



Fusion products of Kirillov–Reshetikhin modules and fermionic multiplicity formulas

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Abstract

We give a complete description of the graded multiplicity space which appears in the Feigin–Loktev fusion product of graded Kirillov–Reshetikhin modules for all simple Lie algebras. This construction is used to obtain an upper bound formula for the fusion coefficients in these cases. The formula generalizes the case of $\mathfrak{g} = A_r$ of our previous paper, where the multiplicities are generalized Kostka polynomials. In the case of other Lie algebras, the formula is the fermionic side of the $X = M$ conjecture. In the cases where the Kirillov–Reshetikhin conjecture, regarding the decomposition formula for tensor products of KR-modules, has been proven in its original, restricted form, our result provides a proof of the conjectures of Feigin and Loktev regarding the fusion product multiplicities.

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1. Introduction

The Feigin–Loktev fusion product [FL99] is a construction which defines a \mathfrak{g} -equivariant grading on the tensor product of \mathfrak{g} -modules. Let V_1, \dots, V_N be finite-dimensional \mathfrak{g} -modules as well as modules over the current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$. Given a set of pairwise distinct

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complex numbers $\{\zeta_1, \dots, \zeta_N\}$, we denote by $V_i(\zeta_i)$ the $\mathfrak{g}[t]$ -module localized at ζ_i (see Section 2.2 for definition of a localized module).

The Feigin–Loktev fusion product of the localized modules $V_i(\zeta_i)$ is a \mathfrak{g} -module and a graded $\mathfrak{g}[t]$ quotient module. As a \mathfrak{g} -module, the fusion product is isomorphic to the usual tensor product of \mathfrak{g} -modules, in sufficiently well-behaved cases. As a graded $\mathfrak{g}[t]$ -module, the decomposition of its n th graded component into \mathfrak{g} -modules is given by the graded multiplicities $M_{\lambda, \{V_i\}}[n]$:

$$V_1 * \dots * V_N(\zeta_1, \dots, \zeta_N) \simeq \bigoplus_{n \geq 0} \bigoplus_{\lambda \in P^+} V_\lambda^{\oplus M_{\lambda, \{V_i\}}[n]},$$

where V_λ are the irreducible \mathfrak{g} -modules, and the symbol $*$ denotes the graded (fusion) product rather than the usual tensor product. The definition of this fusion product depends on the choice of cyclic vectors of V_i and is given in Section 3.1.

There is no a priori reason that the graded multiplicities should be independent of the parameters ζ_i . Nevertheless Feigin and Loktev conjectured [FL99] that they are independent of these localization parameters. They suggested that there is a relation between the generating functions

$$M_{\lambda, \{V_i\}}(q) := \sum_{n \geq 0} M_{\lambda, \{V_i\}}[n] q^n \tag{1.1}$$

and generalized Kostka polynomials [SW99,KS02] (in the cases where a comparison can be made).

Several papers contain results which imply special cases of the Feigin–Loktev conjecture, see e.g. [FJK⁺04,Ked04,CL,AKS06,FKL,FL05]. For example, in [AKS06], we introduced a method to compute the generating function (1.1) in the case of $\mathfrak{g} = A_r$, for an arbitrary sequence of irreducible A_r -modules with highest weights which are integer multiples of the fundamental weights. This provided a proof of the conjectures of Feigin and Loktev by providing explicit, localization parameter-independent formulas for the graded multiplicities. We found that they are indeed related to the generalized Kostka polynomials.

The purpose of this paper is to generalize the results of [AKS06] to other simple Lie algebras \mathfrak{g} . We succeed in applying the techniques of [AKS06] provided that the modules V_i are $\mathfrak{g}[t]$ -modules of the Kirillov–Reshetikhin type [Cha01]. We find that the proof of the Feigin–Loktev conjecture in these cases can be reduced to the proof of the Kirillov–Reshetikhin conjecture. In general, we obtain an upper bound formula for the fusion multiplicities.

Kirillov–Reshetikhin (KR) modules are the finite-dimensional \mathfrak{g} - and $\mathfrak{g}[t]$ -modules which can be deformed to Yangian modules. A KR-module localized at $\zeta \in \mathbb{C}$ is denoted by $\text{KR}_{a\omega_i}(\zeta)$, where $a \in \mathbb{Z}_+$ and ω_i is one of the fundamental weights of \mathfrak{g} . See Definition 2.1 for the definition of these modules in terms of generators and relations. The graded version of this module, $\overline{\text{KR}}_{a\omega_i}$, is independent of ζ . It has the same dimension as $\text{KR}_{a\omega_i}(\zeta)$ and the same \mathfrak{g} -module structure. The modules $\overline{\text{KR}}_{a\omega_i}$ are more commonly referred to as a Kirillov–Reshetikhin modules in the literature [Cha01,CM]. We refer to this version as a graded KR-module here.

The Kirillov–Reshetikhin conjecture concerns the decomposition of KR-modules, or the tensor products of several KR-modules, into irreducible \mathfrak{g} -modules. Let us define the graded multi-

plicities in the fusion product:

$$\text{KR}_{\mu_1}(\zeta_1) * \cdots * \text{KR}_{\mu_N}(\zeta_N) \simeq \bigoplus_{n \geq 0} \bigoplus_{\lambda \in P^+} V_\lambda^{\oplus M_{\lambda, \{\mu_p\}}[n]}.$$

The generating function (1.1) for the graded fusion multiplicities is denoted by $M_{\lambda, \{\mu_p\}}(q)$.

The KR-conjecture [KR90,HKO⁺99] gives the total (ungraded) multiplicities in the form of a restricted sum over binomial coefficients:

$$\dim \left(\text{Hom}_{\mathfrak{g}} \left(V_\lambda, \bigotimes_i \text{KR}_{\mu_i}(\zeta_i) \right) \right) = M_{\lambda, \{\mu_p\}}(1) = \sum_{\{m_a^{(i)} \geq 0: P_a^{(i)} \geq 0\}} \prod_{a,i} \binom{P_a^{(i)} + m_a^{(i)}}{m_a^{(i)}} \quad (1.2)$$

(see Conjecture 2.3 for the definition of the symbols on the right-hand side).

This conjecture implies the completeness of the Bethe ansatz states in the generalized inhomogeneous XXX spin chain. The problem of proving it was addressed in the original work on the subject [KR90], and subsequently was investigated in various forms by [Kle97,Cha01,HKO⁺99, Her,Nak03,FL05] among others. To date, Eq. (1.2) has only been proven only in certain special cases (see Section 2.5 for a complete list).

Our approach in this paper is to describe explicitly the space dual to the multiplicity space. It can be expressed as a space of rational functions with a natural grading. We then find explicit formulas for the upper bound of the graded dimension of this space. The total dimension is, by definition, equal to the multiplicity in the tensor product decomposition. Since our formulas coincide with the right-hand side of (1.2) in the limit $q = 1$, the proof of (1.2) would imply that our upper bound is in fact realized, and is equal the fusion coefficients.

Since the multiplicities depend only on the highest weights $\mu_p = a_p \omega_{i_p}$ and the Cartan matrix C of \mathfrak{g} , and are independent of the localization parameters ζ_p , we thus have a proof of the Feigin–Loktev conjecture in the cases where (1.2) has been proven. That is, the proof of the Feigin–Loktev conjecture can be reduced to the proof of the (restricted form of the) Kirillov–Reshetikhin conjecture. We will also remark below on the way in which a direct proof of the FL-conjecture can be used to prove the KR-conjecture.

The techniques used in this paper are a generalization of the techniques introduced in our previous paper [AKS06] for the case of $\mathfrak{g} = A_r$. In that case, the relations for the KR-modules are precisely the highest-weight conditions for the irreducible representations of A_r with a highest weight which is a positive integral multiple of one of the fundamental weights. Therefore, we were able to compute precisely the fusion Kostka polynomials [FJK⁺04], and show that they are equal to the generalized Kostka polynomials [KSS02]. For other Lie algebras, KR-modules are precisely those which are amenable to the same type of analysis we used in [AKS06]. This is due to the form of the relations which define them, which involve only generators corresponding to simple roots.

In [FJK⁺04], we presented a definition of “fusion Kostka polynomials.” The fusion Kostka polynomials are the graded multiplicities of the fusion of irreducible, finite-dimensional \mathfrak{g} -modules. In [FJK⁺04] we conjectured that these were related to the generalized Kostka polynomials of [KSS02,HKO⁺99]. This is true in the case that $\mathfrak{g} = A_r$, and was proven in [AKS06]. In general there is no known fermionic formula for these coefficients, and we expect that they are different from the fermionic formulas of [HKO⁺99] except in the special cases of fusion products of KR-modules which are also irreducible as \mathfrak{g} -modules, also known as minuscule modules.

The paper is organized as follows. The notation and graded KR-modules localized at ζ are introduced in Section 2. In Section 3 we remind the reader of the notion of the graded tensor product of cyclic $\mathfrak{g}[t]$ -modules, that is, the fusion product of [FL99]. In Section 4, we describe the dual space to the current algebra $U(\mathfrak{n}_-[t, t^{-1}])$ where $\mathfrak{n}_- \subset \mathfrak{g}$ is the nilpotent subalgebra. This is similar to the ideas introduced in [SF94] for simple Lie algebras, and the proofs are a generalization of our constructions in [AKS06] in the case of A_r . The main difference is the Serre relation, which governs the structure of singularities in the dual space. In Section 5, we describe the subspace of matrix elements dual to the multiplicity space. We compute a fermionic formula for the associated graded space, which is an upper bound on these multiplicities. We compare the result at $q = 1$ to the KR-conjecture, giving an equality of coefficients in all cases where Eq. (1.2) has been proven.

2. Notation

2.1. Finite-dimensional \mathfrak{g} -modules

Let \mathfrak{g} be a simple Lie algebra of rank r , $\Pi = \{\alpha_1, \dots, \alpha_r\}$ the set of simple roots and Δ the set of all roots. Let $I_r = \{1, \dots, r\}$ and define the Cartan matrix C with entries $C_{i,j} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$, $i, j \in I_r$. Let $\{\omega_1, \dots, \omega_r\}$ denote the fundamental weights and P^+ the set of dominant integral weights.

The Cartan decomposition of \mathfrak{g} is given by $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, the generators of \mathfrak{n}_- corresponding to simple roots are denoted by f_i , $i \in I_r$, where $f_i \in \mathfrak{g}_{-\alpha_i}$. Similarly, the generators of \mathfrak{n}_+ corresponding to simple roots are denoted by e_i and the generators of \mathfrak{h} by $h_i = [e_i, f_i]$.

Irreducible, finite-dimensional highest weight \mathfrak{g} -modules are parameterized by $\lambda \in P^+$ and denoted by V_λ . The module V_λ is generated by the action of $U(\mathfrak{n}_-)$ on a highest weight vector v_λ , which satisfies the properties

$$e_i v_\lambda = 0, \quad h v_\lambda = \lambda(h) v_\lambda \quad (h_i \in \mathfrak{h}), \quad f_i^{l_i+1} v_\lambda = 0, \tag{2.1}$$

where $\lambda(h_i) = l_i$, and the l_i are determined by $\lambda = \sum_{i=1}^r l_i \omega_i$. We refer to the last condition in (2.1) as the integrability condition with respect to \mathfrak{g} .

2.2. Current algebra modules

Let t be a formal variable, and define the positive current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$. We denote its generators by $x[n] := x \otimes t^n$, with $x \in \mathfrak{g}$ and $n \in \mathbb{Z}_{\geq 0}$. The subalgebra spanned by $x[0]$ with $x \in \mathfrak{g}$ is obviously isomorphic to \mathfrak{g} , and hence we write $\mathfrak{g} \subset \mathfrak{g}[t]$. Therefore, any $\mathfrak{g}[t]$ -module is also a \mathfrak{g} -module.

Consider a complex number $\zeta \in \mathbb{C}P$ and the local variable $t_\zeta = t - \zeta$ ($\zeta \neq \infty$), and $t_\infty = t^{-1}$. There is a corresponding current algebra $\mathfrak{g}[t_\zeta]$, and its generators are denoted by

$$x[n]_\zeta := x \otimes (t_\zeta)^n = x \otimes (t - \zeta)^n.$$

Obviously, $x[n] = x[n]_0$. If $\zeta \neq \infty$ then $\mathfrak{g}[t] = \mathfrak{g}[t_\zeta]$ as vector spaces.

Let $\zeta \in \mathbb{C}$. Given any $\mathfrak{g}[t_\zeta]$ -module $V(\zeta)$, since $\mathfrak{g}[t]$ is isomorphic to $\mathfrak{g}[t_\zeta]$, there is a natural action of $\mathfrak{g}[t]$ on $V(\zeta)$ which is given by expanding about $t_\zeta = 0$. For any $v \in V(\zeta)$,

$$x[n]_0 v = x \otimes t^n v = x \otimes (t_\zeta + \zeta)^n v = \sum_{j \leq n} \binom{n}{j} \zeta^j x[n-j]_\zeta v. \tag{2.2}$$

We use the notation $V(\zeta)$ for both the $\mathfrak{g}[t_\zeta]$ -module and the $\mathfrak{g}[t]$ -module with the action described above. It is called the $\mathfrak{g}[t]$ -module localized at ζ . Usually, a $\mathfrak{g}[t]$ -module localized at 0 is denoted simply by V instead of $V(0)$.

2.3. Grading on cyclic $\mathfrak{g}[t]$ -modules

Let \mathcal{A} be any algebra, then an \mathcal{A} -module is said to be *cyclic* with cyclic vector v if it is generated by the action of $U(\mathcal{A})$ on v . If $V(\zeta)$ has a cyclic vector v with respect to $\mathfrak{g}[t]$ (hence $\mathfrak{g}[t_\zeta]$) then we can choose to endow it with a graded structure with respect to the local variable $t = t_0$, as follows.

The universal enveloping algebra $U(\mathfrak{g}[t])$ is graded by homogeneous degree in t . Although the $\mathfrak{g}[t]$ -action on $V(\zeta)$ is not graded if $\zeta \neq 0$, it is filtered. The action of $U(\mathfrak{g}[t])$ on $V(\zeta)$ inherits a filtration from $U(\mathfrak{g}[t])$: Let $U^{(\leq n)}$ denote the subspace of $U(\mathfrak{g}[t])$ with homogeneous degree in t less than or equal to n . Define

$$\mathcal{F}^{(n)} = U^{(\leq n)} v.$$

Then $\mathcal{F}^{(n)} \subset \mathcal{F}^{(n+1)}$, $\mathbb{C}v = \mathcal{F}^{(0)}$ and $V(\zeta) = \bigcup_{n \geq 0} \mathcal{F}^{(n)}$. Thus, we have defined a filtration of $V(\zeta)$ which depends on the choice of cyclic vector.

The associated graded space is

$$\text{Gr } V(\zeta) = \bigoplus_{n \geq 0} \text{Gr}[n]$$

where

$$\text{Gr}[n] = \mathcal{F}^{(n)} / \mathcal{F}^{(n-1)}.$$

The structure of the graded space depends on the choice of cyclic vector v . We denote the graded $\mathfrak{g}[t]$ -module by $\bar{V}(\zeta)$. It has the structure of a quotient module of $\mathfrak{g}[t]$.

2.4. Kirillov–Reshetikhin modules

The term Kirillov–Reshetikhin (KR) modules originally referred to certain finite-dimensional modules of the Yangian [KR90]. The Yangian is a deformation of the current algebra $\mathfrak{g}[t]$, where \mathfrak{g} is any simple Lie algebra. Thus, one can define KR-modules for the current algebra as a limiting case of the Yangian modules.

In [Cha01] this idea is used to provide a definition of KR-modules for $\mathfrak{g}[t]$ in terms of current generators and relations. The result is a finite-dimensional $\mathfrak{g}[t]$ -module, corresponding to some highest weight of the form $a\omega_i$ where ω_i is one of the fundamental weights of \mathfrak{g} , and $a \in \mathbb{Z}_+$. As a \mathfrak{g} -module, the KR-module is in fact a highest-weight module with highest-weight $a\omega_i$. It is

cyclic with respect to the action of $\mathfrak{g}[t]$. However, KR-modules are not necessarily irreducible as \mathfrak{g} -modules.

We use a definition similar to that found in [Cha01], but we generalize it to the case where the module is localized at an arbitrary value of $\zeta \in \mathbb{C}$ to fit the discussion below. (The definition found in [Cha01] corresponds to the special case $\zeta = 1$.) We also make a distinction between KR-modules and their associated graded space [CM], which we call the graded KR-modules, corresponding to $\zeta = 0$ (this is the KR-module defined in [CM]). As \mathfrak{g} -modules, these modules are isomorphic, but they are different as $\mathfrak{g}[t]$ -modules.

Definition 2.1 (*Ungraded Kirillov–Reshetikhin modules*). Fix $(\zeta, a, i) \in \mathbb{C}^* \times \mathbb{Z}_+ \times I_r$. Consider the $\mathfrak{g}[t]$ -module $\text{KR}_{a\omega_i}(\zeta)$ generated by the action of $U(\mathfrak{g}[t])$ on the vector v with the properties

$$x[n]_\zeta v = 0 \quad \text{if } n \geq 0, x \in \mathfrak{n}_+; \tag{2.3}$$

$$f_j[n]_\zeta v = 0 \quad \text{if } n \geq \delta_{i,j}; \tag{2.4}$$

$$f_i[0]_\zeta^{a+1} v = 0; \tag{2.5}$$

$$h_j[n]_\zeta v = \delta_{n,0} \delta_{i,j} a v. \tag{2.6}$$

This implies, for example, that $f_i[n]_0$ acts on v as

$$f_i[n]_0 v = \zeta^n f_i[0]_\zeta v = \zeta^n f_i[0]_0 v. \tag{2.7}$$

Due to the highest weight conditions (2.3), (2.4) (which imply that $h_j[n]_\zeta v = 0$ if $n > 0$) and the PBW theorem, it is clear that $\text{KR}_{a\omega_i}(\zeta) = U(\mathfrak{n}_-[t])v$.

In the case where $\mathfrak{g} = A_r$, $\text{KR}_{a\omega_i}(\zeta) = V_{a\omega_i}(\zeta)$, which as an A_r -module is the irreducible highest-weight module with highest weight $a\omega_i$. In general, the $\mathfrak{g}[t]$ -evaluation module $V_{a\omega_i}(\zeta)$, corresponding to the irreducible highest weight \mathfrak{g} -module $V_{a\omega_i}$, is a quotient of the KR-module. The weight $a\omega_i$ is the highest \mathfrak{g} -weight in $\text{KR}_{a\omega_i}(\zeta)$.

Definition 2.2 (*Graded Kirillov–Reshetikhin modules*). The graded KR-module $\overline{\text{KR}}_{a\omega_i}(\zeta)$ is the associated graded space of the filtered space $\text{KR}_{a\omega_i}(\zeta)$ defined by the action of $U(\mathfrak{g}[t])$ on the cyclic vector v , as in Section 2.3.

2.5. The Kirillov–Reshetikhin conjecture

The Kirillov–Reshetikhin conjecture [KR90] is an explicit formula for the decomposition of the tensor product of KR-modules into irreducible \mathfrak{g} -modules. (In the literature, there are also other conjectures which are called the KR-conjecture. We will describe one of them below.)

Let \mathbf{R} denote a collection of dominant integral weights of the form $\{a_p \omega_{i_p} : 1 \leq p \leq N\}$, and let $\{\zeta_p : 1 \leq p \leq N\}$ be a collection of distinct nonzero complex numbers. Define $M_{\lambda, \mathbf{R}}$ to be the multiplicity of the irreducible \mathfrak{g} -module V_λ in the tensor product of N KR-modules corresponding to the weights in \mathbf{R} , localized at ζ_p :

$$M_{\lambda, \mathbf{R}} = \dim \left(\text{Hom}_{\mathfrak{g}} \left(\bigotimes_{p=1}^N \text{KR}_{a_p \omega_{i_p}}(\zeta_p), V_\lambda \right) \right). \tag{2.8}$$

Given \mathbf{R} , define $n_a^{(i)}$ to be the number of elements of \mathbf{R} equal to $a\omega_i$ and define $l^{(i)}$ via $\lambda = \sum_{i=1}^r l^{(i)}\omega_i$.

Conjecture 2.3 (The Kirillov–Reshetikhin conjecture).

$$M_{\lambda, \mathbf{R}} = \sum_{\{m_a^{(i)} \geq 0: P_a^{(i)} \geq 0\}} \prod_{a,i} \binom{P_a^{(i)} + m_a^{(i)}}{m_a^{(i)}} \tag{2.9}$$

where the sum is taken over $m_a^{(i)}$ such that

$$\sum_a am_a^{(i)} = \sum_{j,a} C_{i,j}^{-1} an_a^{(j)} - \sum_j C_{i,j}^{-1} l^{(j)},$$

and the “vacancy numbers” $P_a^{(i)}$ are

$$P_a^{(i)} = \sum_b \min(a, b)n_b^{(i)} + \sum_{b,j \neq i} \min(|C_{i,j}|b, |C_{j,i}|a)m_b^{(j)} - 2 \sum_b \min(a, b)m_b^{(i)},$$

with $C_{i,j}$ being the Cartan matrix.

Here,

$$\binom{n}{m} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}. \tag{2.10}$$

This conjecture first appeared in [KR90] for the classical Lie algebras. It has been proven in the following cases:

- When $\mathfrak{g} = A_r$ [KSS02], and the tensor product of any KR-modules.
- When $\mathfrak{g} = D_r$ and \mathbf{R} consists of fundamental weights only [Sch05].
- When \mathfrak{g} is any nonexceptional simple Lie algebra and $\mathbf{R} = \{(a_p\omega_1): a_p \in \mathbb{N}\}$ [OSS03,SS06].

See the review [Sch] for a full status report. In each case, the proof involves a bijection between crystal bases and rigged configurations [KKR86]. We call this version of the Kirillov–Reshetikhin conjecture KR1.

Remark 2.4. The following is also called the KR-conjecture [HKO⁺99,KNT02,Nak03,Her]. It is a modified form of (2.9) where the sum is over $P_a^{(i)} \in \mathbb{Z}$, that is, it is not restricted to be over positive “vacancy numbers.” This means that the sum may contain some negative summands.

We call this second form of the conjecture KR2. The unrestricted form of the conjecture can be proven for any Lie algebra \mathfrak{g} and any set of weights [Her], using an induction system known as the Q -system [Kir83,Nak03,KNT02]. On the basis of numerical evidence, [HKO⁺99] conjectured that the two forms KR1 and KR2 are equal in general. There is no direct proof of this fact and it is a highly nontrivial result.

2.6. Affine algebra modules

The localization procedure of the previous subsection can be extended to the affine algebra associated with \mathfrak{g} . For each $\zeta \in \mathbb{C}P$ there is an inclusion

$$\mathfrak{g}[t_\zeta] \subset \mathfrak{g}[t_\zeta^{-1}, t_\zeta] \subset \mathfrak{g}[t^{-1}, t] = \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t]$$

where $\mathbb{C}[t^{-1}, t]$ is the space of Laurent series in t .

The algebra $\tilde{\mathfrak{g}}_\zeta = \mathfrak{g}[t_\zeta, t_\zeta^{-1}]$ has a canonical central extension by the central c with the cocycle defined by

$$\langle x \otimes f(t), y \otimes g(t) \rangle = \langle x, y \rangle \oint_{t=\zeta} f'(t)g(t) dt, \quad x, y \in \mathfrak{g}, \quad f, g \in \mathbb{C}[t_\zeta, t_\zeta^{-1}]. \quad (2.11)$$

We denote the central extension of $\tilde{\mathfrak{g}}_\zeta$ as $\widehat{\mathfrak{g}}_\zeta$. The usual affine algebra is $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_0$. Note that this central extension can be lifted to the algebra $\mathfrak{g}[t^{-1}, t]$.

The Cartan decomposition of the affine algebra is $\mathfrak{g} = \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_-$ with $\widehat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \oplus (\mathfrak{g} \otimes t^{\pm 1} \mathbb{C}[t^{\pm 1}])$ and $\widehat{\mathfrak{h}} = \mathbb{C}c \oplus \mathfrak{h}$.

In this paper, we consider only $\widehat{\mathfrak{g}}$ -modules which are integrable. Such modules have the property of complete reducibility. Isomorphism classes of irreducible, integrable, highest weight modules of $\widehat{\mathfrak{g}}$ are indexed by a positive integer k and certain dominant integral weights λ of \mathfrak{g} . The integer k is called the *level* of the representation: it is the value of the central element c acting on the irreducible module. The \mathfrak{g} -weight λ is in the subset $P_k^+ \subset P^+$ of weights which have the property $\lambda(\theta) = \sum_{i=1}^r l_i a_i^\vee \leq k$, where θ is the highest root and the a_i^\vee are the co-marks.

Let V be a $\mathfrak{g}[t]$ -module and let \widehat{V} be a $\widehat{\mathfrak{g}}$ -module such that $\widehat{V} = U(\widehat{\mathfrak{n}}_-)V$. (The module V is a *top component* of \widehat{V} , although this definition is not unique.) In the case of irreducible integrable modules, the top component is usually taken to be the finite-dimensional \mathfrak{g} -module V_λ , on which $\mathfrak{g}[t]$ acts by the evaluation representation.

A $\widehat{\mathfrak{g}}$ -module can be localized at ζ in a similar way as a $\mathfrak{g}[t]$ -module, but one must be careful about the appropriate completion of the algebra. We consider modules of $\widehat{\mathfrak{g}}_\zeta$ which are also $\mathfrak{g}[t^{-1}, t]$ -modules (highest-weight modules have this property). Currents in $\mathfrak{g}[t^{-1}, t]$ act on $\widehat{\mathfrak{g}}_\zeta$ -modules by expansion about $t = \zeta$: The formula (2.2) can be used even when $n < 0$. When acting on any vector in a highest weight module, the infinite sums are truncated. The action of the centrally extended current algebra on the module has a cocycle as in (2.11). We call the corresponding module the affine algebra module localized at ζ , denoted by $\widehat{V}(\zeta)$. If $V(\zeta)$ is a $\mathfrak{g}[t]$ -module localized at ζ then $\widehat{V}(\zeta)$ is the affine algebra module induced from it by the action of $\mathfrak{g}[t^{-1}, t]$.

We refer the reader to the Appendix in [FKL⁺01] for a further discussion of tensor (fusion) products of localized affine algebra modules. In this paper we will use only $\mathfrak{g}[t]$ -modules localized at ζ and affine algebra modules localized at infinity.

We will frequently use generating functions for elements in $\mathfrak{g}[t^{-1}, t]$:

$$x(z) = \sum_{n \in \mathbb{Z}} x[n]_0 z^{-n-1}. \quad (2.12)$$

For $\zeta \in \mathbb{C}$ we have

$$x[n]_\zeta = \sum_j x[n-j]_0 (-\zeta)^j \binom{n}{j} = \frac{1}{2\pi i} \oint_{z=\zeta} (z-\zeta)^n x(z) dz,$$

where the contour is taken counter-clockwise around the point ζ . If $\zeta = \infty$, then

$$x[n]_\infty = \frac{1}{2\pi i} \oint_{z=\infty} z^{-n} x(z) dz$$

with a clockwise contour.

3. The fusion product of $\mathfrak{g}[t]$ -modules and matrix elements

The idea of the fusion product comes from the fusion product of integrable $\widehat{\mathfrak{g}}$ -modules in conformal field theory [BPZ84]. It was reformulated as a graded tensor product of finite-dimensional $\mathfrak{g}[t]$ -modules in [FL99]. Strictly speaking, the latter is the restriction of the [BPZ84] fusion product to the top component, or product of primary fields, in the case where the level is sufficiently large. If the last condition is not met, we have the level-restricted fusion product, which will be discussed in a future publication.

3.1. The Feigin–Loktev graded tensor product

Let us recall the definition of the [FL99]-fusion product, specialized to the current context. Let $\{V_i(\zeta_i)\}_{i=1}^N$ be finite-dimensional, cyclic $\mathfrak{g}[t]$ -modules localized at distinct points $\zeta_i \in \mathbb{C}$ and let v_i be the cyclic vectors of the respective modules. Then the tensor product $V_1(\zeta_1) \otimes \cdots \otimes V_N(\zeta_N)$ is generated by the action of $U(\mathfrak{g}[t])$ on the tensor product of cyclic vectors $v_1 \otimes \cdots \otimes v_N$, where the action is by the usual coproduct on the product of localized modules.

If v_i can be chosen to be highest weight vectors with respect to the \mathfrak{g} -action, then the tensor product is generated by the action of the subalgebra $U(\mathfrak{n}_-[t])$ on the tensor product of highest weight vectors.

Since the tensor product of modules localized at distinct points is a cyclic $\mathfrak{g}[t]$ -module, it has a filtration inherited from $U(\mathfrak{g}[t])$, in the same way as the modules described in Section 2.3. The associated graded space is the graded tensor product, or the fusion product, of [FL99]:

$$V_1 * \cdots * V_N(\zeta_1, \dots, \zeta_N) := \text{Gr}(U(\mathfrak{g}[t])v_1 \otimes \cdots \otimes v_N). \tag{3.1}$$

Again the filtration, and hence the grading, is \mathfrak{g} -equivariant, and therefore the graded components are \mathfrak{g} -modules.

Feigin and Loktev conjectured that in certain cases, the graded tensor product should be independent of the complex numbers ζ_i . Our purpose in this paper is to prove this conjecture, in the case where \mathfrak{g} is a simple Lie algebra and the modules V_i are KR-modules with cyclic vectors which are \mathfrak{g} -highest weight vectors.

Our method is to compute explicit formulas for the graded fusion multiplicities. Let $V[n]$ be the graded component in the graded tensor product:

$$V_1 * \cdots * V_N \simeq \bigoplus_{n \geq 0} V[n], \tag{3.2}$$

where $V[n]$ are finite-dimensional \mathfrak{g} -modules. Define

$$V[n] \simeq \bigoplus_{\lambda \in P^+} V_\lambda^{\oplus M_{\lambda, \{V_i\}}[n]} \tag{3.3}$$

as \mathfrak{g} -modules. The generating function for the multiplicities

$$M_{\lambda, \{V_i\}}(q) = \sum_{n \geq 0} M_{\lambda, \{V_i\}}[n] q^n \tag{3.4}$$

is called the graded multiplicity of V_λ in the fusion product. In [FJK⁺04], if $V_i(\zeta_i)$ are taken to be finite-dimensional irreducible \mathfrak{g} -modules, the graded multiplicity was called the fusion Kostka polynomial.

Kirillov–Reshetikhin modules fit the criteria above for the fusion product: they are finite-dimensional cyclic $\mathfrak{g}[t]$ -modules. Moreover their cyclic vector is a highest weight vector with respect to \mathfrak{g} . Choose \mathbf{R} to be a collection of dominant integral highest weights as before, and choose $\{\zeta_1, \dots, \zeta_N\}$ to be distinct nonzero complex numbers. Define $M_{\lambda, \mathbf{R}}(q)$ to be the graded multiplicity of the irreducible \mathfrak{g} -module V_λ in the fusion product of KR-modules:

$$M_{\lambda, \mathbf{R}}(q) = M_{\lambda, \{\text{KR}_{a_p \omega_i_p}\}}(q).$$

Our goal in this paper is to compute the polynomials $M_{\lambda, \mathbf{R}}(q)$.

3.2. Fusion products and matrix elements

It is useful to understand the relation of the graded tensor product (3.1) of finite-dimensional $\mathfrak{g}[t]$ -modules to the fusion product of $\widehat{\mathfrak{g}}$ -modules: The product (3.1) is the graded “top component,” or the space of conformal blocks, of the fusion product of $\widehat{\mathfrak{g}}$ -modules, at sufficiently large level k , together with a grading. This point of view is taken below in order to compute the character. See also the Appendix of [FKL⁺01] and the introduction in [FJK⁺04].

Let $V(\zeta)$ be a graded finite-dimensional cyclic $\mathfrak{g}[t]$ -module localized at ζ , and let $\widehat{V}(\zeta)$ be the $\widehat{\mathfrak{g}}$ -module induced from it at level k . Note that $V(\zeta)$ need not be an irreducible \mathfrak{g} -module. Given a collection of distinct complex numbers $\{\zeta_1, \dots, \zeta_N\}$ and a collection of finite-dimensional, cyclic $\mathfrak{g}[t]$ -modules $\{V_1(\zeta_1), \dots, V_N(\zeta_N)\}$, consider the set of induced modules $\{\widehat{V}_1(\zeta_1), \dots, \widehat{V}_N(\zeta_N)\}$. Their fusion product is denoted by (see Appendix of [FKL⁺01])

$$\widehat{V}_1(\zeta_1) \boxtimes \cdots \boxtimes \widehat{V}_N(\zeta_N). \tag{3.5}$$

This is a $\mathfrak{g} \otimes \mathcal{A}_\zeta$ -module, where \mathcal{A}_ζ is the space of rational functions in t with possible poles at ζ_p for each p . Given a vector $w = w_1 \otimes \cdots \otimes w_N$, an element $x \otimes g(t)$ ($x \in \mathfrak{g}, g(t) \in \mathcal{A}_\zeta$) acts

on the tensor product by the usual coproduct formula, but the action on the p th component is given by the expansion of $g(t)$ about ζ_p .

The algebra $\mathfrak{g} \otimes \mathcal{A}_\zeta \subset \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t]$ has a central extension, with the cocycle given by the sum of the residues at each point. The level of the fusion product is k if the level of each module $\widehat{V}_p(\zeta_p)$ is k . Its decomposition into irreducible level- k -modules of $\widehat{\mathfrak{g}}$ is given by the fusion coefficients or Verlinde numbers. If k is sufficiently large, these coefficients are equal to the multiplicities of the irreducible \mathfrak{g} -modules in the tensor product of the top components $V_p(\zeta)$:

$$\text{mult}_{\widehat{V}_\lambda(0)}(\widehat{V}_1(\zeta_1) \boxtimes \cdots \boxtimes \widehat{V}_N(\zeta_N)) = \text{mult}_{V_\lambda}(V_1(\zeta_1) \otimes \cdots \otimes V_N(\zeta_N)), \quad k \gg 0. \quad (3.6)$$

Remark 3.1. The only role which the level plays here is the following. It is known that any integrable $\widehat{\mathfrak{g}}$ -module is completely reducible, and we use this fact here to argue that the fusion product is completely reducible. Hence, it is isomorphic to a direct sum of irreducible $\widehat{\mathfrak{g}}$ -modules localized at $\zeta = 0$. In fact, we can take $k \rightarrow \infty$ in all of the calculations below.

Thus, the coefficient of the \mathfrak{g} -module V_λ in the decomposition is the dimension of the space $\text{Hom}_{\mathfrak{g}}(V_\lambda, V_1(\zeta_1) \otimes \cdots \otimes V_N(\zeta_N))$. Suppose V_p is generated by a highest weight vector v_p for each p . Then in terms of matrix elements, since

$$V_1(\zeta_1) \otimes \cdots \otimes V_N(\zeta_N) = U(\mathfrak{n}_-[t])v_1 \otimes \cdots \otimes v_N,$$

the coefficient of V_λ is given by the inner product with the lowest weight vector $w_{\lambda^*} \in V_{\lambda^*}$

$$\text{mult}_{V_\lambda}(V_1(\zeta_1) \otimes \cdots \otimes V_N(\zeta_N)) = \dim\{w_{\lambda^*}, U(\mathfrak{n}_-[t])v_1 \otimes \cdots \otimes v_N\}. \quad (3.7)$$

The graded multiplicity of V_λ in graded tensor product is equal to the associated graded space of this space of matrix elements, where the highest weight vectors are taken to have degree 0 and the grading is inherited from the degree in t .

We may consider w_{λ^*} to be a lowest $\widehat{\mathfrak{g}}$ -weight vector in the integrable $\widehat{\mathfrak{g}}$ -module dual to $\widehat{V}_\lambda(0)$, that is, the module localized at infinity $\widehat{V}_{\lambda^*}(\infty)$. This module is a graded module with a grading compatible with the degree in t . The graded fusion coefficients indicate which graded component in a certain finite-dimensional subspace of $\widehat{V}_{\lambda^*}(\infty)$ contain the \mathfrak{g} -module V_λ . One can interpret them as truncated string functions.

4. The dual space to the algebra

In order to explain the structure of the space of matrix elements of the form (3.7), we will use generating functions for elements in this space. This is a space of rational functions, with a certain structure and certain residue properties. In order to explain this structure we proceed in two steps. In this section, we describe the dual space to the universal enveloping algebra of $\mathfrak{n}_-[t^{-1}, t]$. In the next section we will then restrict this space to the subspace of matrix elements (3.7).

Let \mathfrak{g} be a simple Lie algebra with Cartan matrix C with entries given by $C_{\alpha,\beta} = 2(\alpha, \beta)/(\alpha, \alpha)$. The universal enveloping algebra $U = U(\widetilde{\mathfrak{n}}_-) = U(\mathfrak{n}_-[t, t^{-1}])$ is generated by the coefficients of products of generating currents $f_\alpha(x)$ ($\alpha \in \Pi$). These generating currents satisfy two types of operator product expansion (OPE) relations. The first comes from the commutation relation between currents:

$$f_\alpha(x_1)f_\beta(x_2) = \frac{f_{\alpha+\beta}(x_1)}{(x_1 - x_2)} + \text{regular terms} \quad \text{if } \alpha + \beta \in \Delta, \quad (4.1)$$

implying, in particular, that $[f_\alpha(x_1), f_\alpha(x_2)] = 0$. The second relation is a result of the Serre relation for \mathfrak{g} :

$$\text{ad}(f_\alpha)^{m_{\alpha,\beta}} f_\beta = 0, \quad \alpha, \beta \in \Pi,$$

where $m_{\alpha,\beta} = 1 - C_{\alpha,\beta}$. In terms of currents, this means that

$$\prod_{i=1}^{m_{\alpha,\beta}} (x_i^{(\alpha)} - x^{(\beta)}) f_\alpha(x_1^{(\alpha)}) \cdots f_\alpha(x_{m_{\alpha,\beta}}^{(\alpha)}) f_\beta(x^{(\beta)}) \Big|_{x_1^{(\alpha)} = \dots = x_{m_{\alpha,\beta}}^{(\alpha)} = x^{(\beta)}} = 0. \tag{4.2}$$

Since U is a graded space with respect to the Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$, define $U[\mathbf{m}]$ be the subspace consisting of homogeneous components of weight $-C\mathbf{m}$ (in the basis of fundamental weights). Here, $\mathbf{m} = (m^{(\alpha_1)}, \dots, m^{(\alpha_r)})$. That is, $U[\mathbf{m}]$ consists of sums of products of $m^{(\alpha)}$ of each of the currents $f_\alpha(x^{(\alpha)})$, where $\alpha \in \Pi$.

We want to compute the generating function for the d -graded components of $U[\mathbf{m}]$, where $d = t \frac{d}{dt}$ is the homogeneous degree in t . To do this, we introduce the dual space of the algebra, and then introduce a filtration on it. This will be the standard technique for computing characters in everything that follows, and was described in detail for the case of \mathfrak{sl}_{r+1} in [AKS06]. It follows the general idea described in [SF94] for simple \mathfrak{g} .

The following is a description of the dual space $\mathcal{U}[\mathbf{m}]$ to $U[\mathbf{m}]$. It is a space of functions in the set of variables $\{x_i^{(\alpha)} \mid \alpha \in \Pi, 1 \leq i \leq m^{(\alpha)}\}$ with the pairing $\mathcal{U}[\mathbf{m}] \times U[\mathbf{n}] \rightarrow \mathbb{C}$ defined as follows. The pairing is 0 if $\mathbf{m} \neq \mathbf{n}$, and otherwise it is defined inductively by the relations

$$\langle 1, 1 \rangle = 1, \\ \langle g, W f_\beta[n] \rangle = \left\langle \oint_{x_1^{(\beta)}=0} dx_1^{(\beta)} (x_1^{(\beta)})^n g(x_1^{(\beta)}, \dots), W \right\rangle, \quad W \in U[\mathbf{m} - \epsilon_\beta],$$

where ϵ_β is a standard basis vector. The contour integral is taken counter-clockwise around the origin in a contour excluding the singularities at $x_i^{(\alpha)}$ with $(\alpha, i) \neq (\beta, 1)$. Similarly,

$$\langle g, f_\beta[n] W \rangle = \left\langle \oint_{x_1^{(\beta)}=\infty} dx_1^{(\beta)} (x_1^{(\beta)})^n g(x_1^{(\beta)}, \dots), W \right\rangle, \quad W \in U[\mathbf{m} - \epsilon_\beta],$$

where this time the contour is taken clockwise around the point at infinity, excluding all the other points $x_i^{(\alpha)}$.

With this pairing, the OPE relation (4.1) shows that functions in the dual space $\mathcal{U}[\mathbf{m}]$ may have a simple pole whenever $x_i^{(\alpha)} = x_j^{(\beta)}$ if $C_{\alpha,\beta} < 0$. Thus, the dual space is a subspace of the space of rational functions of the form (we impose the usual ordering on the roots)

$$g(\mathbf{x}) = \frac{g_1(\mathbf{x})}{\prod_{\substack{\alpha < \beta \\ C_{\alpha,\beta} < 0}} \prod_{i,j} (x_i^{(\alpha)} - x_j^{(\beta)})}. \tag{4.3}$$

The function $g_1(\mathbf{x})$ is a Laurent polynomial in each of the variables, symmetric with respect to the exchange of variables of the same root type α , as a result of the fact that generators of the same root type commute.

Moreover, due to the Serre-type relation (4.2), the function $g_1(\mathbf{x})$ satisfies the following vanishing condition:

$$g_1(\mathbf{x})|_{x_1^{(\alpha)} = \dots = x_{m_{\alpha,\beta}}^{(\alpha)} = x_1^{(\beta)}} = 0. \tag{4.4}$$

Since these are the only relations among the generating currents corresponding to simple roots in \mathfrak{n}_- , this completes the description of the dual space $\mathcal{U}[\mathbf{m}]$.

At this point, we would like to explicitly point out the differences between the general case \mathfrak{g} discussed in this paper, and the case \mathfrak{sl}_{r+1} which we described in [AKS06]. The main difference is, obviously, the Cartan matrix describing the algebra. This difference has important consequences. On the level of the description it changes the pole structure of the functions (4.3), and the form of the vanishing conditions (4.4) for the nonsimply-laced algebras. More importantly, it changes the properties of the Kirillov–Reshetikhin modules, which are, in general, not irreducible, even though, for \mathfrak{sl}_{r+1} , they are. Nevertheless, the dual space can be described in one, unified way!

4.1. Filtration of the dual space

We introduce a filtration on the space of rational functions $\mathcal{U}[\mathbf{m}]$. The description is very similar to that explained in e.g. [AKS06] for the case of \mathfrak{sl}_{r+1} , except for the vanishing conditions resulting from the Serre relation.

Due to the proliferation of indices, we denote the simple roots by their corresponding index in I_r from this point on. Let $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ be a multipartition of $\mathbf{m} = (m^{(1)}, \dots, m^{(r)})$, that is, $\mu^{(\alpha)}$ is a partition of $m^{(\alpha)}$. Define

$$m_a^{(\alpha)} = \#\{\mu_i^{(\alpha)} = a\} \tag{4.5}$$

that is, the number of parts of length a in the partition $\mu^{(\alpha)}$.

Pick a tableau T of shape $\mu^{(\alpha)}$ on the letters $1, \dots, m^{(\alpha)}$. Let $a(j)$ be the length of the row in T in which j appears, and $i(j) - 1$ be the number of rows above the row in which j appears of the same length $a(j)$. Define the evaluation map

$$\begin{aligned} \varphi_{\boldsymbol{\mu}} : \mathcal{U}[\mathbf{m}] &\rightarrow \mathcal{H}[\boldsymbol{\mu}], \\ x_j^{(\alpha)} &\mapsto y_{a(j), i(j)}^{(\alpha)} \end{aligned} \tag{4.6}$$

extended by linearity, where $\mathcal{H}[\boldsymbol{\mu}]$ is a space of functions in the variables

$$\{y_{a,i}^{(\alpha)} \mid \alpha \in I_r; 1 \leq a \leq m^{(\alpha)}; 1 \leq i \leq m_a^{(\alpha)}\}. \tag{4.7}$$

There is a bijective correspondence between the pairs (a, i) for fixed α and the parts of the partition $\mu^{(\alpha)}$. We order the pairs (a, i) accordingly:

$$(a, i) < (b, j) \quad \text{if } a > b \text{ or if } a = b \text{ and } i < j. \tag{4.8}$$

The vanishing properties (4.4) and poles (4.3) of functions in $\mathcal{U}[\mathbf{m}]$ are independent of the ordering of the variables $x_i^{(\alpha)}$ with the same index α , and therefore the vanishing properties and poles of the functions in the image in $\mathcal{H}(\boldsymbol{\mu})$ under the evaluation map are independent of the particular tableau T chosen. Note that the evaluation maps preserve the homogeneous degree.

A lexicographical ordering on multipartitions $\boldsymbol{\mu} \vdash \mathbf{m}$ is defined in the usual way: partitions are ordered lexicographically, and multipartitions are similarly ordered, $\boldsymbol{\nu} > \boldsymbol{\mu}$ if there exists some $\beta \in I_r$ such that $\nu^{(\alpha)} = \mu^{(\alpha)}$ for all $\alpha < \beta$ and $\nu^{(\beta)} > \mu^{(\beta)}$. Let

$$\Gamma_{\boldsymbol{\mu}} = \bigcap_{\boldsymbol{\nu} > \boldsymbol{\mu}} \ker \varphi_{\boldsymbol{\nu}}, \quad \Gamma'_{\boldsymbol{\mu}} = \bigcap_{\boldsymbol{\nu} \geq \boldsymbol{\mu}} \ker \varphi_{\boldsymbol{\nu}} \subset \Gamma_{\boldsymbol{\mu}}. \tag{4.9}$$

Then $\Gamma_{\boldsymbol{\mu}} \subset \Gamma_{\boldsymbol{\nu}}$ if $\boldsymbol{\mu} < \boldsymbol{\nu}$, $\Gamma'_{\boldsymbol{\mu}} = \{0\}$ if $\boldsymbol{\mu} = ((1^{m^{(1)}}), \dots, (1^{m^{(r)}}))$, and $\Gamma_{\boldsymbol{\mu}} = \mathcal{U}[\mathbf{m}]$ if $\boldsymbol{\mu} = ((m^{(1)}), \dots, (m^{(r)}))$. Thus we have a filtration \mathcal{F} on the space $\mathcal{U}[\mathbf{m}]$ parametrized by all multipartitions of \mathbf{m} :

$$\{0\} \subset \Gamma_{\boldsymbol{\mu}_1} \subset \dots \subset \Gamma_{\boldsymbol{\mu}_t} = \mathcal{U}[\mathbf{m}],$$

where $\boldsymbol{\mu}_i < \boldsymbol{\mu}_{i+1}$ for all i . We can describe the properties of functions in the image of the associated graded space

$$\text{Gr } \mathcal{F} = \bigoplus_{\boldsymbol{\mu} \vdash \mathbf{m}} \Gamma_{\boldsymbol{\mu}} / \Gamma'_{\boldsymbol{\mu}}$$

under the induced evaluation maps.

Theorem 4.1. *The image $\tilde{\mathcal{H}}[\boldsymbol{\mu}]$ of functions in $\Gamma_{\boldsymbol{\mu}} / \Gamma'_{\boldsymbol{\mu}}$ under the evaluation map $\varphi_{\boldsymbol{\mu}}$, is the space of rational functions of the form*

$$h(\mathbf{y}) = \frac{\prod_{\alpha \in I_r} \prod_{(a,i) < (b,j)} (y_{a,i}^{(\alpha)} - y_{b,j}^{(\alpha)})^{2 \min(a,b)}}{\prod_{\substack{\alpha < \beta \\ C_{\alpha,\beta} < 0}} \prod_{i,j,a,b} (y_{a,i}^{(\alpha)} - y_{b,j}^{(\beta)})^{\min(|C_{\alpha,\beta}|b, |C_{\beta,\alpha}|a)}} h_1(\mathbf{y}), \tag{4.10}$$

where $h_1(\mathbf{y})$ is an arbitrary Laurent polynomial in the variables (4.7), symmetric with respect to the exchange of variables $y_{a,i}^{(\alpha)} \leftrightarrow y_{a,j}^{(\alpha)}$. The induced map

$$\bar{\varphi}_{\boldsymbol{\mu}} : \Gamma_{\boldsymbol{\mu}} / \Gamma'_{\boldsymbol{\mu}} \rightarrow \tilde{\mathcal{H}}[\boldsymbol{\mu}]$$

is an isomorphism of graded vector spaces.

We provide a sketch of the proof in Appendix A, since it is very similar to the proof for the given in [AKS06] in the case of $\widehat{\mathfrak{sl}}_{r+1}$ (in the context of integrable modules).

Thus, we have an isomorphism of graded vector spaces,

$$\text{Gr } \mathcal{F} \simeq \bigoplus_{\boldsymbol{\mu} \vdash \mathbf{m}} \tilde{\mathcal{H}}[\boldsymbol{\mu}].$$

We will use this fact below to compute the character of a subspace of $\mathcal{U}[\mathbf{m}]$.

5. The dual space to the fusion product of KR-modules

Our goal is to compute the decomposition of the graded tensor product of several KR-modules. In order to do this, we introduce the space of matrix elements, or conformal blocks corresponding to the fusion product of KR-modules.

5.1. The space of conformal blocks

Once more, let $\{\zeta_p \mid 1 \leq p \leq N\}$ be distinct, nonzero complex numbers, and let \mathbf{R} be a collection of labels (highest weights) for KR-modules

$$\mathbf{R} = \{(a_1\omega_{\alpha_1}), \dots, (a_N\omega_{\alpha_N})\}, \quad a_p \in \mathbb{N}, \alpha_p \in I_r.$$

For each p , let $\text{KR}_p := \text{KR}_{a_p\omega_{\alpha_p}}(\zeta_p)$ be the KR-module localized at the point ζ_p , with highest \mathfrak{g} -weight $a_p\omega_{\alpha_p}$. We denote by $n_a^{(\alpha)}$ the number of KR-modules with a highest weight equal to $a\omega_\alpha$. Define $n^{(\alpha)} = \sum_a an_a^{(\alpha)}$, and $\mathbf{n} = (n^{(1)}, \dots, n^{(r)})$.

The q -multiplicities in the decomposition into irreducible \mathfrak{g} -modules of the graded tensor product of N KR-modules are defined as above:

$$\text{KR}_1 * \dots * \text{KR}_N \simeq \bigoplus M_{\lambda, \mathbf{R}}(q) V_\lambda, \tag{5.1}$$

where the notation is understood to mean that a term $q^n V_\lambda$ denotes an irreducible \mathfrak{g} -module V_λ appearing in the graded component of degree n .

To find the multiplicity of V_λ in the tensor product, we can take the inner product with a distinguished vector w in the dual space V_λ^* which we know to have multiplicity 1. For our purposes here, it is sufficient to take the lowest weight vector $w_{\lambda^*} \in V_\lambda^*$ (see [AKS06]), which is the lowest weight vector of the module $V_{-\omega_0(\lambda)}$ where ω_0 is the longest element in the Weyl group. The vector w_{λ^*} has weight $-\lambda$.

Let us consider the space of matrix elements of the form

$$\langle w_{\lambda^*} f_{\alpha_1}[d_1] \dots f_{\alpha_r}[d_r] v_1 \otimes \dots \otimes v_N \rangle, \tag{5.2}$$

where $\alpha_j \in I_r$, and v_p is the highest weight vector of $\text{KR}_p(\zeta_p)$.

It is convenient to consider the space of generating functions for such matrix elements:

$$\mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}] = \{ \langle w_{\lambda^*} \mid f_{\alpha_1}(x_1^{(\alpha_1)}) \dots f_{\alpha_r}(x_{i_m(\alpha_r)}^{(\alpha_r)}) v_1 \otimes \dots \otimes v_N \rangle \}. \tag{5.3}$$

Here, the variables $x_i^{(\alpha)}$ are formal variables, and the set includes all possible orderings of the currents corresponding to simple roots. Coefficients of fixed powers of each of the variables $x_i^{(\alpha)}$ correspond to matrix elements of the form (5.2). The set of all linearly independent functions in $\mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$ corresponds to distinct matrix elements. We are therefore interested in the graded dimension of the space of such functions, which give rise to the polynomials $M_{\lambda, \mathbf{R}}(q)$, see Theorem 5.2.

In order to understand such matrix elements, one should consider KR_p to be the highest component of a $\widehat{\mathfrak{g}}_\zeta$ -module, similar to the “top component” of an irreducible $\widehat{\mathfrak{g}}_{\zeta_p}$ -module sitting at ζ_p . The induced $\widehat{\mathfrak{g}}_{\zeta_p}$ -module is a (graded) direct sum of several irreducible modules. The

graded tensor product of KR-modules is then the top component of the graded fusion product of several integrable $\widehat{\mathfrak{g}}$ -modules in the usual sense of fusion product of integrable modules.¹

In this sense, it is clear that one should view w_{λ^*} not just as the lowest weight vector of a \mathfrak{g} -module V_{λ^*} , but as a vector in the representation $V_{k,\lambda^*}(\infty)$, the irreducible $\widehat{\mathfrak{g}}_{\infty}$ -module, which is annihilated by the right action of $\mathfrak{n}_-[t]_{\infty}$. This module can also be considered to be a lowest weight right $\widehat{\mathfrak{g}}_0$ -module with lowest weight vector w_{λ^*} . That means that

$$w_{\lambda^*} f_{\alpha}[n]_0 = w_{\lambda^*} f_{\alpha}[-n]_{\infty} = 0 \quad \text{if } n \leq 0. \tag{5.4}$$

Recall that for a module localized at infinity, the local variable is $t_{\infty} = t^{-1}$.

The space of matrix elements of the form (5.3) is a space of functions, and we analyze the structure of this space below.

5.2. The dual space of functions

For convenience, we note the formula for the pairing of a function $g(\mathbf{x})$ in the dual space $\mathcal{U}[\mathbf{m}]$ with a generator localized at $\zeta \neq 0$:

$$\langle g(\mathbf{x}), W f_{\beta}[n]_{\zeta} \rangle = \left\langle \oint_{x_1^{(\beta)}=\zeta} dx_1^{(\beta)} (x_1^{(\beta)} - \zeta)^n g(\mathbf{x}), W \right\rangle, \quad W \in U[\mathbf{m} - \epsilon_{\beta}], \tag{5.5}$$

and

$$\langle g(\mathbf{x}), f_{\beta}[n]_{\infty} W \rangle = \left\langle \oint_{x_1^{(\beta)}=\infty} dx_1^{(\beta)} (x_1^{(\beta)})^{-n} g(\mathbf{x}), W \right\rangle, \quad W \in U[\mathbf{m} - \epsilon_{\beta}]. \tag{5.6}$$

The second integral excludes all poles but the one at infinity. These formulas result from the definitions $x[n]_{\zeta} = x \otimes (t - \zeta)^n$ and $x[n]_{\infty} = x \otimes t^{-n}$.

If $g(\mathbf{x}) \in \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$ then it should be in the orthogonal complement to all of the relations in the algebra, the left ideal of the algebra acting on the tensor product to the left and the right ideal of the algebra acting on the module at infinity to the right. These relations imply the following.

- (1) **Zero weight condition.** Clearly, the correlation function (5.3) will vanish unless the total weight with respect to $\mathfrak{h} \subset \mathfrak{g}$ is 0. Denoting $\lambda = \sum_{\alpha} l^{(\alpha)} \omega_{\alpha}$ and $\mathbf{l} = (l^{(1)}, \dots, l^{(r)})$, this means that $\mathbf{m} = (m^{(1)}, \dots, m^{(r)})$ is determined by the equation

$$\mathbf{m} = C^{-1} \cdot (\mathbf{n} - \mathbf{l}). \tag{5.7}$$

Thus, $\mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}] \subset \mathcal{U}[\mathbf{m}]$ with \mathbf{m} fixed by (5.7).

¹ In this paper, we do not consider level restriction, so we always assume that k is sufficiently large. In practice, this means $k \geq \sum_{\alpha} n^{(\alpha)} a_{\alpha}^{\vee}$ where a_{α}^{\vee} are the comarks. We will consider the more general, “level-restricted” case in a later publication.

- (2) **Relations in the algebra.** Since the space $\mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$ is a subspace of $\mathcal{U}[\mathbf{m}]$, any function $g(\mathbf{x}) \in \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$ is of the form

$$g(\mathbf{x}) = \frac{g_1(\mathbf{x})}{\prod_{\substack{\alpha < \beta \\ C_{\alpha, \beta} < 0}} \prod_{i,j} (x_i^{(\alpha)} - x_j^{(\beta)})}, \tag{5.8}$$

where $g_1(\mathbf{x})$ is a Laurent polynomial which satisfies the Serre-type relations

$$g_1(\mathbf{x})|_{x_1^{(\alpha)} = \dots = x_{m_{\alpha, \beta}}^{(\alpha)} = x_1^{(\beta)}} = 0, \tag{5.9}$$

and which is symmetric with respect to exchange of variables of the same type.

- (3) **Lowest weight condition.** The vector w_{λ^*} is a lowest weight vector with respect to \mathfrak{g} of the module sitting at infinity. The condition at infinity is that $w_{\lambda^*} f_{\alpha}[n]_{\infty} = 0$ if $n \geq 0$, for any α . This means that

$$\oint_{x_1^{(\alpha)} = \infty} dx_1^{(\alpha)} (x_1^{(\alpha)})^{-n} g(\mathbf{x}) = 0, \quad n \geq 0.$$

Changing variables to $x^{-1} = x_1^{(\alpha)}$, we have

$$\oint_{x=0} dx x^{-2+n} g(x^{-1}, \dots) = 0, \quad n \geq 0,$$

thus, if $\deg_x(g(x, \dots)) = m$, then $m \leq -2$ if g is the orthogonal complement of the relations. Therefore we have the degree restriction for $g(\mathbf{x}) \in \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$:

$$\deg_{x_i^{\alpha}} g(\mathbf{x}) \leq -2. \tag{5.10}$$

- (4) **Highest weight conditions.** Each vector v_p is a highest weight vector with weight $\mu_p = m\omega_{\alpha_p}$ sitting at ζ_p . Thus, $f_{\alpha}[n]_{\zeta_p}$ with $n \geq \delta_{\alpha, \alpha_p}$ generate a left ideal which acts trivially on the product $v_1 \otimes \dots \otimes v_N$. This means that for functions $g(\mathbf{z}) \in \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$,

$$0 = \oint_{x_1^{(\alpha)} = \zeta_p} dx_1^{(\alpha)} (x_1^{(\alpha)} - \zeta_p)^n g(\mathbf{x}) \quad \text{if } n \geq \delta_{\alpha, \alpha_p}. \tag{5.11}$$

Here, the contours are taken so that all other poles are excluded. This shows that the function $g(\mathbf{x})$ may have no poles at $x_1^{(\alpha)} = \zeta_p$ if $\alpha \neq \alpha_p$. It may have a simple pole if $\alpha = \alpha_p$. In particular, $g(\mathbf{x})$ has no pole at $x_1^{(\alpha)} = 0$ since there is no module localized at 0.

- (5) **Integrability condition.** If $\alpha \neq \alpha_p$ then the function $g(\mathbf{x})$ may have a pole at $x_i^{(\alpha)} = \zeta_p$. However the condition that

$$(f_{\alpha_p}[0]_{\zeta_p})^{a_p+1} v_p = 0 \tag{5.12}$$

implies that

$$\oint_{x_1^{(\alpha)}=\zeta_p} dx_1^{(\alpha)} \cdots \oint_{x_{a+1}^{(\alpha)}=\zeta_p} dx_{a+1}^{(\alpha)} g(\mathbf{x}) = 0. \tag{5.13}$$

This shows that if we define $g(\mathbf{x}) \in \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$ by (5.8) with

$$g_1(\mathbf{x}) = \frac{g_2(\mathbf{x})}{\prod_{\alpha} \prod_p \prod_{i=1}^{m^{(\alpha)}} (x_i^{(\alpha)} - \zeta_p)^{\delta_{\alpha p, \alpha}}}, \tag{5.14}$$

then the function $g_2(\mathbf{x})$ is a polynomial which vanishes on the following diagonal:

$$g_2(\mathbf{x})|_{x_1^{(\alpha)}=\dots=x_{a_p+1}^{(\alpha)}=\zeta_p} = 0 \tag{5.15}$$

for p such that $\alpha = \alpha_p$.

The space $\mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$ described above has a filtration by homogeneous degree in $x_i^{(\alpha)}$, where we take the degree of the factors $(x_i^{(\alpha)} - \zeta_p)$ in the denominator to be -1 . This is equivalent to setting $\zeta_p = 0$ for all p .

For any graded space V , let $\text{ch}_q V$ denote the generating function of dimensions of the graded components. That is, $\text{ch}_q V = \sum_m q^m \dim V[m]$, where $V[m]$ in the m th graded component of the graded vector space V .

The polynomial $\text{ch}_q \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$ is related to the polynomial $M_{\lambda, \mathbf{R}}(q)$ in Eq. (5.1) as follows. In the definition of the currents $f_{\alpha}(x_i^{(\alpha)})$, the coefficient of the generator $f_{\alpha}[n]$ is $(x_i^{(\alpha)})^{-n-1}$. Therefore,

$$M_{\lambda, \mathbf{R}}(q) = q^{-|\mathbf{m}|} \text{ch}_{q^{-1}} \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]. \tag{5.16}$$

Here, $|\mathbf{m}| = \sum_{\alpha} m^{(\alpha)}$ is fixed by Eq. (5.7).

5.3. Filtration of the space of matrix elements

Consider the filtration \mathcal{F} of the space $\mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}]$ defined using the evaluation mapping, which is inherited from the filtration of $\mathcal{U}[\mathbf{m}]$ introduced in Section 4.1. That is,

$$\tilde{\Gamma}_{\mu} = \Gamma_{\mu} \cap \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}], \quad \tilde{\Gamma}'_{\mu} = \Gamma'_{\mu} \cap \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}].$$

Functions in the image of the associated graded space of \mathcal{F} under the evaluation maps φ_{μ} can be described explicitly, in a manner similar to Theorem 4.1.

Theorem 5.1. *The image of the space $\tilde{\Gamma}_{\mu}$ under the evaluation map φ_{μ} satisfies*

$$\varphi_{\mu}(\tilde{\Gamma}_{\mu}) \subset \mathcal{H}_{\mathcal{C}}[\mu] \subset \mathcal{H}[\mu],$$

where $\mathcal{H}_e[\boldsymbol{\mu}]$ is the space of rational functions $h(\mathbf{y})$ of the form (4.10), where $h_1(\mathbf{y})$ is a function of the form

$$h_1(\mathbf{y}) = \prod_p \prod_{a,i} (y_{a,i}^{(\alpha_p)} - \zeta_p)^{-\min(a_p,a)} \bar{h}(\mathbf{y}), \tag{5.17}$$

where $\bar{h}(\mathbf{y})$ is a polynomial in the variables \mathbf{y} , symmetric under the exchange $y_{a,i}^{(\alpha)} \leftrightarrow y_{a,j}^{(\alpha)}$, such that the total degree of the function $h(\mathbf{y})$ in the variable $y_{a,i}^{(\alpha)}$ is less than or equal to $-2a$.

Again, the proof of this theorem follows the same arguments as the proof in the case of A_r . In fact, the reason one can describe the space dual to the fusion product of KR-modules is that the highest-weight conditions on KR-modules are identical to the highest-weight conditions on A_r -modules with rectangular highest weights (see Eqs. (5.8) and (5.9) in [AKS06]). We note that, as in the case of A_r , we do not know how to directly prove the surjectivity of the map $\varphi_\mu : \tilde{\Gamma}_\mu / \tilde{\Gamma}'_\mu \rightarrow \mathcal{H}_e[\boldsymbol{\mu}]$. Hence, we can only make a statement about the inclusion of spaces.

5.4. Character of the fusion product

The space of functions $\mathcal{H}_e[\boldsymbol{\mu}]$ is a filtered space by homogeneous degree in $y_{a,i}^{(\alpha)}$. The associated graded space is obtained simply by setting $\zeta_p = 0$ for all p in Eq. (5.17). That is, it is isomorphic to the space of functions of the form

$$h(\mathbf{y}) = h_0(\mathbf{y})h_1(\mathbf{y}), \tag{5.18}$$

with

$$\deg_{y_{a,i}^{(\alpha)}} h(\mathbf{y}) \leq -2a,$$

where

$$h_0(\mathbf{y}) = \frac{\prod_\alpha \prod_{(a,i) < (b,j)} (y_{a,i}^{(\alpha)} - y_{b,j}^{(\alpha)})^{2\min(a,b)}}{\prod_{p,a,i} (y_{a,i}^{(\alpha_p)})^{\min(a,\alpha_p)} \prod_{\substack{\alpha < \beta \\ C_{\alpha,\beta} < 0}} \prod_{a,i,b,j} (y_{a,i}^{(\alpha)} - y_{b,j}^{(\beta)})^{\min(|C_{\alpha,\beta}|b, |C_{\beta,\alpha}|a)}}}. \tag{5.19}$$

Define

$$A_{a,b}^{\alpha,\beta} = \delta_{\alpha,\beta} \min(a, b)$$

and

$$B_{a,b}^{\alpha,\beta} = \min(|C_{\alpha,\beta}|b, |C_{\beta,\alpha}|a) \delta_{\alpha \neq \beta}.$$

We have

$$\deg_{y_{a,i}^{(\alpha)}} h_0(\mathbf{y}) = -2a - P_a^{(\alpha)},$$

where

$$P_a^{(\alpha)} = [(B - 2A)\vec{m} + A\vec{n}]_a^{(\alpha)}, \tag{5.20}$$

where $[\vec{m}]_a^{(\alpha)} = m_a^{(\alpha)}$, so that $[A\vec{m}]_a^{(\alpha)} = \sum_{\beta,b} A_{a,b}^{\alpha,\beta} m_b^{(\beta)}$, etc. Recall that $m_a^{(\alpha)}$ is the number of parts of $\mu^{(\alpha)}$ equal to a , whereas $n_a^{(\alpha)}$ is the number of elements in \mathbf{R} of the form (a, α) . The numbers $P_a^{(\alpha)}$ are called vacancy numbers in the literature.

From this, we see that the function $h(\mathbf{y})$ in Eq. (5.18) is a polynomial, symmetric in each set of variables $\{y_{a,i}^{(\alpha)} : i = 1, \dots, a\}$, subject to the degree restriction

$$\deg_{y_{a,i}^{(\alpha)}} h(\mathbf{y}) \leq P_a^{(\alpha)}.$$

Moreover the homogeneous degree of $h_0(\mathbf{y})$ is $Q(\vec{m}, \vec{n}) - \|\vec{m}\|$, where

$$Q(\vec{m}, \vec{n}) = \vec{m}^t A \vec{m} - \frac{1}{2} \vec{m}^t B \vec{m} - \vec{m}^t A \vec{n}$$

and $\|\vec{m}\| = |\mathbf{m}| = \sum_{a,\alpha} am_a^{(\alpha)}$. Thus, we have shown that we have the following upper bound on the character of the fusion product:

Theorem 5.2.

$$M_{\lambda, \mathbf{R}}(q^{-1}) = q^{|\mathbf{m}|} \text{ch}_q \mathcal{C}_{\lambda, \mathbf{R}}[\mathbf{m}] \leq \sum_{\substack{|\vec{m}|=\mathbf{m} \\ C\mathbf{m}=(\mathbf{n}-1)}} q^{Q(\vec{m}, \vec{n})} \begin{bmatrix} \vec{m} + \vec{P} \\ \vec{m} \end{bmatrix}_q, \tag{5.21}$$

where the q -binomial coefficient of vectors is the product of q -binomial coefficients over the entries of the vector,

$$\begin{bmatrix} \vec{n} \\ \vec{m} \end{bmatrix}_q := \prod_{a,\alpha} \begin{bmatrix} n_a^{(\alpha)} \\ m_a^{(\alpha)} \end{bmatrix}_q.$$

The notation $|\vec{m}| = \mathbf{m}$ means that $\sum_a am_a^{(\alpha)} = m^{(\alpha)}$ for all α .

By an upper bound on a polynomial we mean that each of the coefficients is bounded from above by the expression on the right.

Since we have chosen to have no representation localized at 0, the function $h(\mathbf{y})$ has no poles at 0. Thus, the q -binomial coefficient which describes the character of the space is the usual q -binomial coefficient,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q)_n}{(q)_m (q)_{n-m}}, & n \geq m, \\ 0, & \text{otherwise,} \end{cases}$$

where $(q)_n = \prod_{i=1}^n (1 - q^i)$.

The right-hand side of (5.21) is the M -side (that is, the fermionic sum side) of the $X = M$ conjecture in its original form [HKO⁺99]. The fusion coefficients are therefore bounded by $M(\lambda, W, q^{-1})$ in general, where W is the collection of KR-modules labeled by \mathbf{R} .

When $q = 1$, the right-hand side is the fermionic sum of the Kirillov–Reshetikhin conjecture KR1. The conjecture is that the multiplicity of the irreducible module V_λ in the tensor product of KR-modules is equal to the right-hand side, when $q = 1$. This conjecture has been proven to hold for several special cases [KSS02, Sch05, OSS03, SS06]. Thus, we have the equality at least in these cases:

Theorem 5.3. *The equality holds in Eq. (5.21) for the cases of $\mathfrak{g} = A_r$ and \mathbf{R} consisting of arbitrary KR-modules; for the case of $\mathfrak{g} = D_r$ and \mathbf{R} consisting of fundamental weights; and for nonexceptional \mathfrak{g} if the weights in \mathbf{R} are multiples of the weight ω_1 .*

6. Conclusion

By studying the space dual to the fusion product of arbitrary graded Kirillov–Reshetikhin modules of $\mathfrak{g}[t]$ for any simple Lie algebra \mathfrak{g} , we were able to give an upper bound for the fusion coefficients. This upper bound is an equality in the cases where the Kirillov–Reshetikhin conjecture has been proven in the form KR1. We expect that the upper bound provided in this paper is in fact realized in general. One way to prove this claim is to show that the evaluation map φ_μ is surjective. This would also provide an alternative method of proof of the Kirillov–Reshetikhin conjecture.

It is clear from the construction we used that the fusion product of Kirillov–Reshetikhin modules does not depend on the complex numbers ζ_i (the ‘location’ of the modules). Thus, in the cases the Kirillov–Reshetikhin conjecture is proven in the form KR1 (see the list in Section 2.5), we have proven the conjecture by Feigin and Loktev, which states that the fusion product of KR-modules is independent of the ζ_i .

The results obtained in this paper can be used to provide explicit character formulas for arbitrary highest weight modules of arbitrary (nontwisted) affine Lie algebras. In a previous paper [AKS06], we showed how this can be done in the case of general $\widehat{\mathfrak{sl}}_{r+1}$ modules. It is possible to generalize the argument of [AKS06] to arbitrary $\widehat{\mathfrak{g}}$. We will provide the details in a forthcoming publication.

In this paper, we did not consider level restriction, that is, we considered $k \gg 0$. However, the level restricted fusion products of KR-modules are very interesting. We will study these using methods similar to [AKS06] in a future publication.

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Appendix A. Proof of Theorem 4.1

First, we must prove that the evaluation map $\varphi_\mu : \Gamma_\mu \rightarrow \widetilde{\mathcal{H}}[\mu]$, where the space $\widetilde{\mathcal{H}}[\mu]$ is described in Theorem 4.1, is well-defined.

The numerator in Eq. (4.10) describes the vanishing properties of functions in $\varphi_\mu(\Gamma_\mu)$.

Lemma A.1. *Let $g(\mathbf{x}) \in \Gamma_\mu$. Then the function $\varphi_\mu(g(\mathbf{x}))$ has a zero of order at least $2 \min(a, b)$ whenever $y_{a,i}^{(\alpha)} = y_{b,j}$.*

Proof. The proof of this lemma is identical to the proof of Lemma (3.7) of [AKS06]. Consider the two sets of variables, corresponding to two different parts of $\mu^{(\alpha)}$, denoted by R_a and R_b , of a lengths a and b , respectively. Denote the variables corresponding to R_a by $\{x_i \mid 1 \leq i \leq a\}$ and those corresponding to R_b by $\{x'_j \mid 1 \leq j \leq b\}$. Without loss of generality, assume $a \geq b$.

Let $\varphi_\mu = \varphi_2 \circ \varphi_1$, where φ_1 is the evaluation of all the variables except the set corresponding to R_b . Under φ_1 , the variables x_i are all evaluated at y_a . The map φ_2 is the evaluation of the variables x'_j of R_b at y_b .

The image $\bar{g} = \varphi_1(g(\mathbf{x}))$ is vanishes whenever $y_a = x'_j$ for all j , because the evaluation $x'_j = y_a$ corresponds to an evaluation map $\varphi_{\mu'}$ for some $\mu' > \mu$. Functions in Γ_μ are in the kernel of all such evaluation maps, by definition. Thus,

$$\bar{g} = \prod_{j=1}^b (y_a - x'_j) \bar{g}_1.$$

Moreover, since $g(\mathbf{x})$ is a symmetric under the exchange of variables corresponding to the same root, it is easy to see that \bar{g}_1 also vanishes whenever $y_a = x'_j$ for each j . It follows that $\varphi_\mu(g(\mathbf{x})) = \varphi_2(\bar{g})$ has a zero of order at least $2 \min(a, b)$ whenever $y_a = y_b$. The factor $\min(a, b)$ appears because we assumed that $a \geq b$. \square

The denominator in Eq. (4.10) describes the pole structure of functions in the image of the graded component.

Lemma A.2. *The image under the evaluation map φ_μ of any function in $\mathcal{U}[\mathbf{m}]$ has a pole of order at most $\min(|C_{\alpha,\beta}|b, |C_{\beta,\alpha}|a)$ whenever $y_{a,i}^{(\alpha)} = y_{b,j}^{(\beta)}$.*

Proof. The order of the pole is limited by the Serre-type relation (4.4). Let α and β be two connected roots in the Dynkin diagram with $|\alpha| \geq |\beta|$. Then $C_{\beta,\alpha} = -t$ and $C_{\alpha,\beta} = -1$, where $t \in \{1, 2, 3\}$. The Serre relation determines the maximal order of the pole of products of the form

$$f_\alpha(z_1) \cdots f_\alpha(z_a) f_\beta(w_1) \cdots f_\beta(w_b). \tag{A.1}$$

Note that the order of the pole is independent of the ordering of the currents in the product.

We assume that α is the longer root, hence the Serre relation implies that

$$(z_1 - w)(z_2 - w) f_\alpha(z_1) f_\alpha(z_2) f_\beta(w) |_{z_1=z_2=w} = 0$$

and

$$(z - w_1) \cdots (z - w_{t+1}) f_\alpha(z) f_\beta(w_1) \cdots f_\beta(w_{t+1}) |_{z=w_1=\dots=w_{t+1}} = 0.$$

Let $R^{(\alpha)}$ denote a row in $\mu^{(\alpha)}$ of length a , and $R^{(\beta)}$ a row in $\mu^{(\beta)}$ of length b . We visualize a pole as a line connecting box i in $R^{(\alpha)}$ with box j in $R^{(\beta)}$ (corresponding to the factor $z_i - w_j$

in the denominator). The allowed configurations of lines connecting boxes are: each box in $R^{(\beta)}$ has only one line connected to it, and each box in $R^{(\alpha)}$ may be connected to up to t boxes in $R^{(\beta)}$.

With these allowed configurations, it is easy to see that the maximal number of lines connecting $R^{(\alpha)}$ with $R^{(\beta)}$ is $\min(b, ta)$. To see this, first, connect each box in $R^{(\alpha)}$ to one box in $R^{(\beta)}$. This gives $\min(b, a)$ lines. If $t > 1$ and $b > a$ then connect each box in $R^{(\alpha)}$ to an unconnected box in $R^{(\beta)}$. This gives $\min(b, 2a)$ lines. Finally, if $b > 2a$ and $t = 3$, then connect each box in $R^{(\alpha)}$ to an unconnected box in $R^{(\beta)}$. This gives $\min(b, 3a)$ connecting lines.

After evaluating the variables in row $R^{(\alpha)}$ to $y_{a,i}^{(\alpha)}$ and the variables in $R^{(\beta)}$ to $y_{b,j}^{(\beta)}$, the maximal pole when these two variables are equal to each other is of order $\min(|C_{\alpha,\beta}|b, |C_{\beta,\alpha}|a)$. \square

Lemma A.3. *The map $\varphi_\mu : \Gamma_\mu \rightarrow \tilde{\mathcal{H}}[\mu]$ is well defined.*

Proof. The vanishing properties and pole structure in Eq. (4.10) follow from Lemmas A.1 and A.2. Furthermore, since functions in $\mathcal{U}[\mathbf{m}]$ are symmetric with respect to exchange of variables with the same root index α , it follows that their evaluation under the map φ_μ is symmetric under the exchange of variables corresponding to the same row of $\mu^{(\alpha)}$. \square

This shows that the functions described in Theorem 4.1 are in the image $\varphi_\mu(\Gamma_\mu)$. To show that the map $\varphi_\mu : \Gamma_\mu \rightarrow \tilde{\mathcal{H}}[\mu]$ is surjective, we give a function in the pre-image of each function in $\tilde{\mathcal{H}}[\mu]$. The proof of surjectivity is almost identical to the proof of Lemma 3.16 in [AKS06], with the appropriate adjustment made in the pre-image function to account for the different Serre relations for each \mathfrak{g} .

Denote the variables in the preimage of $y_{a,i}^{(\alpha)}$ as follows:

$$\varphi_\mu^{-1}(y_{a,i}^{(\alpha)}) = \{x_{a,i}^{(\alpha)}[1], \dots, x_{a,i}^{(\alpha)}[a]\} \subset \{x_1^{(\alpha)}, \dots, x_m^{(\alpha)}\}.$$

Define

$$g_0(\mathbf{x}) = \frac{\prod_\alpha \prod_{(a,i) < (a',i')} \prod_{j=1}^{a'} (x_{a,i}^{(\alpha)}[j] - x_{a',i'}^{(\alpha)}[j])(x_{a,i}^{(\alpha)}[j+1] - x_{a',i'}^{(\alpha)}[j])}{\prod_{\substack{\alpha < \beta \\ C_{\alpha,\beta} < 0}} \prod_{a,i,a',i'} \prod_{j=1}^{\min(|C_{\alpha,\beta}|b, |C_{\beta,\alpha}|a)} (x_{a,i}^{(\alpha)}[j] - x_{a',i'}^{(\beta)}[j])}, \quad (\text{A.2})$$

where in both the numerator and denominator, we take the argument j to be a cyclic label: That is,

$$x_{a,i}^{(\alpha)}[j+a] := x_{a,i}^{(\alpha)}[j].$$

Recall that according to the ordering (4.8), if $(a, i) < (a', i')$, then $a \geq a'$.

Theorem A.4. *The function $g(\mathbf{x}) = \text{Sym}(g_0(\mathbf{x})g_1(\mathbf{x}))$, where $g_1(\mathbf{x})$ is an arbitrary Laurent polynomial in the variables \mathbf{x} and the symmetrization is over each set of variables with the same label α , is in Γ_μ . Moreover, given a symmetric Laurent polynomial $h_1(\mathbf{y})$, there exists a Laurent polynomial $g_1(\mathbf{x})$ such that*

$$\varphi_\mu(g(\mathbf{x})) = \xi h(\mathbf{y}) \quad (\text{A.3})$$

where $\xi \in \mathbb{R}$ and $h(\mathbf{y})$ is the function of Eq. (4.10).

Proof. See Lemmas 3.11, 3.13 and 3.15 of [AKS06]. The only difference is in the vanishing condition due to the Serre relations. \square

By definition, the induced map $\bar{\varphi}_\mu : \Gamma_\mu / \Gamma'_\mu$ is injective, since $\ker(\varphi_\mu) \cap \Gamma_\mu = \Gamma'_\mu$. Thus, there is an isomorphism:

Lemma A.5. *The induced map*

$$\bar{\varphi}_\mu : \text{Gr}_\mu \Gamma \rightarrow \tilde{\mathcal{H}}[\mu] \quad (\text{A.4})$$

is an isomorphism of graded vector spaces.

This concludes the proof of Theorem 4.1.

References

- [AKS06] Eddy Ardonne, Rinat Kedem, Michael Stone, Fermionic characters and arbitrary highest-weight integrable affine sl_{r+1} modules, *Comm. Math. Phys.* 264 (2) (2006) 427–464.
- [BPZ84] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nuclear Phys. B* 241 (2) (1984) 333–380.
- [Cha01] Vyjayanthi Chari, On the fermionic formula and the Kirillov–Reshetikhin conjecture, *Int. Math. Res. Not.* 2001 (12) (2001) 629–654.
- [CL] Vyjayanthi Chari, Sergei Loktev, Weyl, fusion and Demazure modules for the current algebra of sl_{r+1} , *math.QA/0502165*.
- [CM] Vyjayanthi Chari, Adriano Moura, The restricted Kirillov–Reshetikhin modules for the current and twisted current algebras, *Comm. Math. Phys.* 266 (2006) 431–454.
- [FJK⁺04] B. Feigin, M. Jimbo, R. Kedem, S. Loktev, T. Miwa, Spaces of coinvariants and fusion product. I. From equivalence theorem to Kostka polynomials, *Duke Math. J.* 125 (3) (2004) 549–588.
- [FKL⁺01] B. Feigin, R. Kedem, S. Loktev, T. Miwa, E. Mukhin, Combinatorics of the \widehat{sl}_2 spaces of coinvariants, *Transform. Groups* 6 (1) (2001) 25–52.
- [FKL] B. Feigin, A.N. Kirillov, S. Loktev, Combinatorics and geometry of higher level Weyl modules, *math.QA/0503315*.
- [FL99] B. Feigin, S. Loktev, On generalized Kostka polynomials and the quantum Verlinde rule, in: *Differential Topology, Infinite-Dimensional Lie Algebras, and Applications*, in: *Amer. Math. Soc. Transl. Ser. 2*, vol. 194, Amer. Math. Soc., Providence, RI, 1999, pp. 61–79.
- [FL05] G. Fourier, P. Littelmann, Weyl modules, affine Demazure modules, fusion products and limit constructions, preprint, *math.RT/0509276*.
- [Her] David Hernandez, The Kirillov–Reshetikhin conjecture and solutions of t -systems, *J. reine angew. Math.* 596 (2006) 63–87.
- [HKO⁺99] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Remarks on fermionic formula, in: *Recent Developments in Quantum Affine Algebras and Related Topics*, Raleigh, NC, 1998, in: *Contemp. Math.*, vol. 248, Amer. Math. Soc., Providence, RI, 1999, pp. 243–291.
- [Ked04] Rinat Kedem, Fusion products, cohomology of GL_N flag manifolds, and Kostka polynomials, *Int. Math. Res. Not.* 2004 (25) (2004) 1273–1298.
- [KKR86] S.V. Kerov, A.N. Kirillov, N.Yu. Reshetikhin, Combinatorics, the Bethe ansatz and representations of the symmetric group, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 155 (Differentsialnaya Geometriya, Gruppy Li i Mekh. VIII) (1986) 50–64, 193.
- [Kir83] A.N. Kirillov, Combinatorial identities and completeness of states of the Heisenberg magnet, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 131 (1983) 88–105, Questions in quantum field theory and statistical physics, 4.
- [KS02] Anatol N. Kirillov, Mark Shimozono, A generalization of the Kostka–Foulkes polynomials, *J. Algebraic Combin.* 15 (2002) 27–69.

- [KR90] A.N. Kirillov, N.Yu. Reshetikhin, Representations of Yangians and multiplicities of the occurrence of the irreducible components of the tensor product of representations of simple Lie algebras, *J. Soviet Math.* 52 (3) (1990) 3156–3164.
- [KSS02] Anatol N. Kirillov, Anne Schilling, Mark Shimozono, A bijection between Littlewood–Richardson tableaux and rigged configurations, *Selecta Math. (N.S.)* 8 (1) (2002) 67–135.
- [Kle97] Michael Kleber, Combinatorial structure of finite-dimensional representations of Yangians: The simply-laced case, *Int. Math. Res. Not.* 1997 (4) (1997) 187–201.
- [KNT02] A. Kuniba, T. Nakanishi, Z. Tsuboi, The canonical solutions of the Q -systems and the Kirillov–Reshetikhin conjecture, *Comm. Math. Phys.* 227 (1) (2002) 155–190.
- [Nak03] Hiraku Nakajima, t -Analogues of q -characters of Kirillov–Reshetikhin modules of quantum affine algebras, *Represent. Theory* 7 (2003) 259–274 (electronic).
- [OSS03] Masato Okado, Anne Schilling, Mark Shimozono, A crystal to rigged configuration bijection for nonexceptional affine algebras, in: *Algebraic Combinatorics and Quantum Groups*, World Scientific, River Edge, NJ, 2003, pp. 85–124.
- [Sch] Anne Schilling, $X = M$ theorem: Fermionic formulas and rigged configurations under review, [math.QA/0512161](https://arxiv.org/abs/math.QA/0512161).
- [Sch05] Anne Schilling, A bijection between type $D_n^{(1)}$ crystals and rigged configurations, *J. Algebra* 285 (1) (2005) 292–334.
- [SF94] A.V. Stoyanovskii, B.L. Feĭgin, Functional models of the representations of current algebras, and semi-infinite Schubert cells, *Funktional. Anal. i Prilozhen.* 28 (1) (1994) 68–90, 96.
- [SS06] Anne Schilling, Mark Shimozono, $X = M$ for symmetric powers, *J. Algebra* 295 (2) (2006) 562–610.
- [SW99] Anne Schilling, Ole Warnaar, Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} superpolynomials, *Comm. Math. Phys.* 202 (1999) 359–401.