and seeing that the number of paths obeys the recurrence relation that defines the Fibonacci number. We note that this result, the Fibonacci number f_{n-2}, is the same as for n quasiholes in the $k = 3$ RR states [7]. This is a manifestation of level-rank duality, here between $SU(3)_2$ and $SU(2)_3$.

The final fusion of the spin fields with the parafermions from the particles must produce the identity, in order that the correlator be nonzero. When $d_{n_\uparrow,n_\downarrow}$ is nonzero, necessary and sufficient conditions for this are $N_\uparrow + n_\uparrow \equiv N_\downarrow + n_\downarrow (\mathrm{mod}\,2)$ and $N_\uparrow - n_\downarrow \equiv 0\ (\mathrm{mod}\,2)$. The first condition is the mod 2 version of the condition (24), while the second is equivalent to the condition $(3N+n)/2 \equiv 0\ (\mathrm{mod}\,2)$, on using Eq. (24). Note that, as can be inferred from the fusion rules for the ψ fields (see Table 1), the fusion of the ψ fields from the particle operators does not increase the degeneracy. The abelian properties of the ψ fields correspond to the abelian statistics (Bose or Fermi) of the underlying particles.

These results give the degeneracy $d_{n_\uparrow,n_\downarrow}$ of quasihole states for fixed positions and spins of the quasiholes, which is the basis for non-abelian statistics properties. We see that the result does not depend on how many quasiholes are spin up or spin down, and the sum over all choices of spins gives a further factor of 2^n spin degeneracy for sufficiently large N, when only the positions are fixed. In the following sections, we examine the total degeneracies of quasihole states when both their positions and spins are unrestricted. These are more suitable for numerical checks, and are finite numbers on the sphere (for a disk in the plane, they are infinite, and contain information about edge excitations as well as bulk quasiholes). As in cases studied earlier [12,7], the total degeneracies are not just the numbers found above times a factor for the spatial degeneracy contribution, but involve partitioning the Fibonacci numbers above into a sum of positive integers. We also note here that when a generic Hamiltonian has a ground state in the NASS phase, the degeneracies will not be exact, but will be as given in this section when all quasiholes are asymptotically far separated. This will not be considered further in this paper.

7. Spatial degeneracies

Techniques for calculating degeneracies for a spherical geometry are described in full detail in [12]. On the sphere, the relation between the number of particles and the number of flux quanta for the ground state is given by $N_\phi = \nu^{-1}N - \mathcal{S}$. By increasing the number of flux quanta at fixed N, quasiholes are created. Moreover, when flux have been added, there may be zero-energy states with N not satisfying the conditions required in the ground state, for example that N be even in the MR state [12]. In general, we would define the number of flux "added" as $\Delta N_\phi = N_\phi - \nu^{-1}N + \mathcal{S}$. This is defined relative to a ground state at the same number of particles, even though such a zero-energy state for N not divisible by $2k$ (or k for the RR states) would require a non-integer number of flux, and does not exist. Consequently, while our N_ϕ is always an integer, as discussed in Section 5, ΔN_ϕ does not have to be an integer. The number of quasiholes n can then be defined as $n = k\Delta N_\phi$ (for the RR states), or $n = 2k\Delta N_\phi$ for the NASS states considered here, in agreement with our formula for N_ϕ in Eq. (27).

To explain the spatial degeneracies, we use the Laughlin case as an example, and follow the discussion of [12]. The Laughlin wave function for N particles in the presence of n quasiholes can be written as [15]

$$\widetilde{\Psi}^M_{\mathrm{L,qh}}(z_1,\ldots,z_N;w_1,\ldots,w_n) = \prod_{i<j}(z_i-z_j)^M \prod_{i,k}(z_i-w_k). \tag{30}$$

For this state, $n = \Delta N_\phi$ (adding one quantum of flux creates one quasihole), so we have $N_\phi = M(N-1) + n$, where we used that $\mathcal{S} = M$ for the Laughlin state. To continue, we calculate the degeneracy due to the presence of the quasiholes, by expanding the factor $\prod_{i,k}(z_i - w_k)$ in sums of products of the elementary symmetric polynomials

$$e_m = \sum_{i_1 < i_2 < \cdots < i_m} w_{i_1} w_{i_2} \cdots w_{i_m}. \tag{31}$$

Viewing the coordinates w_i as coordinates of bosons, we find that n bosons are to be placed in $N+1$ orbitals. The dimension of the space of available states (linearly-independent wave functions) equals the number of ways in which one can put n bosons in $N+1$ orbitals, which is

$$\binom{N+n}{n}. \tag{32}$$

This is the spatial degeneracy we are after, although for the simple case of the Laughlin states.

The situation for the MR state is discussed in detail in [12]. For the MR state, there is an additional complication, namely quasihole states in which there are unpaired fermions are possible; this is the origin of the degeneracies for fixed quasihole positions, already discussed. We will denote the number of unpaired particles by F, with the requirement that $N - F$ be even, so that the number of unbroken pairs $(N - F)/2$ is an integer. For n sufficiently large, N need not be even in the zero-energy states. It was found that the spatial degeneracy depends on the number of unbroken pairs; in fact, for the MR state, it was given by [12]

$$\binom{\frac{N-F}{2} + n}{n}. \tag{33}$$

For the clustered state of [7] (in which the particles form clusters of order k rather than pairs), the spatial degeneracy is given similarly by [7,13]

$$\binom{\frac{N-F}{k} + n}{n}, \tag{34}$$

where $N - F$ must be divisible by k (there appears to be no known general analytic proof of this formula for $k > 2$).

Based on these earlier results, and because the NASS wave functions involve clusters of up particles and separately of downs, we expect that the spatial degeneracy for the NASS states is just a product of two binomial coefficients, involving the F_1, F_2 "unclustered" particles (or parafermions) for spins up and down, respectively:

$$\binom{\frac{N_\uparrow - F_1}{k} + n_\uparrow}{n_\uparrow} \binom{\frac{N_\downarrow - F_2}{k} + n_\downarrow}{n_\downarrow}. \tag{35}$$

Again, $N_\uparrow - F_1$ and $N_\downarrow - F_2$ must be divisible by k. Notice that these numbers never depend on M.

The case $k=1$, $F_1=F_2=0$ gives the spatial degeneracies for the general Halperin (m',m',m) abelian states (as was mentioned briefly in Ref. [12] for a particular case). For the spin-singlet cases $(m+1,m+1,m)$ $(m=M+1)$ of interest here, the conditions that correspond to zero-energy states with only the particle and quasihole numbers, N and n, fixed are that $N_\uparrow+n_\uparrow=N_\downarrow+n_\downarrow$. For N sufficiently large, these allow any choice of spin, up or down, for each quasihole, giving a factor of 2^n spin degeneracy for fixed positions (this holds for all k). Such degeneracy does not contribute to the degeneracy on which non-abelian statistics depends, and the statistics is abelian in the present $k=1$ case (as of course was expected). The full degeneracy in this case $k=1$ is obtained by summing (35) over $N_\uparrow$, $N_\downarrow$, $n_\uparrow$, $n_\downarrow$, subject to the conditions just stated, with $F_1=F_2=0$. These imply that the summation is over only the possible values of $S_z=(N_\uparrow-N_\downarrow)/2$, and the result is

$$\binom{N+n}{n}. \tag{36}$$

Note that this number includes the spin degeneracy. If we take the ratio to the number

$$\binom{(N+n)/2}{n} \tag{37}$$

of quasihole states with, say, $n_\downarrow=0$, the result tends to 2^n as $N\to\infty$, which is again the spin degeneracy for fixed positions. In the remainder of this paper, we concentrate exclusively on $k=2$.

With the orbital degeneracies in hand, we need to know how to break up the degeneracies stemming from the fusion rules of the spin fields. So in fact we need to know the number of unpaired particles of either spin, F_1, F_2, for each possible path on the Bratteli diagram. The next section will treat this problem, by partitioning the numbers $d_{n_\uparrow,n_\downarrow}$ in the following way

$$d_{n_\uparrow,n_\downarrow}=f_{n_\uparrow+n_\downarrow-2}=\sum_{F_1,F_2}\begin{Bmatrix} n_\uparrow & n_\downarrow \\ F_1 & F_2\end{Bmatrix}_2. \tag{38}$$

The symbol $\left\{\begin{smallmatrix} n_\uparrow & n_\downarrow \\ F_1 & F_2\end{smallmatrix}\right\}_2$ is interpreted as the number of zero-energy states containing $n_\uparrow$, $n_\downarrow$ quasiholes at fixed positions and F_1, F_2 unpaired parafermions. The symbol vanishes if the conditions that $F_1-n_\downarrow$, $F_2-n_\uparrow$ be even are not satisfied; these conditions are equivalent to the two mod 2 conditions discussed at the end of Section 6. The subscript 2 indicates that we are dealing with the case $k=2$.

8. Counting $\mathbf{SU(3)_2/[U(1)]^2}$ parafermion states

It was explained in [12] that the state counting for the MR state involves the systematics of Majorana fermions, which act as BCS quasiparticles (unpaired fermions) occupying zero-energy states [20]. For the more general RR (spin-polarized) states with order k clustering [7], the Majorana fermion is replaced by an $SU(2)_k/U(1)$ parafermion [7,13].

We recall that for the NASS state at level $k=2$, the role of the Majorana fermion is taken over by the parafermions that are associated to $SU(3)_2/[U(1)]^2$. The (spin-up or -down) quasiholes correspond to (two different) spin fields $\sigma_{\uparrow,\downarrow}$ of this parafermion theory (see

Section 5). In Section 6, we calculated the number of different quantum states (conformal blocks for the correlators) that can result from introducing $n_\uparrow\sigma_\uparrow$ and $n_\downarrow\sigma_\downarrow$ spin fields (and also varying the number of particles $N_\uparrow$, $N_\downarrow$) with the result $d_{n_\uparrow,n_\downarrow} = f_{n_\uparrow+n_\downarrow-2}$. The degeneracy results from the presence of varying numbers of particles that are not members of clusters, which in the correlators can be identified with the fundamental parafermions ψ_1, ψ_2 of the parafermion theory. F_1 (F_2) is the number of ψ_1 (ψ_2) excitations. These numbers are subject to the conditions that $F_1 \equiv n_\downarrow \pmod 2$, $F_2 \equiv n_\uparrow \pmod 2$, otherwise the number of zero-energy states is zero; these conditions come from those discussed in Section 6. In the previous section, we found that the orbital degeneracy depends on the numbers F_1 and F_2, see Eq. (35). We now turn to the calculation of the symbols

$$\left\{ \begin{matrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{matrix} \right\}_2, \tag{39}$$

which partition the degeneracies $d_{n_\uparrow,n_\downarrow}$ in the correct way. We will also keep track of the angular momentum (L) multiplet structure associated with these parafermion states. To do this, we have to go through a series of steps.

First of all, we consider the infinite (chiral) character corresponding to the full parafermionic CFT (see [25])

$$\mathrm{ch}(x_1, x_2; q) = \sum_{F_1,F_2} \frac{q^{(F_1^2+F_2^2-F_1F_2)/2}}{(q)_{F_1}(q)_{F_2}} x_1^{F_1} x_2^{F_2}, \tag{40}$$

where $(q)_a = \prod_{j=1}^{a}(1-q^j)$ for integer a. Here F_1 and F_2 are unrestricted non-negative integers, and x_1, x_2, q are indeterminates.

What is needed for our purposes here is the truncation of this expression to (a sum of) polynomials $Y_{n_\uparrow,n_\downarrow}(x_1, x_2; q)$ that describes the states that occur when $n_\uparrow$, $n_\downarrow$ spin fields (quasiholes) are present. The approach is described in Refs. [14,26], see also Ref. [13], and details for the present case will be given in Ref. [27]. We find that these polynomials satisfy the following recursion relations

$$\begin{aligned} Y_{(n_\uparrow,n_\downarrow)} &= Y_{(n_\uparrow-2,n_\downarrow)} + x_1 q^{(n_\uparrow-1)/2} Y_{(n_\uparrow-2,n_\downarrow+1)}, \\ Y_{(n_\uparrow,n_\downarrow)} &= Y_{(n_\uparrow,n_\downarrow-2)} + x_2 q^{(n_\downarrow-1)/2} Y_{(n_\uparrow+1,n_\downarrow-2)} \end{aligned} \tag{41}$$

with initial conditions

$$\begin{aligned} &Y_{(1,0)} = Y_{(0,1)} = 0, \\ &Y_{(0,0)} = Y_{(2,0)} = Y_{(0,2)} = 1, \\ &Y_{(1,1)} = q^{1/2} x_1 x_2. \end{aligned} \tag{42}$$

Recursion relations similar to the above (but lacking the $x_{1,2}$ dependence), have been considered in the mathematical literature on special polynomials associated to $\mathrm{SU}(3)_2$, see for instance [28]. The coefficient of $x_1^{F_1} x_2^{F_2}$ in the polynomial $Y_{(n_\uparrow,n_\downarrow)}$ is a polynomial in q with the sum of the coefficients equal to the symbol (39), that is

$$Y_{(n_\uparrow,n_\downarrow)}(x_1, x_2, 1) = \sum_{F_1,F_2} x_1^{F_1} x_2^{F_2} \left\{ \begin{matrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{matrix} \right\}_2. \tag{43}$$

We notice that the recursion relations preserve the conditions on the parities of F_1, F_2 (the exponents of x_1, x_2) that are part of the definition of the symbol (39). In the limit where $(n_\uparrow, n_\downarrow) \to (\infty, \infty)$, the sum of these polynomials over the four choices, $n_\uparrow$ and $n_\downarrow$ each either even or odd, approaches the expression $\mathrm{ch}(x_1, x_2; q)$ given above.

The coefficient of $x_1^{F_1} x_2^{F_2}$ in the polynomial $Y_{n_\uparrow, n_\downarrow}$ has a special form, which allows us to extract information on the L quantum numbers of the parafermion states: after multiplying with a factor $q^{-(n_\uparrow F_1 + n_\downarrow F_2)/4}$, we recognize a sum of terms of the form q^{l_z}, which together form a collection of angular momentum (L) multiplets with quantum numbers $L_z = l_z$ [13].

To illustrate the above, we present the polynomials for the case of two added flux quanta, giving eight quasiholes. The polynomials are

$$
\begin{aligned}
Y_{(8,0)} &= 1 + \left(q^2 + q^3 + 2q^4 + q^5 + q^6\right)x_1^2 + q^8 x_1^4 \\
&\quad + \left(q^6 + q^7 + q^8 + q^9 + q^{10}\right)x_1^4 x_2^2, \\
Y_{(7,1)} &= \left(q^{\frac{1}{2}} + q^{\frac{3}{2}} + q^{\frac{5}{2}} + q^{\frac{7}{2}}\right)x_1 x_2 + \left(q^{\frac{7}{2}} + 2q^{\frac{9}{2}} + 2q^{\frac{11}{2}} + 2q^{\frac{13}{2}} + q^{\frac{15}{2}}\right)x_1^3 x_2 \\
&\quad + q^{\frac{19}{2}} x_1^5 x_2^3, \\
Y_{(6,2)} &= 1 + \left(q^2 + q^3 + q^4\right)x_1^2 + \left(q^2 + q^3 + 2q^4 + q^5 + q^6\right)x_1^2 x_2^2 \\
&\quad + \left(q^6 + q^7 + q^8\right)x_1^4 x_2^2, \\
Y_{(5,3)} &= \left(q^{\frac{1}{2}} + 2q^{\frac{3}{2}} + 2q^{\frac{5}{2}} + q^{\frac{7}{2}}\right)x_1 x_2 + \left(q^{\frac{7}{2}} + q^{\frac{9}{2}} + q^{\frac{11}{2}}\right)x_1^3 x_2 \\
&\quad + \left(q^{\frac{9}{2}} + q^{\frac{11}{2}} + q^{\frac{13}{2}} + q^{\frac{15}{2}}\right)x_1^3 x_2^3, \\
Y_{(4,4)} &= 1 + q^2 x_1^2 + q^2 x_2^2 + \left(q^2 + 2q^3 + 3q^4 + 2q^5 + q^6\right)x_1^2 x_2^2 + q^8 x_1^4 x_2^4, \\
&\text{etc.}
\end{aligned}
\tag{44}
$$

From the polynomial $Y_{(5,3)}$ (as an example), we read off the following nonzero values of the symbols

$$
\begin{aligned}
&\begin{Bmatrix} 5 & 3 \\ 1 & 1 \end{Bmatrix}_2 = 6 \quad \left(L = \frac{3}{2}, L = \frac{1}{2}\right), \\
&\begin{Bmatrix} 5 & 3 \\ 3 & 1 \end{Bmatrix}_2 = 3 \quad (L = 1), \\
&\begin{Bmatrix} 5 & 3 \\ 3 & 3 \end{Bmatrix}_2 = 4 \quad \left(L = \frac{3}{2}\right).
\end{aligned}
\tag{45}
$$

In fact, it is possible to write the polynomials $Y_{(n_\uparrow, n_\downarrow)}$ in a closed form [29],

$$
Y_{(n_\uparrow, n_\downarrow)}(x_1, x_2; q) = \sum_{F_1, F_2}{}' q^{(F_1^2 + F_2^2 - F_1 F_2)/2} x_1^{F_1} x_2^{F_2} \begin{bmatrix} \frac{n_\uparrow + F_2}{2} \\ F_1 \end{bmatrix} \begin{bmatrix} \frac{n_\downarrow + F_1}{2} \\ F_2 \end{bmatrix}, \tag{46}
$$

where $\begin{bmatrix} a \\ b \end{bmatrix}$ is the q-deformed binomial (q-binomial), defined as $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(q)_a}{(q)_b (q)_{a-b}}$, and theprime on the sum denotes the restriction on F_1, F_2 values. Using the property that in the limit $q \to 1$ the q-binomials become ordinary binomials, we find the following explicit formula for $\{\}_2$ (under the same conditions on F_1, F_2, otherwise it vanishes):

$$\left\{\begin{matrix} n_\uparrow & n_\downarrow \\ F_1 & F_2 \end{matrix}\right\}_2 = \binom{\frac{n_\uparrow+F_2}{2}}{F_1}\binom{\frac{n_\downarrow+F_1}{2}}{F_2}. \tag{47}$$

Note that if we take the sum over all F_1 and F_2, we indeed find the correct value, namely a Fibonacci number

$$\sum_{F_1,F_2}{}' \binom{\frac{n_\uparrow+F_2}{2}}{F_1}\binom{\frac{n_\downarrow+F_1}{2}}{F_2} = f_{n_\uparrow+n_\downarrow-2}. \tag{48}$$

While Eq. (47) gives us just the number, we can also obtain the angular momentum content easily. The binomial $\binom{a}{f}$ is interpreted as the number of possible ways of putting f fermions in a boxes which have quantum numbers $L_z = -(a-1)/2, \ldots, (a-1)/2$ assigned to them. In this way, an angular momentum multiple structure is assigned to the binomials (see [12]). The angular momentum content of the symbols $\{\}_2$ is obtained by adding the angular momenta associated to the binomials in the usual way.

9. Final counting formula

We are now in the position to write down the formula for the total degeneracy of zero-energy quasihole states of the $k=2$ non-abelian spin-singlet states. Recall that there are two conditions on the numbers of quasiholes (see Section 5). The first condition is $N_\uparrow + n_\uparrow = N_\downarrow + n_\downarrow$, which implies that the correlator is a spin-singlet, or that the wave functions have total spin determined by the spin-1/2 quasiholes added. The other condition was that $(3N+n)/2$ be even, to ensure that N_ϕ is an integer, where $N = N_\uparrow + N_\downarrow$, and $n = n_\uparrow + n_\downarrow = 4\Delta N_\phi$, which relates the number of excess flux quanta and the number of quasiholes added. These imply that $N_\uparrow - n_\downarrow = N_\downarrow - n_\uparrow$ must be even.

The result of the previous few sections is now that the total number, summed over all spin components, of zero-energy states as a function of the number of particles and added flux quanta is

$$\#(N,\Delta N_\phi) = \sum_{N_{\uparrow,\downarrow};n_{\uparrow,\downarrow};F_{1,2}}{}' \binom{\frac{n_\uparrow+F_2}{2}}{F_1}\binom{\frac{n_\downarrow+F_1}{2}}{F_2}\binom{\frac{N_\uparrow-F_1}{2}+n_\uparrow}{n_\uparrow}\binom{\frac{N_\downarrow-F_2}{2}+n_\downarrow}{n_\downarrow}, \tag{49}$$

where the prime on the sum indicates that it is restricted to values obeying all the conditions just mentioned, and to $N_\uparrow - F_1$ and $N_\downarrow - F_2$ even as discussed in Section 7. Note again that these conditions imply that $n_\uparrow + F_2$ and $n_\downarrow + F_1$ are even.

In addition, the orbital angular momentum decomposition of the states can be obtained, by combining the angular momenta found in the orbital and parafermion factors in the preceding two sections. The spin quantum number of any given state is simply $S_z = (N_\uparrow - N_\downarrow)/2$ and one readily recognizes the multiplet structure for the SU(2) spin symmetry. (We remark that the parafermion theory by itself does not have a proper SU(2) spin symmetry.)

In Table 2, we present counting results for $N = 4, 8, 12$ and $\Delta N_\phi = 1, 2, 3, 4$. We specify the number of states as a function of the L and S quantum numbers. In Table 3

Table 2
Counting results for the NASS states at $k=2$. N is the number of electrons; ΔN_ϕ is the number of excess flux quanta. The results are given as a function of the L (angular momentum) and S (total spin) quantum numbers. The total number of states is also indicated

	$\Delta N_\phi=1$		$\Delta N_\phi=2$		$\Delta N_\phi=3$		$\Delta N_\phi=4$	
$N=4$	$\#=20$	$S=0\ 1\ 2$	$\#=104$	$S=0\ 1\ 2$	$\#=321$	$S=0\ 1\ 2$	$\#=755$	$S=0\ 1\ 2$
	$L=0$	1 0 1	$L=0$	1 0 1	$L=0$	2 0 1	$L=0$	2 0 1
	$L=1$	0 1 0	$L=1$	0 2 0	$L=1$	0 2 0	$L=1$	0 3 0
	$L=2$	1 0 0	$L=2$	2 1 1	$L=2$	2 2 2	$L=2$	3 2 2
			$L=3$	0 1 0	$L=3$	1 3 0	$L=3$	1 4 1
			$L=4$	1 0 0	$L=4$	2 1 2	$L=4$	3 3 2
					$L=5$	0 1 0	$L=5$	1 3 0
					$L=6$	1 0 0	$L=6$	2 1 1
							$L=7$	0 1 0
							$L=8$	1 0 0
$N=8$	$\#=105$	$S=0\ 1\ 2$	$\#=1719$	$S=0\ 1\ 2\ 3\ 4$				
	$L=0$	2 0 1	$L=0$	4 1 3 0 1				
	$L=1$	0 2 0	$L=1$	1 7 2 1 0				
	$L=2$	2 1 1	$L=2$	7 7 6 1 0				
	$L=3$	0 1 0	$L=3$	3 9 3 1 0				
	$L=4$	1 0 0	$L=4$	6 6 4 0 0				
			$L=5$	2 5 1 0 0				
			$L=6$	3 2 1 0 0				
			$L=7$	0 1 0 0 0				
			$L=8$	1 0 0 0 0				
$N=12$	$\#=336$	$S=0\ 1\ 2$						
	$L=0$	3 0 1						
	$L=1$	0 3 0						
	$L=2$	3 3 2						
	$L=3$	1 3 0						
	$L=4$	2 1 1						
	$L=5$	0 1 0						
	$L=6$	1 0 0						

we give some results for N not a multiple of four. Notice that for $n=1$, there are no zero-energy states, as expected from Section 6. The results listed in Tables 2 and 3 are for the cases we checked numerically, as we discuss in the next section, and are in full agreement with those results.

10. Numerical methods and results

We next turn to some numerical studies of the NASS states. We consider only cases where the particles are fermions, to represent electrons. We have studied the $k=2$, $M=1$ ($\nu=4/7$) case in both the toroidal (PBC) and spherical geometries. We first present the results for the sphere. As discussed before the flux-charge relation for this state is $N_\phi =$

Table 3
Counting results for the NASS states at $k = 2$ with fractional ΔN_ϕ (symbols as in Table 2)

	$\Delta N_\phi = 1/2$		$\Delta N_\phi = 3/2$	
$N = 2$	$\# = 3$	$S = 0\ 1$	$\# = 10$	$S = 0\ 1$
	$L = 0$	0 1	$L = 0$	1 0
			$L = 1$	0 1
$N = 6$	$\# = 10$	$S = 0\ 1$	$\# = 175$	$S = 0\ 1\ 2\ 3$
	$L = 0$	1 0	$L = 0$	0 2 0 1
	$L = 1$	0 1	$L = 1$	2 1 1 0
			$L = 2$	0 3 1 0
			$L = 3$	2 1 0 0
			$L = 4$	0 1 0 0
	$\Delta N_\phi = 1/4$		$\Delta N_\phi = 5/4$	
$N = 5$	$\# = 0$		$\# = 48$	$S = 1/2\ 3/2$
			$L = 1/2$	1 1
			$L = 3/2$	1 1
			$L = 5/2$	1 0
	$\Delta N_\phi = 3/4$		$\Delta N_\phi = 7/4$	
$N = 3$	$\# = 4$	$S = 1/2$	$\# = 28$	$S = 1/2\ 3/2$
	$L = 1/2$	1	$L = 0$	0 0
			$L = 1$	1 1
			$L = 2$	1 0

$7N/4 - 3$. The number of single-particle orbitals (the lowest LL degeneracy) is $N_\phi + 1$. In order to make contact with the results on more conventional geometries the radius R of the sphere has to be chosen so that the number of flux is $N_\phi = 2R^2$ (where the magnetic field strength B is fixed, such that the magnetic length is 1 in our units), so $R = \sqrt{N_\phi/2}$ [16]. The filling factor is $\nu = N/N_\phi = 2\pi\bar{n}$, where $\bar{n} = N/(4\pi R^2)$ is the particle number density.

For numerical purposes, it is best to re-express the interaction Hamiltonian in terms of projection operators onto different values of the total angular momentum for different groups of particles [16]. For the $M = 1$, $k = 2$ case of the NASS states, the required Hamiltonian can be written as

$$H = U \sum_{i<j<k} P_{ijk}(3N_\phi/2 - 3, 3/2) + V' \sum_{i<j} P_{ij}(N_\phi, 0), \tag{50}$$

with $U, V' > 0$. Here $P_{ijk}(L, S)$ ($P_{ij}(L, S)$) are projection operators for the three (respectively, two) particles specified onto the given values of total angular momentum L and spin S for the three (respectively, two) particles. Each projection is normalized to $P^2 = P$. To see that this is the required Hamiltonian, that corresponds to the short range δ-function interaction for $M = 0$, and gives the same numbers of zero-energy states found above, note the following. First, the maximal angular momentum for several particles corresponds to the closest approach of those particles [16]. In particular, the two-body term is a contact interaction, and $V' = V_0$, the zeroth Haldane pseudo-potential [16]. The two-body term implies that any zero-energy states must have no component with total

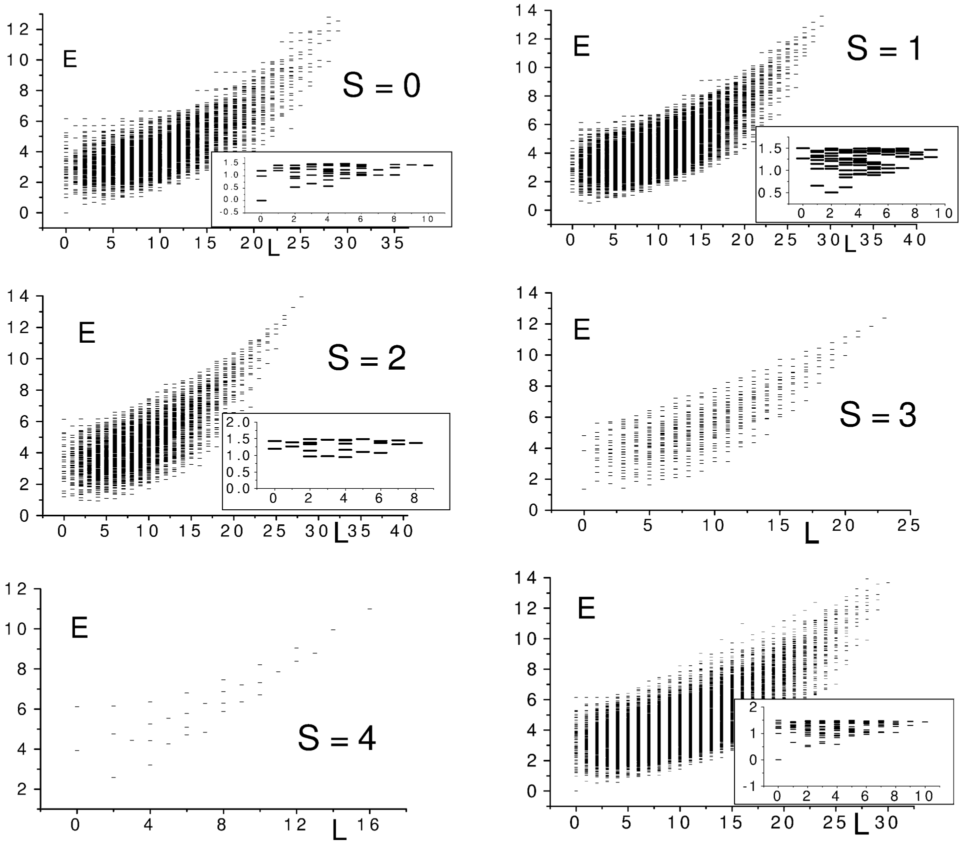

Fig. 2. The spectrum of the NASS model ground state for $N = 8$ and $4/7$ filling. The last panel shows all S values combined. The insets are the low lying levels.

angular momentum N_ϕ and total spin zero, which, since we are dealing with spin $1/2$ fermions, means the wave function must vanish when any two particles coincide. The wave function must therefore contain a factor $\widetilde{\Psi}_L^1$; multiplication by this factor defines a one-one mapping of the full space of states of spin $1/2$ bosons in the lowest LL, with N_ϕ reduced by $N-1$, onto the subspace of states of the fermions that is annihilated by the two-body term in H. Under this mapping, the three-body Hamiltonian for the $M = 0$ case corresponds to the three-body term in H, and selects the corresponding states as zero-energy states. In particular, the total spin of the three bosons when they coincide (and hence of the fermions) must be $3/2$. Hence the zero-energy eigenstates of the present Hamiltonian are given by the results derived earlier. Note also that H can be rewritten in terms of δ-functions and their derivatives. The zero-energy eigenstates of this Hamiltonian were found for various N and N_ϕ values, and analyzed in terms of L and S. The results are shown in Tables 2 and 3, and agree with the counting formulas presented above.

Next we discuss the full spectra of the Hamiltonian. In Fig. 2 we show the excitation spectrum of an $N = 8$, $N_\phi = 11$ system classified by the total spin $S = 0$ to $S = 4$.

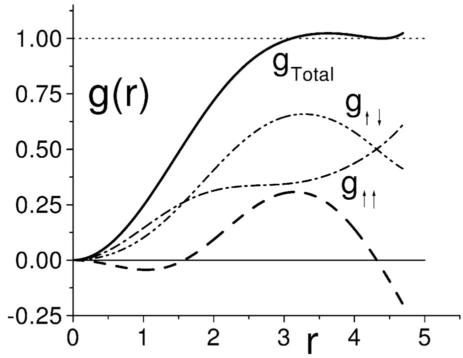

Fig. 3. The spin up–up and spin up–down pair correlation functions, together with their sum (solid line) and difference (dashed line), versus the chord distance, calculated in the ground state for $N = 8$, $N_\phi = 11$ ($\nu = 4/7$).

Whenever necessary we have shown the low-lying spectrum in an inset. The frame in the lower right hand corner shows the entire spectrum irrespective of the total spin quantum number of the state. The choice of U and V' is immaterial to the ground state, which is always the unique zero-energy eigenstate of H. Obviously, the excitation spectrum will depend on the choice of these coefficients. In producing Fig. 2 we chose $U = V' = 1$ so that H is a sum of projection operators. There appears to be a well-defined gap, suggesting that the system is incompressible (for spin as well as charge) in the thermodynamic limit, as assumed in the preceding analysis. In this connection, we may point out that, as well as the quantized Hall conductivity for charge, our system then has a quantized Hall conductivity for spin, given by $k/4\pi$ in natural ($\hbar = 1$) units, which is associated with the $\mathrm{SU}(2)_k$ subalgebra of the chiral algebra (see, e.g., Ref. [20] and references therein). Collective modes with $S = 0$ ($L = 2, 3, 4$) and $S = 1$ ($L = 1, 2, 3$) below the continuum can be tentatively identified in the spectra (see insets in Fig. 2). That is, these may be finite-size dispersion curves of single neutral excitations in plane-wave (spherical harmonics on the sphere) wave functions, which would be charge and spin modes, respectively. We shall not address the precise nature of these neutral modes here.

In Fig. 3 we show the various pair correlation functions of interest, $g_{\uparrow\uparrow}(r)$, $g_{\uparrow\downarrow}(r)$, as well as $g_{\mathrm{Total}} = g_{\uparrow\uparrow}(r) + g_{\uparrow\downarrow}(r)$ and $g_{\uparrow\downarrow}(r) - g_{\uparrow\uparrow}(r)$. The widely-different correlations between like and opposite spins is no doubt magnified by finite-size effects.

The 8-electron system is the first non-trivial size and is probably too small for any meaningful comparisons of the overlap with the 2-body Coulomb potential problem. Nonetheless we found a nontrivial overlap-squared (about 55%) with the ground state of the Coulomb potential for particles in the lowest LL (again, with no Zeeman term), at the same N, N_ϕ. By modifying the value of the lowest pseudo-potential for the Coulomb interaction this overlap-squared can be improved to 93% (and probably beyond) without any intervening phase transition (an energy gap with the ground state is maintained at all times while the pseudo-potentials are varied). However, in the lowest LL we do not expect to produce a better trial wave function than one constructed by the composite fermion (CF) method [30], in which two flux quanta per particle are attached (in the opposite direction

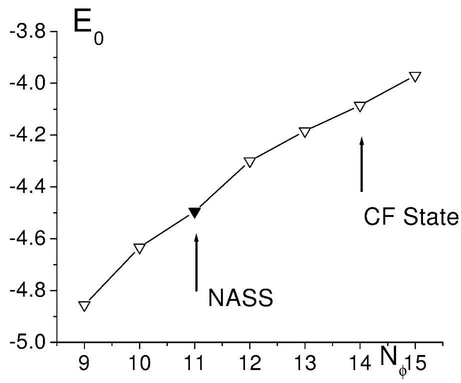

Fig. 4. The ground state energy of the pure Coulomb potential in the lowest Landau level versus N_ϕ, at $N = 8$. The numbers of flux N_ϕ for the NASS and the spin-singlet CF states are marked.

to the background magnetic field) and the resulting CFs fill completely the first two LLs of CFs (with both spins). By construction this is a uniform ($L = 0$) spin-singlet state. We have not constructed this state, as it occurs at a different flux for a given N ($N_\phi = 7N/4$), making a direct comparison with our NASS state difficult. We note, however, that for $N = 8$ the CF state corresponds to a spin-singlet Fermi-liquid-like state, as at $\nu = 1/2$. That is because the net effective field of the CFs is zero for this size (states that lie in sequences for different filling factors can coincide at finite size on the sphere, because the shifts $\mathcal{S}$ may be different — see, e.g., [31]). We expect that, as usual, this CF state will have a very large overlap with the exact ground state of the Coulomb potential. However, our numerical data for $N = 8$ shows a much stronger cusp at the N_ϕ of the NASS state than at the N_ϕ of the state obtained by the hierarchy/CF construction, where in fact no cusp can be discerned. See Fig. 4.

We have also studied the $N = 8$ size on the torus. Unfortunately, as on the sphere this size is too small for any meaningful comparison (e.g., there are only four distinct many-body $\mathbf{k}$ vectors for this size; one is at the zone center, the other three at the zone boundary). We would just like to point out that, for the analog of H on the torus, the degeneracy for the 4/7 state is 2 (excluding the 7-fold center of mass degeneracy), in agreement with the count in Section 2, since the number of particles is divisible by 4. These are two $\mathbf{k} = \mathbf{0}$ states. For the pure Coulomb potential in the lowest LL the state at 4/7 is in fact compressible: The total spin is $S = 1$ and its $\mathbf{k}$ vector varies with the geometry of the PBC unit cell. However, one obtains an incompressible state by increasing V_0 or V_1 and we obtained overlap-squared as large as 50% when we compared the lowest two states (which happen to be both $S = 0$, $\mathbf{k} = \mathbf{0}$ states) to the model NASS states. Note that the shift is zero on the torus, and there can be no interference from $\nu = 1/2$ here. We have not performed any further or systematic studies of such issues because, as in the case of the sphere, we suspect that the CF-based state will be closer to that of the Coulomb potential. That is, we expect that the system with Coulomb interaction in the lowest LL at $\nu = 4/7$ is in fact in the hierarchy/composite-fermion phase (whether spin-singlet or not), not the NASS phase considered in this paper. We will return to more comprehensive studies of larger sizes in the future.

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References

[1] R.L. Willett, J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard, J.H. English, Phys. Rev. Lett. 59 (1987) 1776.
[2] W. Pan, J.-S. Xia, V. Shvarts, D.E. Adams, H.L. Störmer, D.C. Tsui, L.N. Pfeiffer, K.W. Baldwin, K.W. West, Phys. Rev. Lett. 83 (1999) 3530.
[3] For a short review, see N. Read, cond-mat/0011338.
[4] R. Morf, Phys. Rev. Lett. 80 (1998) 1505;
E.H. Rezayi, F.D.M. Haldane, Phys. Rev. Lett. 84 (2000) 4685.
[5] W. Pan, H.L. Stormer, D.C. Tsui, L.N. Pfeiffer, K.W. Baldwin, K.W. West, cond-mat/0103144.
[6] G. Moore, N. Read, Nucl. Phys. B 360 (1991) 362.
[7] N. Read, E. Rezayi, Phys. Rev. B 59 (1999) 8084.
[8] H. Cho, J.B. Young, W. Kang, K.L. Chapman, A.C. Gossard, M. Bichler, W. Wegschneider, Phys. Rev. Lett. 81 (1998) 2522.
[9] E. Ardonne, K. Schoutens, Phys. Rev. Lett. 82 (1999) 5096.
[10] E. Ardonne, F.J. M van Lankvelt, A.W.W. Ludwig, K. Schoutens, Separation of spin and charge in paired spin-singlet quantum Hall states, cond-mat/0102072.
[11] C. Nayak, F. Wilczek, Nucl. Phys. B 479 (1996) 529.
[12] N. Read, E. Rezayi, Phys. Rev. B 54 (1996) 16864.
[13] V. Gurarie, E. Rezayi, Phys. Rev. B 61 (2000) 5473.
[14] K. Schoutens, Phys. Rev. Lett. 79 (1997) 2608.
[15] R.B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395.
[16] F.D.M. Haldane, Phys. Rev. Lett. 51 (1983) 605;
F.D.M. Haldane, in: R.E. Prange, S.M. Girvin (Eds.), The Quantum Hall Effect, 2nd edn., Springer-Verlag, New York, 1990.
[17] M. Greiter, X.-G. Wen, F. Wilczek, Phys. Rev. Lett. 66 (1991) 3205;
M. Greiter, X.-G. Wen, F. Wilczek, Nucl. Phys. B 374 (1992) 567.
[18] D. Gepner, Nucl. Phys. B 290 (1987) 10.
[19] B. Halperin, Helv. Phys. Acta 56 (1983) 75.
[20] N. Read, D. Green, Phys. Rev. B 61 (2000) 10267.
[21] D. Gepner, E. Witten, Nucl. Phys. B 278 (1986) 493.
[22] A.B. Zamolodchikov, V.A. Fateev, Yad. Phys. 43 (1986) 1031;
A.B. Zamolodchikov, V.A. Fateev, Sov. J. Nucl. Phys. 43 (1986) 657.
[23] M. Hamermesh, Group Theory and Its Application to Physical Problems, Dover, New York, 1989, Chapter 7.
[24] X.-G. Wen, Adv. Phys. 44 (1995) 405.
[25] A. Kuniba, T. Nakanishi, J. Suzuki, Mod. Phys. Lett. A 8 (1993) 1649;
G. Georgiev, Combinatorial constructions of modules for infinite-dimensional Lie algebras, II. Parafermionic space, q-alg/9504024.
[26] P. Bouwknegt, K. Schoutens, Nucl. Phys. B 547 (1999) 501.
[27] E. Ardonne, K. Schoutens, to be published.
[28] A. Schilling, S.O. Warnaar, Commun. Math. Phys. 202 (1999) 359.

[29] P. Bouwknegt, Multipartitions, generalized Durfee squares and affine Lie algebra characters, math.CO/0002223;
E. Ardonne, P. Bouwknegt, K. Schoutens, J. Stat. Phys. 102 (2001) 421.
[30] J.K. Jain, Phys. Rev. Lett. 63 (1989) 199.
[31] E. Rezayi, N. Read, Phys. Rev. Lett. 72 (1994) 900;
E. Rezayi, N. Read, Phys. Rev. Lett. 73 (1994) 1052 (C).