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Classification of metaplectic modular categories [☆]

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A B S T R A C T

We obtain a classification of metaplectic modular categories: every metaplectic modular category is a gauging of the particle–hole symmetry of a cyclic modular category. Our classification suggests a conjecture that every weakly-integral modular category can be obtained by gauging a symmetry (including the fermion parity) of a pointed (super-)modular category.

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1. Introduction

Achieving a classification of modular categories analogous to the classification of finite abelian groups is an interesting mathematical problem [4,5]. In this note, we classify metaplectic modular categories. Our classification suggests a close connection between finite abelian groups and weakly-integral modular categories via gauging, thus leads to a potential approach to proving the Property F conjecture for weakly-integral modular categories [2,6,14].

A simple object X is weakly-integral if its squared quantum dimension d_X^2 is an integer. A modular category is weakly-integral if every simple object is weakly-integral. Inspired by the applications to physics and topological quantum computation, we focus on weakly-integral modular categories [7,8]. An important class of weakly-integral modular categories is the class of metaplectic modular categories—unitary modular categories with the fusion rules of $SO(N)_2$ for some odd integer $N > 1$ [11,12]. The metaplectic modular categories first appeared in the study of parafermion zero modes, which generalize the Majorana zero modes. The name *metaplectic* comes from the fact that the resulting braid group representations from the generating simple objects in $SO(N)_2$ are the metaplectic representations, which are the symplectic analogues of the spinor representations. Our main result is a classification of metaplectic modular categories: every metaplectic modular category is a gauging of the particle–hole symmetry of a cyclic modular category.

The property F conjecture says that all braid group representations afforded by a weakly-integral simple object have finite images. For $SO(N)_2$, the property F conjecture follows from [15]. It is possible that all weakly-integral modular categories can be obtained by gauging symmetries of pointed modular categories including fermion parities of pointed super-modular categories [3]—categories with all simple objects having their quantum dimension equal to 1. Our classification supports this possibility. If this is true, then a potential approach to the property F conjecture for all weakly-integral modular categories would be to prove that gauging preserves property F.

2. Cyclic modular categories

Definition 2.1. Let $\mathbb{Z}_n$ be the cyclic group of n elements. A $\mathbb{Z}_n$-cyclic modular category is a modular category whose fusion rule is the same as the cyclic group $\mathbb{Z}_n$ for some integer n.

A $\mathbb{Z}_n$-cyclic modular category is determined by a non-degenerate quadratic form $q : \mathbb{Z}_n \to \mathbb{U}(1)$ (see [13] and [10, Appendix D]). We will denote the $\mathbb{Z}_n$-cyclic modular category determined by such a quadratic form q as $\mathcal{C}(\mathbb{Z}_n, k)$ for $q(j) = e^{2\pi i s_j}$, $s_j = \frac{kj^2}{n}$, $0 \leq j \leq n-1$, $(k, n) = 1$. We will mostly be interested in the case n odd, for which there is always a symmetric bicharacter β such that $q(j) = \beta(j, j)$, from which the braiding on $\mathcal{C}(\mathbb{Z}_n, k)$ is obtained.

First for M, N relatively prime, $\mathcal{C}(\mathbb{Z}_{MN}, k)$ is a direct product of $\mathcal{C}(\mathbb{Z}_M, kN)$ and $\mathcal{C}(\mathbb{Z}_N, kM)$. The simple object types, j, of $\mathcal{C}(\mathbb{Z}_{MN}, k)$ can be labeled by pairs (a, b), where $j = aM + bN$ and $0 \leq a \leq N-1$, $0 \leq b \leq M-1$. The fusion rules are

$$j_1 \times j_2 = (a_1, b_1) \times (a_2, b_2) = ([a_1 + a_2]_N, [b_1 + b_2]_M), \tag{2.1}$$

and the topological twists are $\theta_j = e^{2\pi i s_j}$:

$$s_j = \frac{kj^2}{MN} = \frac{k(aM + bN)^2}{MN} = \frac{kMa^2}{N} + \frac{kNb^2}{M} + 2abk. \tag{2.2}$$

Therefore, we have shown that $\mathcal{C}(\mathbb{Z}_{MN}, k) = \mathcal{C}(\mathbb{Z}_M, kN) \boxtimes \mathcal{C}(\mathbb{Z}_N, kM)$.

Next we find all distinct $\mathbb{Z}_{p^a}$-cyclic modular categories, where p is an odd prime.

For $\mathcal{C}(\mathbb{Z}_{p^a}, k)$, write $k = p^l m$, where $p \nmid m$. Note that if $l \geq 1$, the resulting category is not modular (since the form $q(x) = e^{2\pi i k x^2/p^a}$ is degenerate). Therefore, we must assume $(k, p) = 1$. The twist of the j-th simple object is $e^{\frac{2\pi i k}{p^a} j^2}$. If for n_1 and n_2, the categories are isomorphic, it means that one can solve the congruent equation

$$\frac{n_1}{p^a} \equiv \frac{n_2 j^2}{p^a} (\text{mod } 1), \tag{2.3}$$

for some j such that $p \nmid j$ (so that j is a generator of $\mathbb{Z}_{p^a}$). We need to solve $j^2 \equiv n_2^{-1} n_1 \,(\text{mod } p^a)$, which is solvable when $\left(\frac{n_1}{p^a}\right) = \left(\frac{n_2}{p^a}\right)$. Therefore, there are two distinct theories.

Braided tensor auto-equivalences of the $\mathbb{Z}_n$-cyclic-modular categories are group isomorphisms of $\mathbb{Z}_n$ which preserve the quadratic form q [10]. The **particle–hole symmetry** of a $\mathbb{Z}_n$-cyclic modular category with n odd is the categorical symmetry $\mathbb{Z}_2$ of $\mathcal{C}(\mathbb{Z}_n, k)$, where the non-trivial element of $\mathbb{Z}_2$ acts on $\mathcal{C}(\mathbb{Z}_n, k)$ via the braided tensor auto-equivalence that sends j to $n - j$.

3. Metaplectic modular categories

The unitary modular categories $SO(N)_2$ for odd $N > 1$ has 2 simple objects X_1, X_2 of dimension $\sqrt{N}$, two simple objects $\mathbf{1}, Z$ of dimension 1, and $\frac{N-1}{2}$ objects Y_i, $i = 1, \dots, \frac{N-1}{2}$ of dimension 2. The fusion rules are:

(1) $Z \otimes Y_i \cong Y_i$, $Z \otimes X_i \cong X_{i+1}$ (modulo 2), $Z^{\otimes 2} \cong \mathbf{1}$,
(2) $X_i^{\otimes 2} \cong \mathbf{1} \oplus \bigoplus_i Y_i$,
(3) $X_1 \otimes X_2 \cong Z \oplus \bigoplus_i Y_i$,
(4) $Y_i \otimes Y_j \cong Y_{\min\{i+j, N-i-j\}} \oplus Y_{|i-j|}$, for $i \neq j$ and $Y_i^{\otimes 2} = \mathbf{1} \oplus Z \oplus Y_{\min\{2i, N-2i\}}$.

The fusion rules for the subcategory generated by Y_1 (with simple objects $\mathbf{1}, Z$ and all Y_i) are precisely those of the dihedral group of order $2N$.

Definition 3.1. A metaplectic modular category is a unitary modular category $\mathcal{C}$ with the same fusion rules as $SO(N)_2$ for some odd $N > 1$.

From a modular category $\mathcal{C}$ admitting an action of a finite group G by braided auto-equivalences one may sometimes construct a new modular category called the *gauging* of the symmetry (see [6]). This is a two step process: first one extends $\mathcal{C}$ to a G-crossed braided fusion category $\mathcal{C}_G^\times$ (a G-graded fusion category having $\mathcal{C}$ as its identity component), then one takes the G-equivariantization to obtain a new modular category of dimension $|G|^2 \dim \mathcal{C}$.

Theorem 3.2.

(1) *Suppose $\mathcal{C}$ is a metaplectic modular category with fusion rules $SO(N)_2$, then $\mathcal{C}$ is a gauging of the particle–hole symmetry of a $\mathbb{Z}_N$-cyclic modular category.*
(2) *For $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ with distinct odd primes p_i, there are exactly 2^{s+1} many inequivalent metaplectic modular categories.*

To prove the theorem, we start with two lemmas.

Lemma 3.3. *The object Z is a boson: $\theta_Z = 1$.*

Proof. Let Y be any of the $\frac{N-1}{2}$ simple objects of dimension 2. By orthogonality of the rows of the S-matrix, we find that $S_{YZ} = 2$. Observing that $Y \otimes Z \cong Y$, we apply the balancing equation (see e.g. [1]):

$$S_{ij}\theta_i\theta_j = \sum_{k=0}^{r-1} N_{i^*j}^k d_k \theta_k$$

to obtain: $2\theta_Y\theta_Z = S_{YZ}\theta_Y\theta_Z = \theta_Y d_Y = 2\theta_Y$. It follows that $\theta_Z = 1$. $\square$

Since Z is a boson (i.e. $\dim(Z) = 1$ and $\theta_Z = 1$), we may condense ("de-equivariantize" in the categorical language) to obtain a $\mathbb{Z}_2$-graded category [10]. Since Z interchanges $\mathbf{1} \leftrightarrow Z$ and $X_1 \leftrightarrow X_2$ and fixes the Y_i the resulting condensed category $\mathcal{D} := \mathcal{C}^{\mathbb{Z}_2}$ has N objects of quantum dimension 1 in the identity sector $\mathcal{D}_0$ and one object of dimension $\sqrt{N}$ in the non-trivial sector $\mathcal{D}_1$ (see [2]). Clearly, the fusion rules of $\mathcal{D}_0$ must be identical to those of some abelian group A of order N. In the following, we show that $A \cong \mathbb{Z}_N$. As an aside, we point out that the category $\mathcal{D}$ is a Tambara–Yamagami category [16].

Lemma 3.4. *The fusion rules of $\mathcal{D}_0$ are the same as $\mathbb{Z}_N$.*

Proof. It is enough to find a tensor generator for $\mathcal{D}_0$, that is, a simple object U so that $\{U^{\otimes i} : i \geq 0\}$ contains all simple objects in $\mathcal{D}_0$. Now under condensation each object Y_i becomes the sum of two invertible simple objects in $\mathcal{D}_0$. The image of Y_i under condensation is $Y_i^1 \oplus Y_i^2$, a sum of invertible simple objects in $\mathcal{D}_0$. We denote by $\mathbf{1}_0$ the image of $\mathbf{1}$ and Z under condensation (i.e. the unit object in $\mathcal{D}_0$). We will proceed to show that Y_1^1 is a tensor generator for $\mathcal{D}_0$.

From $Y_1^{\otimes 2} \cong \mathbf{1} \oplus Z \oplus Y_2$ we obtain

$$(Y_1^1)^{\otimes 2} \oplus (Y_1^2)^{\otimes 2} \oplus 2Y_1^1 \otimes Y_1^2 = 2\mathbf{1}_0 \oplus Y_2^1 \oplus Y_2^2.$$

This implies $Y_1^{1*} = Y_1^2$, so that Y_1^2 appears as some tensor power of Y_1^1. Thus Y_1^1 is a tensor generator provided each $Y_i^{(j)}$ appears in some tensor power of $(Y_1^1 \oplus Y_1^2)$. Since every Y_i appears in some tensor power of Y_1 the result follows. □

Proof of Theorem 3.2. (1) By Lemmas 3.3, 3.4, each metaplectic modular category is obtained from gauging a $\mathbb{Z}_2$-symmetry of a cyclic modular category. But the particle–hole symmetry is the only non-trivial $\mathbb{Z}_2$-symmetry of a cyclic modular category. (2) As discussed above, there are exactly two cyclic modular categories for each prime power factor in N. When gauging the particle–hole symmetry, there is an additional choice parameterized by $H^3(\mathbb{Z}_2; U(1)) \cong \mathbb{Z}_2$ [9,2,6]. Therefore, the number of metaplectic modular categories is 2^{s+1}.

4. Witt classes and open problems

Gauging preserves Witt classes [6]. Therefore, the Witt classes of metaplectic modular categories are the same as those of the corresponding cyclic modular categories.

Proposition 4.1. *The modular category $\mathcal{C}(\mathbb{Z}_{p^{2a}}, q)$ is a quantum double $\mathcal{Z}(\mathrm{Vec}^{\omega}_{\mathbb{Z}_{p^a}})$.*

Proof. It is easy to see that regardless of the quadratic form q, the simple objects $[np^a]$ are all bosons, for $n = 0, 1, \ldots, p^a - 1$. They form a $\mathbb{Z}_{p^a}$ fusion category. In fact, one can define a Lagrangian subalgebra $\bigoplus_{n=0}^{p^a-1} [np^a]$ of $\mathcal{C}(\mathbb{Z}_{p^{2a}}, q)$. This shows that $\mathcal{C}(\mathbb{Z}_{p^{2a}}, q)$ is indeed a quantum double. Now let us condense this subalgebra, which identifies $[j]$ with $[j + np^a]$. Therefore, one can label the distinct simple objects after condensation by $[j]$, $j = 0, 1, \ldots, p^a - 1$. Hence $\mathcal{C}(\mathbb{Z}_{p^{2a}}, q)$ must be a quantum double of $\mathbb{Z}_{p^a}$, generally twisted by a class in H^3 [10]. □

One open question is to prove property F for all metaplectic modular categories. Another one is to construct universal computing models from metaplectic modular categories by supplementing braidings with measurements [8].

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