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On the particle entanglement spectrum of the Laughlin states

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Abstract

The study of the entanglement entropy and entanglement spectrum has proven to be very fruitful in identifying topological phases of matter. Typically, one performs numerical studies of finite-size systems. However, there are few rigorous results in this regard. We revisit the problem of determining the rank of the 'particle entanglement spectrum' (PES) of the Laughlin states. We reformulate the problem into a problem concerning the ideal of symmetric polynomials that vanish under the formation of several clusters of particles. We introduce an explicit generating set of this ideal, and we prove that polynomials in this ideal have a total degree that is bounded from below. We discuss the difficulty in proving the same bound on the degree of any of the variables, which is necessary to determine the rank of the PES.

Keywords: entanglement spectrum, quantum Hall effect, Laughlin state

1. Introduction

The study of topological phases of matter has benefited greatly from considering the entanglement properties of the ground states of topological phases. The work of Kitaev and Preskill [1] and of Levin and Wen [2] revealed that the entanglement entropy is a good probe of the topological nature of a system and provides a measure for the particle content of the topological phase [3]. The entanglement entropy of a pure quantum state $|\Psi\rangle$ relative to a bipartite partition of the total Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ provides a measure of the entanglement of $|\Psi\rangle$. The entanglement entropy is defined as the Von Neumann entropy S of the reduced density matrix of either one of the two parts. For instance

$$S = -\mathrm{Tr}\left(\rho_A \log \rho_A\right), \tag{1}$$

where $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$.

In the context of the fractional quantum Hall (FQH) effect, various ways to partition the Hilbert space were proposed [4]. Of particular importance, is the spatial partitioning scheme in which the system is split into two regions A and B separated by a real-space cut of length L. For a system exhibiting topological order the real-space entanglement entropy is of the form [2, 3]

$$S = \alpha L - \gamma + \cdots, \tag{2}$$

where $\cdots$ stands for subdominant terms as L becomes large. The subdominant term γ is universal, and depends only on the nature of the topological phase. It bears the name *topological entanglement entropy*, and is a measure for the particle content of the topological phase [3]. The first term αL, while non-universal, means that the amount of entanglement is proportional to the length of the boundary separating the two regions. This property called *area law* has appeared in various areas of physics, such as black-hole physics and quantum information. For a quantum many-body state, this property is of particular importance since it opens the way to extremely efficient numerical simulations such as the density matrix renomalization group [5] and matrix product states [6] methods. For FQH state this avenue of research was successfully undertaken [7–9] and opened the way to a reliable microscopic calculation of quasi-holes properties such as radius and braiding [10].

Although the real-space cut is of paramount importance in the study of topological phases of matter, there are other natural ways to partition a quantum Hall system: the *orbital cut*, and the *particle cut* [4]. While, in principle, the entanglement entropy behaves according to the area law equation (2) only for real-space cuts, it was numerically observed [11] that the area law is also valid for orbital cuts. In this paper we will concentrate on the particle cut, in which one numbers the (identical) particles constituting the phase (for instance, the electrons in the quantum Hall case), and one declares the particles numbered 1, 2, …, N_A to belong to subsystem A, while the remaining particles numbered $N_A + 1$, $N_A + 2$, …, N belong to subsystem B. The spectrum of the reduced density matrix obtained by tracing out the particles in subsystem B is the 'particle entanglement spectrum' (PES) [12].

While the entanglement entropy S provides a good probe of topological order, the topological entanglement entropy γ does not determine unambiguously the universality class of the topological state. Li and Haldane [13] realized that the spectrum of $H_A = -\log \rho_A$, which is called the entanglement Hamiltonian, itself contains much more information than the entanglement entropy. They proposed to use the low lying part of this *entanglement spectrum* as a 'fingerprint' of the topological phase. To be more specific, under a bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a pure quantum state $|\Psi\rangle$ admits a Schmidt decomposition

$$|\Psi\rangle = \sum_i e^{-\xi_i/2} \left|\psi_i^A\right\rangle \otimes \left|\psi_i^B\right\rangle, \tag{3}$$

where the $e^{-\xi_i/2}$'s are positive numbers called the Schmidt singular values, while $|\psi_i^A\rangle$ and $|\psi_i^B\rangle$ form orthonormal sets in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The reduced density matrix is then simply

$$\rho_A = \sum_i e^{-\xi_i} \left|\psi_i^A\right\rangle \left\langle\psi_i^A\right| = e^{-H_A}. \tag{4}$$

The entanglement spectrum is the set of all *entanglement energies* ξ_i. The bipartition can be chosen to preserve as much symmetry as possible, which in turn yields quantum numbers for the ξ_i's, such as the momentum along the cut. Li and Haldane observed that—per momentum

sector—the number of entanglement energies reproduces exactly the number of gapless edge modes. They proposed that tracing out the degrees of freedom of part B introduces a virtual edge for part A. The Li-Haldane conjecture is therefore two-fold. For a FQH state in the thermodynamic limit:

(i) the entanglement energies and edge modes have the same counting,
(ii) the entanglement spectrum is proportional to the (edge) CFT spectrum.

It is now understood that (ii) can only hold in the case of a real-space cut, which maintains locality along the cut [14–16]. For an orbital cut the entanglement Hamiltonian H_A has no reason to be local. On the other hand the point (i) holds irrespective of the cut for model wave functions that can be written as correlation functions in a CFT. Such wave functions are precisely of MPS form [17], and the CFT Hilbert space provides a one-to-one mapping [18] between edge modes and entanglement energies.

While the agreement between the counting of the number of modes in the entanglement spectrum and the counting of the edge excitations is well understood, in practice, this fingerprint is used for finite-size systems. The entanglement counting develops finite-size effects, which naively have no structure. However, it has been conjectured and numerically substantiated [19] that there is a counting principle underlying the finite-size entanglement counting of model states. Before we turn to the main topic of this paper, namely the PES for the Laughlin states, we mention that the entanglement entropy and spectra are actively studied, see [20–26] for some studies in the context of the quantum Hall effect.

We now consider the ground state $|\Psi\rangle$ of a model quantum Hall state, such as the Laughlin [27] or Moore–Read [28] state, that are the exact zero energy states of a model Hamiltonian. For a given number N of particles, this is the unique zero-energy state of a model Hamiltonian that occurs at the following number of flux quanta

$$N_\phi = \frac{1}{\nu} N - \mathcal{S}, \tag{5}$$

where ν is the filling fraction and $\mathcal{S}$ is the shift. Now suppose that the N particles are divided into two groups, group A containing N_A of the particles, and group B containing the rest $N_B = N - N_A$ of them. Without any loss of generality, we can assume that $N_A \leqslant N_B$. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_{N_A})$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_{N_B})$ be the coordinates of particles in A and B, respectively. The Schmidt decomposition of the wave function $\Psi(\boldsymbol{z})$, that is

$$\Psi(\boldsymbol{z}) = \sum_i e^{-\xi_i/2} \psi_i^A(\boldsymbol{x}) \psi_i^B(\boldsymbol{y}), \tag{6}$$

involves wave functions $\psi_i^A(\boldsymbol{x})$ for N_A particles. After tracing out the particles of part B, we are left with a reduced system of N_A particles, but the amount of flux remains the same, namely

$$N_\phi = \frac{1}{\nu} N_A - \mathcal{S} + \Delta N_\phi, \tag{7}$$

where $\Delta N_\phi = \nu^{-1} N_B$. The presence of this excess flux ΔN_ϕ indicates that we should view the reduced system as one with N_A particles, in the presence of quasi-hole excitations. For a *real* system with N_A particles, and ΔN_ϕ excess flux quanta the number of zero-energy states of the model Hamiltonian (which we will call the number of quasi-hole states) can often be obtained exactly [29–33]. For instance, in the case of the $\nu = 1/m$ Laughlin case, quasi-hole states of N_A particles in ΔN_ϕ excess flux are of the form

$$\psi_i(\boldsymbol{x}) = P_i(\boldsymbol{x}) \prod_{i<k} (x_i - x_k)^m, \tag{8}$$

where $P_i(\boldsymbol{x})$ is a symmetric polynomial in N_A variables with degree in each variable at most ΔN_ϕ. The number of quasi-hole states is therefore

$$\begin{pmatrix} \Delta N_\phi + N_A \\ N_A \end{pmatrix}. \tag{9}$$

This number forms an upper bound for the rank of the reduced density matrix [4, 11, 23].

From numerical investigations, it is known that in all cases considered, this upper bound is in fact reached[5]. This observation has led to the 'rank saturation' conjecture, which can be thought of as a finite-size version of the Li-Haldane conjecture, namely, the entanglement level counting of the PES of a model state is equal to the number of bulk quasi-hole states. This means that the states $\psi_i^A(\boldsymbol{x})$ appearing in the Schmidt decomposition of $\Psi(\boldsymbol{z})$ span all the quasi-hole states of N_A particles in ΔN_ϕ excess flux. Proving analytically that this upper bound is indeed reached has proven to be a difficult problem.

In this paper, we revisit this problem for the general $\nu = \frac{1}{m}$ Laughlin states. We start by considering the $\nu = 1$ Laughlin state, which is simply the Slater determinant of the completely filled lowest Landau level. We explain how to obtain the rank of the reduced density matrix of the PES in this case. To do so, we will make some use of the properties of symmetric polynomials. To get a grip on the $\nu = \frac{1}{m}$ Laughlin states, we then use the following strategy. After partitioning the particles into two sets A and B, we 'split' the N_B particles in part B into mN_B particles, and consider the $\nu = 1$ Laughlin state of the system thus obtained. For this system, we already obtained the rank of the reduced density matrix. If one can show that clustering the mN_B particles into groups of size m, does not lead to a smaller rank of the reduced density matrix, one deduces the rank of the reduced density matrix for the $\nu = \frac{1}{m}$ Laughlin state, and shows that the upper bound is indeed reached.

The hard step in the strategy outlined above is to show that the clustering of the mN_B particles into N_B groups of size m does not reduce the rank of the reduced density matrix. Proving this statement turns out to be highly non-trivial. As we explain in the main text, one has to show that there is no (non-zero) symmetric polynomial in mN_B variables that vanishes under the formation of N_B groups of variables each of size m, and whose degree in any of the variables is N_B or less. Although we did not fully succeed in proving this statement, we did make substantial progress. In particular, we constructed an explicit generating set of the ideal of polynomials that vanish under this clustering. Using this construction we were able to show that a non-zero symmetric polynomial in mN_B variables that vanishes under the formation of N_B groups of variables each of size m must have a *total* degree at least $N_B + 1$. Proving this weaker statement is already a non-trivial result, mainly because the positions of the various clusters can be arbitrary, which means that the clustering condition is non-local. Moreover, we were able to prove that all polynomials in the generating set have a degree at least $N_B + 1$.

The outline of the article is as follows. In section 2, we introduce the notion of partitions, and several types of symmetric polynomials, that we make use of throughout the article. The PES of the $\nu = 1$ Laughlin state is discussed in section 3. We continue in section 4 by explaining how the result for $\nu = 1$ can be used to make progress for the $\nu = 1/m$ Laughlin states, and recast the problem in terms of clustering properties of symmetric polynomials. In section 5, we prove that the *total* degree of polynomials that vanish under clustering is

[5] Provided no additional trivial constraint, coming from the Hilbert space of the reduced system, is present. The particles of the simplest systems for which this can happen have spin $S = 1$.

bounded from below, and provide an explicit construction for such polynomials in general. In section 7, we make some comments on why it is much harder to prove that not only the total degree, but also the degree of any variable for polynomials that vanish under the clustering is bounded from below. In addition, we provide a proof for the statement in the case where one forms two clusters of size m. Finally, we discuss our results in section 8. In the appendix A, we derive some properties of the polynomials which are used in section 5, and in appendix B we provide an alternate set of polynomials that can be used in the proof of section 5.

2. Some notations

In this section we introduce some definitions and notations that are used in what follows. We start by introducing the notion of partitions, which play a central role in the theory of symmetric polynomials. For a general introduction to the subject of partitions, we refer to [35] and for the theory of symmetric polynomials to [34].

2.1. Partitions

For a positive integer $\mathcal{D}$, a non-increasing sequence $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_r)$ of strictly positive integers $\lambda_1, \lambda_2, ..., \lambda_r$ is called an *r-partition* of $\mathcal{D}$ if $\sum_{i=1}^{r} \lambda_i = \mathcal{D}$. The λ_i's are the *parts* of $\boldsymbol{\lambda}$, and r is called the *length* of $\boldsymbol{\lambda}$, which is denoted by $l(\boldsymbol{\lambda})$. We call $\mathcal{D}$ the *weight* of $\boldsymbol{\lambda}$, which is denoted by $|\boldsymbol{\lambda}|$. We write $\boldsymbol{\lambda} \vdash \mathcal{D}$ to indicate that $\boldsymbol{\lambda}$ is a partition of $\mathcal{D}$. By convention, $\boldsymbol{\lambda} = \varnothing$ is the only partition of zero which we call the *empty* partition. The number of parts of partition $\boldsymbol{\lambda}$ that are equal to a given integer j is denoted by $n_j(\boldsymbol{\lambda})$ or simply n_j. We also define

$$z_{\boldsymbol{\lambda}} = \prod_{j=1}^{\lambda_1} j^{n_j} n_j!. \tag{10}$$

Finally, the set of all partitions of $\mathcal{D}$ is denoted by $\mathrm{Par}(\mathcal{D})$. It is not too hard to convince oneself (see [35]) that the number of partitions with at most r parts and each part at most d is equal to $\binom{r+d}{r}$.

2.2. Symmetric polynomials

In what comes, we will be dealing with the ring Λ_N of *symmetric polynomials* in N variables. A polynomial P is called a symmetric polynomial in N variables, if for all permutations σ of $\{1, \ldots, N\}$,

$$P\left(x_{\sigma(1)}, ..., x_{\sigma(N)}\right) = P(x_1, ..., x_N). \tag{11}$$

The degree d of a symmetric polynomial is simply the degree in one of its variables.

A polynomial $P(x_1, ..., x_N)$ is called homogeneous of total degree $\mathcal{D}$, if for any real number l,

$$P\left(l\, x_1, \ldots, l\, x_N\right) = l^{\mathcal{D}} P(x_1, ..., x_N). \tag{12}$$

For instance, the polynomial $P(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$ is a homogeneous symmetric polynomial of degree $d = 2$ and total degree $\mathcal{D} = 3$.

There are different bases that one can consider for Λ_N. A natural one, is given by the set of, so-called, *symmetric monomials*. Given a partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $r \leqslant N$, the symmetric monomial $m_{\boldsymbol{\lambda}}(x_1, \ldots, x_N)$ is defined as

$$m_\lambda(x_1, \ldots, x_N) \coloneqq \sum_\sigma x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(r)}^{\lambda_r} x_{\sigma(r+1)}^0 \cdots x_{\sigma(N)}^0, \tag{13}$$

where the sum is over all *distinct* permutations σ of the parts of λ, and it is defined to be 1 if λ is the empty partition. On the other hand if $r > N$ we set $m_\lambda(x_1, \ldots, x_N) = 0$. For example,

$$\begin{aligned} m_{(2,1,1)}(x_1, x_2, x_3) &= x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2, \\ m_{(2,1)}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2, \\ m_{(2,1,1)}(x_1, x_2) &= 0. \end{aligned} \tag{14}$$

When studying rank saturation of the PES for the Laughlin state, finite-size effects imply an upper bound for the degree of polynomials. We will therefore be led to consider the space Λ_N^d of symmetric polynomials in N variables, with degree (in each of the variables) at most d. A basis for this space is given by the symmetric monomials $m_\lambda(x_1, \ldots, x_N)$ corresponding to partitions λ with at most N parts and each part at most d. Therefore, we have

$$\dim\left(\Lambda_N^d\right) = \binom{N + d}{N}. \tag{15}$$

Another important family of symmetric polynomials is the set of *elementary symmetric polynomials*. The elementary symmetric polynomials that are labelled by an integer n are defined in terms of symmetric monomials as $e_n \coloneqq m_{\underbrace{(1,\ldots,1)}_{n \text{ ones}}}$. For instance,

$$\begin{aligned} e_0(x_1, x_2, x_3) &= 1, \\ e_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3, \\ e_2(x_1, x_2, x_3) &= x_1 x_2 + x_1 x_3 + x_2 x_3, \\ e_3(x_1, x_2, x_3) &= x_1 x_2 x_3, \\ e_{n \geqslant 4}(x_1, x_2, x_3) &= 0. \end{aligned} \tag{16}$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, the elementary symmetric polynomial e_λ is defined as $e_\lambda \coloneqq e_{\lambda_1} \cdots e_{\lambda_r}$. As an example,

$$\begin{aligned} e_{(2,1,1)}(x_1, x_2) &= e_2(x_1, x_2) e_1(x_1, x_2) e_1(x_1, x_2) \\ &= x_1 x_2 (x_1 + x_2)^2. \end{aligned} \tag{17}$$

It is known that the set of all polynomials $e_\lambda(x_1, \ldots, x_N)$, where λ is a partition with at most d parts and each part at most N, forms a basis of the space Λ_N^d.

Lastly, the *power sum symmetric polynomials*, defined as

$$p_i(x_1, \ldots, x_N) \coloneqq x_1^i + \cdots + x_N^i, \tag{18}$$

are of special importance. In fact, the set $\{p_1, p_2, \ldots, p_N\}$ generates Λ_N. This means that any symmetric polynomial P in N variables can be written as a polynomial in $(p_1, \ldots, p_N)$. In other words, the set of polynomials $p_\lambda \coloneqq p_{\lambda_1} \cdots p_{\lambda_r}$, where λ is a partition with each part at most N, forms a basis of Λ_N. For example,

$$e_2 = \sum_{i<j} x_i x_j = \frac{p_1^2 - p_2}{2}, \tag{19}$$

independent of the number of variables N. Most importantly, the decomposition of any symmetric polynomial P in N variables as a polynomial in $(p_1, \ldots, p_N)$ is unique. One should note that, unlike for the symmetric monomials m_λ and the elementary symmetric polynomials e_λ, there is no natural restriction on λ such that the corresponding p_λ's form a basis for Λ_N^d.

3. The $\nu = 1$ state

To obtain the rank of the reduced density matrix of the Laughlin states in the case of the 'particle cut', we start by considering the simplest case, the $\nu = 1$ Laughlin state, which is just a single Slater determinant,

$$\Psi_{\nu=1}(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j), \tag{20}$$

up to a geometry-dependent Gaussian factor. For instance, the plane and sphere geometry give rise to different Gaussian factors, inherited from the respective inner products. However, for our purposes the precise form of the Gaussian factor is irrelevant. The results presented in this paper involves only the notion of linear independence, and does not refer to the notion of orthogonality. As a consequence, the underlying inner product plays no role and our result is equally valid on the plane, sphere, and cylinder.

Now suppose that the N particles are divided into two groups A, containing N_A of the particles, and B containing $N_B = N - N_A$ particles. At this stage we do not assume $N_A \leqslant N_B$. Let us rename the coordinates of particles in A and B to $\boldsymbol{x} = (x_1, x_2, \ldots, x_{N_A})$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_{N_B})$, respectively. The rank of the reduced density matrix in the case of such a particle cut can be obtained from a decomposition of the wave function $\Psi(z)$ of the form

$$\Psi(z) = \sum_i \psi_i^A(\boldsymbol{x}) \psi_i^B(\boldsymbol{y}), \tag{21}$$

where the set of wave functions ψ_i^A (resp. ψ_i^B) are independent. Note that this is not quite a Schmidt decomposition since we do not demand the ψ_i^A's to form an orthonormal set. Although this is not a Schmidt decomposition, the number of terms in the sum is equal to the Schmidt rank, or equivalently, to the rank of the reduced density matrix. Therefore, we will call the decomposition (21) a Schmidt decomposition, although strictly speaking this is an abusive notation.

Before we explicitly write the $\nu = 1$ Laughlin state in such a 'Schmidt-decomposed' form, we note that we can obtain the rank of the reduced density matrix in the $\nu = 1$ case in a straightforward way. This state is simply obtained by filling the Landau orbitals from 0 up to $N_\Phi = N - 1$

$$\left|\Psi_{\nu=1}\right\rangle = |111\cdots111\rangle. \tag{22}$$

The Schmidt decomposition relative to the particle cut scheme described above amounts to choose N_A out of the N particles

$$\begin{aligned}\left|\Psi_{\nu=1}\right\rangle \propto |\underbrace{111\cdots11}_{N_A}00\cdots0\rangle \otimes |000\cdots0011\cdots1\rangle \\ + |\underbrace{111\cdots10}_{N_A}10\cdots0\rangle \otimes |000\cdots0101\cdots1\rangle + \cdots \\ + |\underbrace{000\cdots00}_{N_A}11\cdots1\rangle \otimes |111\cdots1100\cdots0\rangle,\end{aligned} \tag{23}$$

which means that the rank of the reduced density matrix is given by the number of ways in which the N_A particles of system A can be divided over the number of orbitals. The number of orbitals is given by $N = N_A + N_B$, so we obtain that the rank of the reduced density matrix is

given by $\begin{pmatrix} N_A + N_B \\ N_A \end{pmatrix}$. We note that the same result can be obtained directly from the wave function [15], which is a single Slater determinant $\Psi_{\nu=1}(z) = \prod_{i<j} (z_i - z_j)$.

It is instructive to perform a more explicit Schmidt decomposition of the $\nu = 1$ Laughlin wave function. We start by writing the state explicitly in terms of the variables x_i and y_i of groups A and B, respectively. Dropping the exponential factors, we have

$$\Psi_{\nu=1}(z) = \Psi_{\nu=1}(\boldsymbol{x})\left(\prod_{i,j}\left(x_i - y_j\right)\right)\Psi_{\nu=1}(\boldsymbol{y}). \tag{24}$$

We are going to use the following result [34]

$$\prod_{i=1}^{N_A}\prod_{j=1}^{N_B}\left(1 + x_i y_j\right) = \sum_{\lambda} m_\lambda(\boldsymbol{x}) e_\lambda(\boldsymbol{y}), \tag{25}$$

where the sum is over all partitions $\boldsymbol{\lambda}$ with maximally N_A parts, and each part being maximally N_B, i.e., all partitions which fit in a rectangle of height N_A and width N_B. Thus

$$\prod_{i,j}\left(x_i - y_j\right) = \sum_{\lambda}(-1)^{|\lambda|} m_{\bar{\lambda}}(\boldsymbol{x}) e_\lambda(\boldsymbol{y}). \tag{26}$$

Here, we used the relation

$$\left(\prod_i x_i^{N_B}\right) m_\lambda(-1/\boldsymbol{x}) = (-1)^{|\lambda|} m_{\bar{\lambda}}(\boldsymbol{x}), \tag{27}$$

where the partition $\bar{\boldsymbol{\lambda}}$ stands for the complement of $\boldsymbol{\lambda}$ with respect to the rectangle of height N_A and width N_B. In addition, $(-1/\boldsymbol{x})$ is shorthand for $(-1/x_1, \ldots, 1/x_{N_A})$. As an example, it is shown in figure 1 that for $N_A = 3$, $N_B = 4$ and $\boldsymbol{\lambda} = (2, 1)$, one finds $\bar{\boldsymbol{\lambda}} = (4, 3, 2)$. We then obtain a Schmidt decomposition for the $\nu = 1$ Laughlin state

$$\Psi_{\nu=1}(z) = \sum_{\lambda}(-1)^{|\lambda|} q_\lambda^A(\boldsymbol{x}) q_\lambda^B(\boldsymbol{y}), \tag{28}$$

where

$$q_\lambda^A(\boldsymbol{x}) = m_{\bar{\lambda}}(\boldsymbol{x})\Psi_{\nu=1}(\boldsymbol{x}), \tag{29}$$

and

$$q_\lambda^B(\boldsymbol{y}) = e_\lambda(\boldsymbol{y})\Psi_{\nu=1}(\boldsymbol{y}). \tag{30}$$

A few remarks on this formula are in order here. The number of terms in the sum is most important here. The sum over $\boldsymbol{\lambda}$, is over all partitions with maximally N_A parts, and each part being maximally N_B, i.e., all partitions which fit in a rectangle of height N_A and width N_B. There are $\begin{pmatrix} N_A + N_B \\ N_A \end{pmatrix}$ such partitions. The rank of the reduced density matrix of the $\nu = 1$ Laughlin state is thus given by

$$\begin{pmatrix} N_A + N_B \\ N_A \end{pmatrix}, \tag{31}$$

and we recover the dimension of $\Lambda_{N_A}^{N_B}$. It is straightforward to check that this is also the dimension of the set of anti-symmetric polynomials in N_A variables with maximum degree

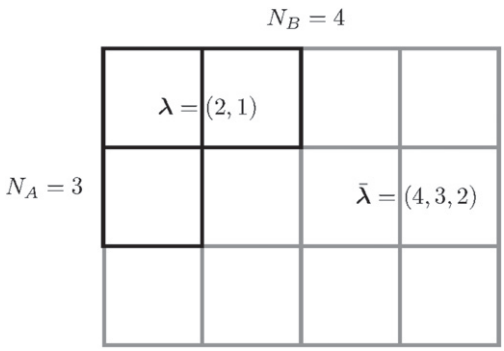

Figure 1. The relation between the partition λ and its complement $\bar{\lambda}$ for given N_A and N_B.

$N = N_A + N_B$, which is nothing but the space of 'quasi-hole' states for the non-interacting $\nu = 1$ case. The set of polynomials q_λ^A (resp. q_λ^B) forms a basis for the space of anti-symmetric polynomials in N_A (resp. N_B) variables with maximum degree N. Note that this result is symmetric under exchange of A and B, and in particular it holds whether or not $N_A \leqslant N_B$. This is a particularity of the $\nu = 1$ case and it will no longer be true for the $\nu = \frac{1}{m}$ Laughlin state with $m > 1$.

4. Schmidt decomposition of the $\nu = \frac{1}{m}$ Laughlin state

We are now going to compute the rank of the reduced density matrix for the generic $\nu = 1/m$ Laughlin state

$$\Psi_m(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m. \tag{32}$$

As usual we divide the particles into two groups A and B, containing N_A and $N_B = N - N_A$ of them, respectively, and we assume that $N_A \leqslant N_B$. We are interested in obtaining a Schmidt decomposition of this state. As for the $\nu = 1$ case, we can write

$$\Psi_m(\boldsymbol{z}) = \Psi_m(\boldsymbol{x}) \left(\prod_{i=1}^{N_A} \prod_{j=1}^{N_B} \left(x_i - y_j \right)^m \right) \Psi_m(\boldsymbol{y}). \tag{33}$$

Proving that the rank of the PES for the $\nu = 1/m$ Laughlin state is saturated boils down to finding a Schmidt decomposition for the wave function (33) and counting the number of terms in the decomposition. As in the $\nu = 1$ case, one needs to take care of only the middle term of the wave function (33)

$$\Phi_m(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^{N_A} \prod_{j=1}^{N_B} \left(x_i - y_j \right)^m. \tag{34}$$

To do so we start from

$$\Phi_1(\boldsymbol{x}, \boldsymbol{w}) = \prod_{i=1}^{N_A} \prod_{j=1}^{mN_B} (x_i - w_j), \tag{35}$$

for $\boldsymbol{w} = (w_1, \ldots, w_{mN_B})$. From (26),

$$\Phi_1(\boldsymbol{x}, \boldsymbol{w}) = \sum_{\lambda} (-1)^{|\lambda|} m_{\bar{\lambda}}(\boldsymbol{x}) e_{\lambda}(\boldsymbol{w}), \tag{36}$$

but this time the sum is over all partitions $\boldsymbol{\lambda}$ which fit in a rectangle of height N_A and width mN_B. We then relate Φ_m to Φ_1 through the clustering transformation, which is a linear transformation from Λ_{mN_B} to Λ_{N_B} defined as follow.

To a symmetric polynomial $P(\boldsymbol{w})$ in mN_B variables, we associate the polynomial $C_m(P(\boldsymbol{y}))$ in N_B variables obtained by regrouping the particles into clusters of m, i.e.

$$\big(C_m P\big)\Big(y_1, y_2, \ldots, y_{N_B}\Big) = P\left(\underbrace{y_1, y_1, \ldots, y_1}_{m}, \underbrace{y_2, y_2, \ldots, y_2}_{m}, \ldots, \underbrace{y_{N_B}, y_{N_B}, \ldots, y_{N_B}}_{m}\right). \tag{37}$$

It is easy to see that after clustering Φ_1 becomes Φ_m, i.e.

$$C_m\big(\Phi_1\big) = \Phi_m. \tag{38}$$

Applying the clustering transformation to both sides of equation (36) results in

$$\Phi_m(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\lambda} (-1)^{|\lambda|} m_{\lambda}(\boldsymbol{x}) C_m(e_{\lambda})(\boldsymbol{y}). \tag{39}$$

As mentioned earlier, the sum is over all partitions $\boldsymbol{\lambda}$ with maximally N_A parts, and each part being maximally mN_B. There are $\begin{pmatrix} N_A + mN_B \\ N_A \end{pmatrix}$ such partitions. This is precisely the number of Laughlin quasi-hole states for N_A particles in $\Delta N_\Phi = mN_B$ extra fluxes, and we recover the usual upper bound for the rank of the reduced density matrix.

Rank saturation of the PES for the $\nu = 1/m$ Laughlin state boils down to the following, non-trivial result: the polynomials $C_m(e_\lambda)$ are independent. More precisely, one has to prove that the linear transformation

$$\begin{aligned} \Lambda^{N_A}_{mN_B} &\longrightarrow \Lambda^{mN_A}_{N_B} \\ P(\boldsymbol{w}) &\longrightarrow C_m(P)(\boldsymbol{y}) \end{aligned}$$

is injective as long as $N_A \leqslant N_B$. Since $\dim \Lambda^{mN_A}_{N_B} \geqslant \dim \Lambda^{N_A}_{mN_B}$, it is sufficient to show that this linear map has a trivial kernel. Namely, besides $P = 0$, no polynomial in mN_B variables and maximum degree N_A can vanish under the clustering transformation.

5. Clustering properties of symmetric polynomials

In this section we are going to describe the ideal of symmetric polynomials in $q = m\,N$ variables that vanish under the clustering transformation (37). In particular, we are going to construct a generating set of this subspace, and prove that a non-zero symmetric polynomial in $q = m\,N$ variables that vanishes under the clustering transformation has a *total* degree $\mathcal{D}$ of at least $N + 1$. We are also going to prove that this symmetric polynomial of minimal total degree is unique (up to a scaling numerical factor).

The statement that the total degree of a symmetric polynomial in $q = m\,N$ variables that vanishes under the clustering transformation is at least $N + 1$ is a weaker statement than stating that the degree of each variable is at least $N + 1$, but easier to prove. After finishing

the proof of the statement on the total degree, we comment on how one might prove the stronger statement, limiting the degree of the polynomials.

As a warmup, we start with two simple examples, which we will come back to after the proof. We start with the case $m = 2$ and $N = 1$, i.e., we are looking for a symmetric polynomial in two variables y_1 and y_2, of total degree 2, that vanishes when $y_1 = y_2$. It is easy to see that the polynomial has degree two, namely $(y_1 - y_2)^2$.

The case $m = 2$ and $N = 2$ is already more complicated. With some thought, one can construct a total degree 3 symmetric polynomial in four variables $y_1, \ldots, y_4$, that vanishes when $y_1 = y_2$ and $y_3 = y_4$, namely $(y_1 + y_2 - y_3 - y_4)(y_1 - y_2 + y_3 - y_4)(y_1 - y_2 - y_3 + y_4)$. It is already less trivial to convince oneself that no lower degree symmetric polynomial with the same vanishing properties exists. Upon increasing the m and N, even finding polynomials with the correct vanishing properties becomes a hard problem, which is caused by the non-locality of their defining property. Namely, polynomials have to vanish, independent of the position of the various clusters. As we indicated above, we solve this problem in a constructive way.

Our construction is motivated by the following observation. The ring Λ_q of symmetric polynomials in $q = mN$ variables is generated by $\{p_1, \ldots, p_q\}$, and the power sum polynomials p_i have a very simple behaviour under the clustering transformation (37), namely

$$C_m(p_i) = m\, p_i. \tag{40}$$

However, after the clustering transformation there are only N variables left. This means that $\mathcal{A}_N = \{p_1, \ldots, p_N\}$ forms a minimal set of generators, and the polynomials $\{p_{N+1}, \ldots, p_q\}$ are no longer independent after being clustered. The generators p_i are not very convenient to describe the clustering transformation, and this is why we introduce a new set of generators $\tilde{\mathcal{A}}_q = \{\tilde{p}_1, \ldots, \tilde{p}_q\}$ of Λ_q as

$$\tilde{p}_n = \sum_{\boldsymbol{\mu} \vdash n} (-1)^{|\boldsymbol{\mu}|} \left(-\frac{1}{m}\right)^{l(\boldsymbol{\mu})} \frac{p_{\boldsymbol{\mu}}}{z_{\boldsymbol{\mu}}}. \tag{41}$$

Alternatively, the polynomials $\tilde{p}_n$ can be defined in terms of the polynomials $r_n^{(x)}$ of appendix A through $\tilde{p}_n = \dfrac{(-1)^n}{n!} r_n^{(1/m)}$. The main property of these new polynomials is that they behave nicely under clustering:

$$C_m\left(\tilde{p}_n\right) = e_n, \qquad n = 1, \ldots, N, \tag{42}$$

$$C_m\left(\tilde{p}_n\right) = 0, \qquad n > N, \tag{43}$$

as inherited from the properties of $r_n^{(x)}$ described in appendix A. We also introduce the $\tilde{p}_\lambda$ in the usual way as $\tilde{p}_{\boldsymbol{\lambda}} = \tilde{p}_{\lambda_1} \ldots \tilde{p}_{\lambda_r}$, where r is the length of $\boldsymbol{\lambda}$. In terms of these modified power sums $\tilde{p}_n$, it is now relatively easy to describe the ideal of polynomials in Λ_q that vanish under the clustering transformation (37):

Theorem 1. *The set* $\{\tilde{p}_\lambda | N + 1 \leqslant \lambda_1 \leqslant q\}$ *forms a basis for the ideal of symmetric polynomials in* $q = mN$ *variables that vanish under the clustering transformation. In other words, this ideal is generated by the set* $\{\tilde{p}_{N+1}, \tilde{p}_{N+2}, \ldots, \tilde{p}_q\}$.

Proof. Suppose that P is a symmetric polynomial in q variables. Because $\tilde{\mathcal{A}}_q$ is a generating set, there exists a polynomial R in q variables such that

$$P = R\left(\tilde{p}_1, \ldots, \tilde{p}_q\right).$$

Generically, there are two kinds of monomials in the polynomial R. Those that depend only on the first N variables $\tilde{p}_1, \ldots, \tilde{p}_N$, and the ones that depend on at least one of the $\tilde{p}_n$, with $n > N$. Accordingly, R can be decomposed uniquely into a sum of two polynomials

$$R\left(\tilde{p}_1, \ldots, \tilde{p}_q\right) = A\left(\tilde{p}_1, \ldots, \tilde{p}_N\right) + B\left(\tilde{p}_1, \ldots, \tilde{p}_q\right).$$

Thus, by construction, $C_m(B) = 0$. It is now straightforward to check that $C_m(R) = 0$ if and only if $A = 0$, since

$$C_m(P) = A(e_1, \ldots, e_N),$$

and the $\{e_1, \ldots, e_N\}$ are algebraically independent in N variables. Therefore the set $\{\tilde{p}_\lambda | N + 1 \leqslant \lambda_1 \leqslant q\}$ is a basis of the kernel of the clustering transformation. ■

Corollary 1. *The only symmetric polynomial P in mN variables with total degree* N *or less that vanishes under the clustering transformation is $P = 0$. Moreover, $\tilde{p}_{N+1}$ is the unique (up to an overall factor) symmetric polynomial in q variables and total degree $N + 1$ that vanishes under this clustering.*

Since the modified power sum $\tilde{p}_n$ has total degree n, this corollary follows directly from theorem 1. Let us illustrate this with two simple examples.

Example 1. Consider the simplest non-trivial case where $q = 2$, $N = 1$, and $m = 2$. In this case, the clustering condition is $y_1 = y_2$. The definition of $\tilde{p}_2$ yields

$$\tilde{p}_2 = \frac{1}{8}\left(p_1^2 - 2p_2\right) = -\frac{1}{8}(y_1 - y_2)^2, \tag{44}$$

which reproduces the expected result.

Example 2. As another example, let $q = 4$, $N = 2$, and $m = 2$. This time, clustering conditions are $y_1 = y_2$ and $y_3 = y_4$. We have

$$\tilde{p}_3 = \frac{1}{48}\left(p_1^3 - 6p_2p_1 + 8p_3\right). \tag{45}$$

For $q = 4$ variables, this is

$$\tilde{p}_3 = \frac{1}{16}\left(y_1 + y_2 - y_3 - y_4\right)\left(y_1 - y_2 + y_3 - y_4\right)\left(y_1 - y_2 - y_3 + y_4\right). \tag{46}$$

We should note that these two examples are not representative for the general case, in the sense that the polynomials $\tilde{p}_{N+1}$ do not generically factorize to a simpler form. For instance, for $N = 3$ and $m = 2$, we have

$$\tilde{p}_4 = \frac{1}{384}\left(p_1^4 - 12p_2p_1^2 + 12p_2^2 + 32p_1p_3 - 48p_4\right),$$

which does not have a simple factorized form when restricting to $q = 6$ variables.

Conjecture 1. There is no non-zero symmetric polynomial P in mN variables with degree N or less that vanishes under the clustering transformation.

While we know that the modified power sum $\tilde{p}_n$ has degree n (see appendix A), this is not sufficient to prove this conjecture.

6. *SU*(2) invariance

In the context of the FQH effect, there is a natural action of $SU(2)$ on $\Lambda_N^{N_\Phi}$ coming from the rotational invariance of the sphere. The angular momentum operators on the sphere [36] are

$$L^- = \sum_{i=1}^{N} \frac{\partial}{\partial z_i}, \tag{47}$$

$$L^3 = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} - \frac{N_\Phi}{2} \right), \tag{48}$$

$$L^+ = \sum_{i=1}^{N} \left(z_i N_\Phi - z_i^2 \frac{\partial}{\partial z_i} \right). \tag{49}$$

Every polynomial P in $\Lambda_N^{N_\Phi}$ has a $SU(2)$ symmetric $\Omega(P)$ with opposite angular momentum L^3 given by

$$\Omega(P)(z_1, \ldots, z_N) = \left(\prod_{i=1}^{N} z_i^{N_\Phi} \right) P\left(1/z_1, \ldots, 1/z_N\right). \tag{50}$$

Under this $\mathbb{Z}_2$ operation, L^- and L^+ are exchanged and $L^3 \to -L^3$.

These linear operators are compatible with the clustering, in the sense that $C_m L^i = L^i C_m$. Note that in these identities the $SU(2)$ operators in the left-hand side (lhs) act in $\Lambda_{mN}^{N_\Phi}$, while in the rhs they act in $\Lambda_N^{mN_\Phi}$:

$$\begin{array}{ccc} \Lambda_{mN}^{N_\Phi} & \xrightarrow{L^i} & \Lambda_{mN}^{N_\Phi} \\ \downarrow C_m & & \downarrow C_m \\ \Lambda_N^{mN_\Phi} & \xrightarrow{L^i} & \Lambda_N^{mN_\Phi} \end{array} \tag{51}$$

The same is true for the $\mathbb{Z}_2$ operation Ω. These commuting properties are straightforward to check for L^3, L^-, and for Ω. Therefore, it also holds for $L^+ = \Omega L^- \Omega$. For instance, the clustering transformation clearly preserves the total degree, hence the action of clustering commutes with L^3. Likewise, L^- being the generator of global translations, it commutes with the clustering. The following theorem follows immediately:

Theorem 2. *The ideal of symmetric polynomials in mN variables that vanish under the clustering transformation is invariant under the action of* L^i *and* Ω.

Corollary 2. *The polynomial* $\tilde{p}_{N+1}$ *is translationally invariant.*

The polynomial $\tilde{p}_{N+1}$ vanishes under clustering, and therefore so does $L^-\tilde{p}_{N+1}$. If this last polynomial was non-zero, it would have a total degree N, which is forbidden by corollary 1. Therefore $L^-\tilde{p}_{N+1} = 0$ and $\tilde{p}_{N+1}$ is translationally invariant.

In fact, it is possible to directly calculate $L^{-}\tilde{p}_i$, and one gets

$$L^{-}\tilde{p}_i = (N + 1 - i)\tilde{p}_{i-1}. \tag{52}$$

Note that this results only hold for $q = mN$ variables. This follows from the behaviour of $r_n^{(x)}$ under translations, which is given in appendix A.4.

Since the kernel of C_m is invariant under the action of L^i, it can be decomposed into irreducible representations of $SU(2)$. In order to prove that non-zero polynomials that vanish under the clustering have degree at least $N + 1$, it is therefore sufficient to prove it for lowest weights, that is to say translation invariant polynomials. Therefore conjecture 1 is equivalent to the following:

Conjecture 2. The only translationally invariant symmetric polynomial P in mN variables with degree N or less that vanishes under the clustering transformation is $P = 0$.

7. A possible road towards finishing the proof

As we saw in the previous section, we were able to prove that the *total* degree of a symmetric polynomial is at least $N + 1$, if the polynomial vanishes under the clustering transformation (37). However, we would like to show that the the maximum degree of any of the variables (i.e., the number of fluxes N_ϕ) is at least $N + 1$. Proving this statement turns out to be much harder than it looks at first. One of the reasons is that the clustering we consider is a non-local process. Namely, the positions of the various clusters are arbitrary. Therefore, proving that the total degree is bounded from below is already a non-trivial result. What makes proving a bound on the total degree more tractable in comparison to proving a bound on the degree, is that upon taking linear combinations, the total degree of the polynomials does not change, provided the resulting polynomial does not vanish. The degree of the polynomial, however, can be lowered by taking linear combinations.

To show that the rank of the reduced density matrix for the particle cut does indeed satisfy the upper bound given in the introduction, it suffices to prove that the clustering map $C_m : \Lambda_{mN_B}^{N_A} \longrightarrow \Lambda_{N_B}^{mN_A}$, is injective if $N_A \leqslant N_B$. In the case $N_A = N_B$, the map C_m would then actually be bijective. One possible route in trying to prove this, is to find two suitable bases for Λ_{mN}^{N} and Λ_{N}^{mN} in which the map C_m acts in an upper-triangular way, and then check that all diagonal elements are non-zero. We did not, however, succeed in finding suitable bases.

A completely different route to prove that the rank of the reduced density matrix is given by the upper bound is to try to make use of the results for the Read–Rezayi states [37]. These states are defined by the property that they vanish if $k + 1$ particles are put at the same location (in their simplest bosonic incarnation). In particular, it is known exactly that how many symmetric polynomials there are, that satisfy this clustering condition, for an arbitrary number of particles, and arbitrary degree [32, 33]. In addition, there are explicit expressions for these polynomials [33], see also [38]. Using these results, we can prove the wanted result for $N = 2$ and arbitrary m. That is, we can show that any symmetric polynomial in $2m$ variables, that vanishes if two clusters of m variables each are formed, has degree at least three.

To do so, assume that P is a polynomial in $2m$ variables that vanishes under the clustering, $C_m P = 0$. We know that the total degree of this polynomial is at least 3, and we want to show that the minimal value of the degree is three as well.

To show this, we note that the polynomial P also vanishes if we make one big cluster of $2m$ variables. From the results on the Read–Rezayi states, we know that such symmetric

polynomials have degree at least two (it vanishes, so it should vanish quadratically), and that it is unique (up to an overall factor). In addition, we know an explicit form of this polynomial P', namely

$$P'(z_1, \ldots, z_{2m}) = S\big[(z_1 - z_3)(z_2 - z_3)\big], \tag{53}$$

where S denotes the complete symmetrization over all $2m$ variables. In this case, by inspection, one can convince oneself that P' does not vanish if one makes two clusters of m variables. Thus, the minimal degree of a polynomial P in $2m$ variables that does vanish under C_m has degree at least three. In fact, it is not too hard to find an expression similar to the one for P', namely

$$P(z_1, \ldots, z_{2\,m}) = S\big[(z_1 - z_4)(z_2 - z_4)(z_3 - z_4)\big]. \tag{54}$$

It is not completely obvious that this vanishes under the clustering for $m > 2$, but one can convince oneself that after symmetrization, one indeed does get zero.

Though it is not going to be easy, one could try to proceed in this way. Constructing the next case, namely polynomials that vanish for three clusters of m variables, is already more involved. Writing down an explicit form similar to the ones above is not straightforward, but one can for instance symmetrize the following combination

$$\begin{aligned} P(z_1, \ldots, z_{3m}) = S\big[&(z_1 - z_5)(z_2 - z_5)(z_3 - z_5)(z_4 - z_5) \\ &- (z_1 - z_5)(z_2 - z_5)(z_2 - z_6)(z_3 - z_6)\big]. \end{aligned} \tag{55}$$

This polynomial is the unique polynomial (up to a constant factor) in $3m$ variables, of degree and total degree 4, that vanishes under formation of three clusters of m variables. We stress, however, that this alone does not imply that there are no polynomials of degree three, that vanish under the same clustering conditions.

The lowest degree polynomial for $N = 4$ and arbitrary m can still be written by symmetrizing an expression like the one in equation (55), i.e., two terms only, but it seems likely that these expressions become more complicated upon increasing N. In addition, having these explicit expressions does not help in excluding the existence of lower degree polynomials with the same clustering conditions.

8. Discussion

In this paper, we revisited the study of the PES for the $\nu = 1/m$ Laughlin states, in particular the rank of the associated reduced density matrix. To determine this rank, we make use of the rank of the reduced density matrix for the $\nu = 1$ Laughlin state. We showed that to relate the rank for the $\nu = 1/m$ Laughlin state to the case $\nu = 1$, one has to prove a bound on the degree of symmetric polynomials that vanish under the formation of certain clusters. Though we were not able to finish the proof of this statement, we made substantial progress by explicitly constructing a set of polynomials that vanish under the clustering, and we proved that the *total* degree of these polynomials is bounded from below.

We commented on a possible, though most likely rather hard, route towards finishing the proof. In this paper, we concentrated on the Laughlin states. It would be interesting to see if similar methods can be used to make progress on different model states, such as the Moore–Read and Read–Rezayi states, that exhibit excitations obeying non-Abelian statistics.

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Appendix A. Properties of the polynomials $r_n^{(x)}$

In this appendix we introduce a family of symmetric polynomials $\{r_n^{(x)}\}$ defined through the generating function

$$\exp\left(-x\sum_{k=1}^{\infty}p_k\frac{t^k}{k}\right)=\sum_{n=0}^{\infty}r_n^{(x)}\frac{t^n}{n!}. \tag{A.1}$$

The key property of the $r_n^{(x)}$'s is their behaviour under the clustering transformation C_m:

$$C_m\left(r_n^{(x)}\right)=r_n^{(m\,x)}, \tag{A.2}$$

which is a direct consequence of their definition. Further properties follow from the generating function, namely

- $r_n^{(1)}=(-1)^n\,n!\,e_n$,
- $r_n^{(-1)}=n!\,h_n$ with $h_n=\sum_{\lambda\vdash n}m_\lambda$,
- $\partial_x r_n^{(x)}\big|_{x\,=\,0}=-(n-1)!p_n$ for $n\geqslant 1$,

where the first two relations are obtained by comparison to the generating functions for the e_n and h_n, see for instance [34]. Therefore, this family of polynomials interpolates between power sums p_n, elementary symmetric polynomials e_n, and complete homogeneous symmetric polynomials h_n. We give one additional property,

$$r_n^{(x+y)}=\sum_{k=0}^{n}\binom{n}{k}r_k^{(x)}r_{n-k}^{(y)}, \tag{A.3}$$

that follows from the definition.

A.1. Explicit formulas for $r_n^{(x)}$

The generating function can be expanded using Bell's polynomials [39], yielding an explicit expression for $r_n^{(x)}$, that is

$$r_n^{(x)}=n!\sum_{\lambda\vdash n}(-x)^{l(\lambda)}\frac{p_\lambda}{z_\lambda}. \tag{A.4}$$

Alternatively, this expression can be obtained by acting with C_x on Newton's identity expressing elementary symmetric polynomials in terms of power sums (here, we allow x to be real, and set $C_x p_i=xp_i$, for an infinite number of variables). Another explicit expression is given in terms of a determinant of power sums

$$r_n^{(x)} = (-1)^n \begin{vmatrix} xp_1 & 1 & 0 & \cdots & 0 \\ xp_2 & xp_1 & 2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ xp_{n-1} & xp_{n-2} & \cdots & xp_1 & n-1 \\ xp_n & xp_{n-1} & \cdots & xp_2 & xp_1 \end{vmatrix}. \tag{A.5}$$

The first few polynomials are given by

$$\begin{aligned} r_0^{(x)} &= 1, \\ r_1^{(x)} &= -xp_1, \\ r_2^{(x)} &= x^2 {p_1}^2 - xp_2, \\ r_3^{(x)} &= -x^3 {p_1}^3 + 3x^2 p_1 p_2 - 2xp_3, \\ r_4^{(x)} &= x^4 {p_1}^4 - 6x^3 {p_1}^2 p_2 + 3x^2 {p_2}^2 + 8x^2 p_1 p_3 - 6xp_4, \\ r_5^{(x)} &= -x^5 {p_1}^5 + 10x^4 {p_1}^3 p_2 - 15x^3 p_1 {p_2}^2 - 20x^3 {p_1}^2 p_3 \\ &\quad + 20x^2 p_2 p_3 + 30x^2 p_1 p_4 - 24xp_5. \end{aligned}$$

From triangularity it follows that the family $\{r_1^{(x)}, \ldots, r_n^{(x)}\}$ algebraically spans all symmetric polynomials in n variables as long as $x \neq 0$.

A.2. Degree of $r_n^{(x)}$

Let us consider the monomial decomposition of $r_n^{(x)}$

$$r_n^{(x)} = \sum_{\lambda \vdash n} q_n^{(\lambda)}(x)\, m_\lambda. \tag{A.6}$$

We are going to compute the first coefficient, namely $q_n^{(\lambda)}(x)$ for $\lambda = (n)$. If this first coefficient is non-zero, the degree of $r_n^{(x)}$ is n. This coefficient is a polynomial of degree n in x, since from (A.4) we have $r_n^{(x)}(z, 0, 0, \ldots) \sim (-x)^n z^n$ as x goes to infinity.

We now take $x = m$ to be an integer, and write $r_n^{(m)} = C_m\left(r_n^{(1)}\right) = (-1)^n n!\; C_m\left(e_n\right)$. By considering the definition of e_n, it follows that $m \geqslant n$ in order for $r_n^{(m)}(z, 0, 0, \ldots)$ to be non-zero, because $m_\lambda = 0$ if the number of variables is less than $l(\lambda)$.

Therefore, $q_n^{(\lambda=(n))}(x)$ vanishes for $x = 0, 1, \ldots, n-1$. It follows from the asymptotic behaviour given above that

$$q_n^{(\lambda=(n))}(x) = (-1)^n x(x-1)(x-2)\cdots(x-n+1). \tag{A.7}$$

Hence, the degree of $r_n^{(x)}$ is n as long as $x \neq 0, 1, \ldots, n-1$.

A.3. Monomial decomposition of $r_n^{(x)}$

In this appendix we quote the full monomial decomposition of $r_n^{(x)}$, namely

$$q_n^{(\lambda)}(x) = (-1)^n \binom{n}{\lambda_1, \ldots, \lambda_n} \prod_{i=0}^{n-1} (x-i)^{\lambda_{i+1}^t}, \tag{A.8}$$

where λ^t stands for the transpose of λ. The parts of λ^t are given by $\lambda_i^t = l(\lambda) - \sum_{j=1}^{i-1} n_j(\lambda)$, or equivalently, $\lambda_i^t - \lambda_{i+1}^t = n_i(\lambda)$. Below, we sketch how this can be established.

Lemma 1. *The monomial expansion of* $\partial_x^j r_n^{(x)}\big|_{x=0}$ *involves only partitions* $\boldsymbol{\lambda}$ *such that* $\lambda_{j+1} \leqslant 0$, *i.e.,* $\lambda_i = 0$ *for* $i \geqslant j+1$.

Proof. From the generating function for the $r_n^{(x)}$ we get

$$\partial_x^j r_n^{(x)}\,|_{x=0} = (-1)^j n! \sum_{\substack{k_1,\ldots,k_j \geqslant 1 \\ k_1+\cdots+k_j=n}} \prod_i \frac{p_{k_i}}{k_i}. \tag{A.9}$$

In this expression each term $\prod_i p_{k_i}$ has a monomial decomposition that involve only partitions $\boldsymbol{\lambda}$ with a length $l(\boldsymbol{\lambda}) \leqslant j$, from which lemma 1 follows. ■

Lemma 2. *The monomial expansion of* $\partial_x^j r_n^{(x)}\big|_{x=i}$, *with* i *an integer, involves only partitions with* $\lambda_{j+1} \leqslant i$.

Proof. The case $i = 0$ boils down to lemma 1. Lemma 2 can be proven by induction on i using

$$\partial_x^j r_n^{(x)}\,|_{x=i} = n! \sum_p \frac{(-1)^{n-p}}{p!} e_{n-p}\left(\partial_x^j r_p^{(x)}\,|_{x=i-1}\right), \tag{A.10}$$

which follows from taking ∂_x^j in (A.3), namely

$$\partial_x^j r_n^{(x+y)} = \sum_p \binom{n}{p} \partial_x^j r_p^{(x)} r_{n-p}^{(y)}, \tag{A.11}$$

and then choosing $x = i-1$ and $y = 1$. ■

Corollary 3. *The coefficient* $q_n^{(\lambda)}(x)$ *is of the form*

$$q_n^{(\lambda)}(x) = c_n^{(\lambda)} \prod_i (x-i)^{\lambda_{i+1}^t}. \tag{A.12}$$

Proof. Lemma 2 is equivalent to stating that $\partial_x^j q_n^{(\lambda)}(x) = 0$ for $x = 0, 1, \ldots, \lambda_{j+1} - 1$. Thus, $x = i$ is a root with degeneracy λ_{i+1}^t of the polynomial in x $q_n^{(\lambda)}(x)$, because λ_{i+1}^t is the number of parts of $\boldsymbol{\lambda}$ that are bigger or equal to $i+1$. Since this is true for all $i \in \mathbb{N}$, we have a total of $\sum_i \lambda_i^t = n$ zeros. Since $q_n^{(\lambda)}(x)$ is of degree at most n, corollary 3 follows. ■

Lemma 3. *The coefficients* $c_n^{(\lambda)}$ *are given by*

$$c_n^{(\lambda)} = (-1)^n \frac{n!}{\lambda_1!\cdots\lambda_n!} = (-1)^n \binom{n}{\lambda_1, \ldots, \lambda_n}. \tag{A.13}$$

Proof. The asymptotic behaviour of (A.6) for x going to infinity yields

$$(-1)^n p_1^n = \sum_{\boldsymbol{\lambda} \vdash n} c_n^{(\lambda)}\, m_{\boldsymbol{\lambda}}. \tag{A.14}$$

Lemma 3 follows by expanding the lhs using the multinomial theorem (assuming the number of variables $p \geqslant n$)

$$\left(\sum_{i=1}^{p} x_i\right)^n = \sum_{\substack{k_1,\ldots,k_p\geqslant 0 \\ k_1+\cdots+k_p=n}} \frac{n!}{k_1!\cdots k_p!} \prod_i x_i^{k_i}, \tag{A.15}$$

and then gathering the terms of the rhs into symmetric monomials. ■

A.4. Behaviour of $r_n^{(x)}$ under translations

Translations are well defined in the case of finitely many variables $\{x_1, \ldots, x_r\}$, in which case we set $p_0 = r$, i.e., the number of variables. In that case L^- is the generator of translations

$$L^- = \sum_{i=1}^{r} \frac{\partial}{\partial x_i}. \tag{A.16}$$

By Leibniz's rule, its action on $p_\lambda = \prod_j p_{\lambda_j}$ is

$$L^- p_\lambda = \sum_{j\geqslant 1} n_j(\lambda) j p_{j-1} p_{\lambda\setminus\{j\}}, \qquad p_0 = r, \tag{A.17}$$

where the $n_j(\lambda)$ is the number of parts of λ that equal j, and $\lambda\setminus\{j\}$ denotes the partition derived from λ by deleting one part that equals j. We can now act on $r_n^{(x)}$,

$$L^- r_n^{(x)} = n! \sum_{\lambda\vdash n} \frac{(-x)^{l(\lambda)}}{z_\lambda} \sum_{j=1}^{n} n_j(\lambda) j p_{j-1} p_{\lambda\setminus\{j\}}. \tag{A.18}$$

We can change the summation variable from λ to $\boldsymbol{\mu} = \lambda\setminus\{j\}$, after noticing that $\boldsymbol{\mu}$ is in one-to-one mapping with (λ, j), since

$$j = n - |\boldsymbol{\mu}|, \qquad \lambda = \boldsymbol{\mu} \bigcup \{j\}. \tag{A.19}$$

In particular $l(\lambda) = l(\boldsymbol{\mu}) + 1$, $n_j(\lambda) = n_j(\boldsymbol{\mu}) + 1$ and $z_\lambda = z_{\boldsymbol{\mu}}(n_j(\boldsymbol{\mu}) + 1)j$. We find

$$L^- r_n^{(x)} = n! \sum_{\boldsymbol{\mu}\vdash 0}^{n-1} \frac{(-x)^{l(\boldsymbol{\mu})+1}}{z_{\boldsymbol{\mu}}} p_{n-1-|\boldsymbol{\mu}|} p_{\boldsymbol{\mu}}. \tag{A.20}$$

We now split the sum into two parts. First the term $|\boldsymbol{\mu}| = n - 1$ is simply

$$n! \sum_{\boldsymbol{\mu}\vdash n-1} \frac{(-x)^{l(\boldsymbol{\mu})+1}}{z_{\boldsymbol{\mu}}} p_0 p_{\boldsymbol{\mu}} = -xnp_0 r_{n-1}^{(x)}. \tag{A.21}$$

Now the reminder is

$$n! \sum_{\boldsymbol{\mu}\vdash 0}^{n-2} \frac{(-x)^{l(\boldsymbol{\mu})+1}}{z_{\boldsymbol{\mu}}} p_{n-1-|\boldsymbol{\mu}|} p_{\boldsymbol{\mu}} = n(n-1) r_{n-1}^{(x)}, \tag{A.22}$$

as can be seen from the determinant formula (A.5) by repeatedly developing along the last column until the matrix is 1×1. These last 'determinants' correspond to the factor $p_{n-1-|\boldsymbol{\mu}|}$ in the sum above. Finally we get

$$L^- r_n^{(x)} = n\left(n - 1 - xp_0\right) r_{n-1}^{(x)}, \tag{A.23}$$

from which equation (52) in the main text follows.

Appendix B. An alternate set of generators

In this appendix, we briefly introduce an alternate set of generators p_i', that could be used in the proof in section 5. This set of generators of $\Lambda_{q=mN}$ is constructed to satisfy $p_i' = p_i$ for $i \leqslant N$, and $C_m\left(p_i'\right) = 0$ for $i > N$, and have total degree i. In the construction, to keep track of the number of variables of the power sums, we introduce an additional index N, so

$$p_{i,N}(x_1,...,x_N) = x_1^i + \cdots + x_N^i. \tag{B.1}$$

Recall that $\mathcal{A}_N = \{p_{1,N}, \ldots, p_{N,N}\}$ are algebraically independent and generates all symmetric polynomials in N variables. In particular for each $i \in \mathbb{N}^*$ there exists a unique polynomial $T_{i,m,N}$ in N variables, such that

$$m\, p_{i,N} = T_{i,m,N}\left(m\, p_{1,N},...,m\, p_{N,N}\right). \tag{B.2}$$

The $p_{i,q}'$ are now defined as follows

$$p_{i,q}' = \begin{cases} p_{i,q}, & 1 \leqslant i \leqslant N, \\ p_{i,q} - T_{i,m,N}\left(p_{1,q},...,p_{N,q}\right), & N < i \leqslant q. \end{cases} \tag{B.3}$$

By construction, they obey $C_m\left(p_{i,q}'\right) = 0$ for $i \geqslant N+1$, are non-zero, and have total degree i. In addition, they form an alternate generating set $\mathcal{A}_q' = \{p_{1,q}', \ldots, p_{q,q}'\}$ of Λ_q, because the $p_{i,q}$ of the generating set $\mathcal{A}_q$ can be expressed in terms of the $p_{i,q}'$ in equation (B.3).

Since, as is shown in corollary 1, $\tilde{p}_{N+1}$ is the unique (up to a scale factor) polynomial that vanishes under the clustering C_m, we find that $p_{N+1}' \propto \tilde{p}_{N+1}$, and that $L^- p_{N+1}' = 0$. In addition, by a direct calculation, one finds that $L^- p_{i,q}' = \mathrm{i} p_{i-1,q}'$ for $1 \leqslant i < N$ and $N+1 < i \leqslant q$.

Namely, by acting on both sides of the definition of $T_{i,m,N}$, equation (B.2) with L^- gives

$$\begin{aligned} L^- T_{i,m,N}\left(mp_{1,N}, \ldots, mp_{N,N}\right) &= qT_{i,m,N;1}\left(mp_{1,N}, \ldots, mp_{N,N}\right) \\ &\quad + \sum_{j=2}^{N} m\, j\, p_{j-1,N}\, T_{i,m,N;j}\left(mp_{1,N}, \ldots, mp_{N,N}\right) \\ &= \mathrm{i}mp_{i-1,N} = \mathrm{i}T_{i-1,m,N}\left(mp_{1,N}, \ldots, mp_{N,N}\right), \end{aligned}$$

where $T_{i,m,N;j}$ denotes the derivative of $T_{i,m,N}$ with respect to its jth argument. In particular by setting $X_j = mp_{j,N}$, we find that for general arguments,

$$\mathrm{i}T_{i-1,m,N}(X_1, \ldots, X_N) = qT_{i,m,N;1}(X_1, \ldots, X_N) + \sum_{j=2}^{N} jX_{j-1}\, T_{i,m,N;j}(X_1, \ldots, X_N). \tag{B.4}$$

We can now act with L^- on both sides of the definition of $p_{i,q}'$, equation (B.3). Using the relation equation (B.4), we find that, for $i > N+1$,

$$\begin{aligned} L^- p_{i,q}' &= L^- p_{i,q} - L^- T_i\left(p_{1,q}, \ldots, p_{N,q}\right) \\ &= \mathrm{i}p_{i-1,q} - \mathrm{i}T_{i-1}\left(p_{1,q}, \ldots, p_{N,q}\right) \\ &= \mathrm{i}p_{i-1,q}', \end{aligned} \tag{B.5}$$

which is what we wanted to show. We note that in the case that $i = N + 1$, the only thing that changes in the argument above is that the lhs of equation (B.4) is replaced by iX_N, leading to the result $L^{-}p'_{N+1,q} = 0$, which we showed in the main text using a different method.

Finally, we mention that it is also possible to prove that the degree of $p'_{i,q}$ equals i for $N + 1 \leqslant i \leqslant 2N + 1$, directly from the definition. We believe that the degree of $p'_{i,q}$ also equals i for $2N + 1 < i \leqslant q$, but did not find a proof of this.

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