Non-abelian statistics in quantum Hall systems

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Abstract. It has long been known that, under the laws of quantum mechanics, particles in two spatial dimensions may exhibit *non-abelian braid statistics*. In this note we briefly discuss particular states of matter, so-called paired and clustered quantum Hall states, which offer a concrete realization of this scenario: quasi-holes over these quantum Hall states satisfy non-abelian braid statistics. The associated degeneracies have their origin in combinatorial properties of (para-)fermions in conformal field theory. We also discuss the exclusion statistics of edge excitations in these quantum Hall systems.

1. Introduction: non-abelian statistics in the quantum Hall arena

The fractional quantum Hall effect has unveiled states of matter that can be characterized as incompressible quantum fluids with topological order. Such states are formed in a two-dimensional electron gas, at very low temperature and in the presence of a strong perpendicular magnetic field. It has been recognized early on [**13, 5**] that the excitations over fractional quantum Hall states obey fractional braid statistics: a configuration of N quasi-holes over a fractional quantum Hall ground state forms a one-dimensional representation of the braid group B_N, where the braiding of two quasi-holes is typically represented by $e^{i\alpha\pi}$, with α a rational but non-integer number. The requirement that particle states have to represent the braid group rather than the permutation group is special for two dimensions: the braid group is the fundamental group of the configuration space of identical particles only in two dimensions. On general grounds it is known that, for two-dimensional quantum systems, higher dimensional representations of the braid group B_N are allowed (see [**9**] for an early reference). In such a situation, the braiding of particles is represented by matrices, and since matrices in general do not commute, this leads to the notion of non-abelian statistics.

It is now believed that the 'non-abelian statistics scenario' is realized in novel types of quantum Hall states, which are characterized by a pairing or clustering of electrons under quantum Hall conditions. There exists concrete experimental [**28**] and numerical [**17**] evidence that the simplest of these states, the pfaffian state proposed by Moore and Read [**18**], exists in nature. It is expected that, more generally, these states can be realized in multi-layer quantum Hall systems with sufficiently strong interlayer tunneling. In the literature, other approaches to

1991 *Mathematics Subject Classification.* Primary 81V70; Secondary 81T40,81R10.

Research supported in part by the foundation FOM of the Netherlands.

construct non-abelian quantum Hall states have been proposed [**27**]. The relation to the paired and clustered states that are studied in this paper remains unclear at the moment.

In this contribution to the proceedings of the International Congress of Mathematical Physics 2000, we briefly review these fascinating developments. We first recall some properties of the fractional quantum Hall states, stressing the role of conformal field theory in the theoretical description. After that we present the paired and clustered states and briefly discuss the statistics properties of their quasi-hole excitations.

2. The fractional quantum Hall effect

The discovery of the fractional quantum Hall (fqH) effect [**24**] was truly remarkable and unanticipated. At fractional filling fraction ν a quantum Hall (qH) plateau was observed. The filling fraction is defined as the ratio of the number of electrons and the number of available states in the lowest Landau level: $\nu = N/N_\phi$, where N_ϕ is the number of flux quanta piercing the sample, and N the number of electrons. This observation implies that a gap is formed within a Landau level, and that the fundamental charge carriers have fractional charge. Soon after the discovery, Laughlin made a fundamental break-through by proposing his by now famous wave functions, which describe the qH effect at filling fraction $\nu = \frac{1}{m}$, where m is an odd integer [**16**]

$$\widetilde{\Psi}_{\mathrm{L}}^{m}(z_1,\dots,z_N) = \prod_{i<j}(z_i - z_j)^m \ . \tag{2.1}$$

Here and below we display reduced qH wave functions $\widetilde{\Psi}(z)$, which are related to the actual wave functions $\Psi(z)$ via $\Psi(z) = \widetilde{\Psi}(z) \exp\left(-\sum_i \frac{|z_i|^2}{4l^2}\right)$ with $l = \sqrt{\frac{\hbar c}{eB}}$ the magnetic length.

Although the qH effect occurs at relatively high magnetic fields, it was soon realized that the electron spin can indeed play an important role. The spin-polarized Laughlin states were generalized by Halperin, who proposed a set of spin-singlet wave functions [**14**]

$$\begin{aligned} &\widetilde{\Psi}_{\mathrm{SS}}^{m+1,m+1,m}(z_1^{\uparrow},\dots,z_N^{\uparrow};z_1^{\downarrow},\dots,z_N^{\downarrow}) = \\ &\qquad \prod_{i<j}(z_i^{\uparrow} - z_j^{\uparrow})^{m+1} \prod_{i<j}(z_i^{\downarrow} - z_j^{\downarrow})^{m+1} \prod_{i,j}(z_i^{\uparrow} - z_j^{\downarrow})^{m} \ , \end{aligned} \tag{2.2}$$

where $z_i^{\uparrow}$ and $z_j^{\downarrow}$ are the coordinates of the spin-up and spin-down electrons, respectively. The state Eq. (2.2) has filling fraction $\nu = 2/(2m+1)$.

2.1. The qH effect-CFT connection.

2.1.1. *Bulk connection.* Following Moore and Read [**18**] one observes that it is natural to view (lowest Landau level) qH wave functions as conformal blocks of electron-type operators in a suitable chiral conformal field theory (CFT) in 2+0 dimensions. This point of view is related to the fundamental role of Chern-Simons field theories for qH systems (compare with [**29**], where an explicit link between Chern-Simons theory and CFT is established).

As an example, the Laughlin ground state wave function (2.1) is obtained as

$$\widetilde{\Psi}_{\rm L}^{m} = \lim_{z_\infty \to \infty} (z_\infty)^{mN^2} \langle V_e(z_1)\dots V_e(z_N) : e^{-i\sqrt{m}N\varphi}(z_\infty) :\rangle \ , \tag{2.3}$$

with $V_e(z) =: \exp(i\sqrt{m}\varphi)$: a chiral vertex operator in the $c=1$ chiral CFT describing a single scalar field φ compactified on a radius $R^2 = m$.

2.1.2. *Edge connection.* While bulk excitations over a qH fluid are gapped, one expects gapless excitations at the edge of a sample. Following Wen [**25**], one observes that the edge excitations are described by a chiral Luttinger Liquid or chiral CFT in 1+1 dimensions. In the example of the $\nu = \frac{1}{m}$ Laughlin states, one again has the scalar field theory at $R^2 = m$. The neutral operator $\rho = i\sqrt{m}\partial\varphi$ is identified with edge density waves, while vertex operators of type $V^q(z) =: \exp(iq\sqrt{m}\varphi) :$ represent charged edge excitations, the charge being equal to qe with $-e$ the charge of the electron and $\frac{e}{m}$ of the fundamental quasi-holes.

2.2. Fractional statistics in the fqH effect. In the case of an abelian quantum Hall state, changing the magnetic field by one flux quantum $\Phi_0 = \frac{h}{e}$ results in the creation of a quasi-hole (or particle, depending on the sign of the change). These quasi-holes can have remarkable properties, such as a fractional charge. Also, the quasi-holes over the Laughlin fqH states are anyons, i.e. they realize fractional braid statistics [**13, 5**]. The fundamental phase for the braiding of two such excitations is given by $e^{i\frac{\pi}{m}}$.

Closely related to this are the fractional exclusion statistics of these same excitations [**12, 15**]. Focusing on edge excitations, one can show that the gapless, charged edge excitations of an abelian qH state satisfy a form of exclusion statistics closely related to that of Haldane [**12**]. A particularly natural choice of basis for the edge excitations employs edge electrons and quasi-holes as the fundamental quanta [**7**]. In this basis, the exclusion statistics parameter matrix is diagonal with self-exclusion parameters equal to m (for the edge electrons) and $\frac{1}{m}$ (for the edge quasi-holes).

For general abelian qH states, one may argue [**1**] that the statistics matrix G of edge excitations (in a specific basis) is of the form

$$\mathbf{G} = \mathbf{K}_e \oplus \mathbf{K}_e^{-1}, \tag{2.4}$$

where $\mathbf{K}_e$ is the so-called K-matrix that characterizes the topological order of the quantum Hall state (see for instance [**26**]).

3. Paired and clustered quantum Hall states

Prompted by the observation of a qH effect at filling fraction $\nu = \frac{5}{2}$ [**28**] a number of novel qH states have been proposed. Among these is the Moore-Read state or pfaffian state, which is characterized by a p-wave pairing of the fundamental electrons [**18, 10**]. A generalization, where the pairing is replaced by a clustering of order k was proposed by Read and Rezayi [**21**]. In [**3**] the present authors made a further generalization to a class of spin singlet qH states characterized by a clustering into $(2k)$-plets of electrons.

The wave function of the (spin polarized) pfaffian state is given by

$$\widetilde{\Psi}_{\mathrm{Pf}}(z_i) = \mathrm{Pf}\Big(\frac{1}{z_i - z_j}\Big) \prod_{i<j} (z_i - z_j)^{M+1} \ , \tag{3.1}$$

where $\mathrm{Pf}(M_{i,j}) = \frac{1}{2^{N/2}(N/2)!} \sum_\sigma \mathrm{sgn}(\sigma) \prod_{r=1}^{N/2} M_{\sigma(2r-1),\sigma(2r)}$ is the pfaffian of an antisymmetric matrix. For the wave function to be antisymmetric (we are describing electrons) M needs to be an odd integer, which implies an even-denominator filling fraction $\nu = \frac{1}{M+1}$.

The pfaffian wave function can be viewed as a correlator in a $c = 3/2$ CFT, consisting of a free scalar field and a Majorana fermion. The electron operator becomes $\psi(z) : \exp(i\sqrt{M+1}\,\varphi_c) : (z)$. The correlator of N of electron operators (and a suitable background charge) splits into a product of vertex operators, giving the Laughlin part of the wave function, and a product of fermion fields, which gives the pfaffian factor.

Upon generalizing the Majorana fermion to the $\mathbb{Z}_k$ parafermions [**30**] associated to the coset $\frac{\widehat{\mathrm{su}}(2)_k}{\widehat{\mathrm{u}}(1)}$, one obtains the clustered states of [**21**]. Their wave functions are constructed in the same way as the pfaffian wave function, with explicit parafermion factors brought in by the electron operator. The result is a state in which the electrons form clusters of order k rather than pairs. The filling fraction takes the form $\nu = \frac{k}{kM+2}$, with M an odd integer. Note that for $k = 1$ the Laughlin states (with $m = M + 2$) are recovered, while $k = 2$ gives the pfaffian states.

As stated before, the Halperin states are spin-singlet analogues of the Laughlin states. In the same way, the states of [**3**] are spin-singlet analogues of the Read-Rezayi states. To construct their wave functions, two boson fields are needed (for charge and spin) in addition to the 'higher rank' parafermions associated to $\frac{\widehat{\mathrm{su}}(3)_k}{[\widehat{\mathrm{u}}(1)]^2}$ [**8**]. The non-abelian spin-singlet states have filling fraction $\nu = \frac{2k}{2kM+3}$ with M an odd integer. For $k = 1$ the Halperin states (2.2), with $m = M + 1$, are recovered.

4. Quasi-holes over paired and clustered qH states

In a BCS superconductor, where electrons are paired up, the fundamental flux quantum is reduced to $\frac{1}{2}\Phi_0 = \frac{h}{2e}$. The same phenomenon occurs in the paired and clustered qH states, and this means that inserting a single flux quantum Φ_0 creates more than a single quasi-hole. For the spin-polarized states of [**20**] the number of quasi-holes is given by $n = k\Delta N_\phi$, where ΔN_ϕ is the number of excess flux quanta[1]. For the spin-singlet states of [**3**], this relation becomes $n^\uparrow + n^\downarrow = 2k\Delta N_\phi$, were $n^{\uparrow,\downarrow}$ denotes the number of spin-up and down quasi-holes, respectively.

The quasi-holes over the paired and clustered qH states carry fractional charge and satisfy non-abelian braid statistics. They can be studied with the help of the associated CFT. The wave functions of states in the presence of quasi-holes are obtained by inserting into the CFT correlators the appropriate quasi-hole operators, which consist of a vertex operator part and a spin field of the parafermion theory. In the case of the pfaffian, this is the spin field σ of the Ising model and the quasi-hole operator becomes $\sigma(w) : \exp(\frac{i}{2\sqrt{M+1}}\varphi_c) : (w)$.

[1]Note that we adopt a slightly different notation than the one used in [**20, 11**]. Here, n denotes the number of quasi-holes, rather than the number of excess flux quanta.

The non-abelian statistics have their origin in the non-trivial fusion rules of the parafermion spin fields. In general, there is more than one way to fuse the fields in the correlator to the identity; for n spin fields the number of ways will be denoted by d_n. The braiding of quasi-holes is then represented by a matrix of size $d_n \times d_n$.

4.1. Braid statistics. The simplest example that exhibits the non-abelian braiding is the situation where 4 quasi-holes are added to the pfaffian state. For the pfaffian $d_4 = 2$, so there are two distinct states which, following [**19**], we write as $\Psi^{(4\mathrm{qh},0)}$ and $\Psi^{(4\mathrm{qh},\frac{1}{2})}$. Starting from the state $\Psi^{(4\mathrm{qh},0)}$, and braiding two of the particles, we find the following transformation

$$\Psi^{(4\mathrm{qh},0)} \to \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}} \left(\Psi^{(4\mathrm{qh},0)} + \Psi^{(4\mathrm{qh},\frac{1}{2})} \right) . \tag{4.1}$$

Wave functions for the pfaffian state with n quasi-holes can be written as [**20**]

$$\widetilde{\Psi}_{\mathrm{Pf,qh}}(z_1,\ldots,z_N; w_1,\ldots,w_n) = \frac{1}{2^{(N-F)/2}(N-F)/2!} \prod_{i<j}(z_i - z_j)^{M+1}$$
$$\times \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \prod_{k=1}^{F} z_{\sigma(k)}^{m_k} \prod_{l=1}^{(N-F)/2} \frac{\Phi(z_{\sigma(F+2l-1)}, z_{\sigma(F+2l)}; w_1,\ldots,w_n)}{(z_{\sigma(F+2l-1)} - z_{\sigma(F+2l)})} , \tag{4.2}$$

where

$$\Phi(z_1,z_2;w_1,\ldots,w_n) = \frac{1}{((n/2)!)^2} \sum_{\tau \in S_n} \prod_{r=1}^{n/2} (z_1 - w_{\tau(2r-1)})(z_2 - w_{\tau(2r)}) . \tag{4.3}$$

The integers $m_1,\ldots,m_F$ must be chosen such that they satisfy $0 \leq m_1 < m_2 < \cdots < m_F \leq \frac{n}{2} - 1$, giving rise to a degeneracy $d_n^{(F)} = \begin{pmatrix} \frac{n}{2} \\ F \end{pmatrix}$. The number F is interpreted as the number of unpaired electrons in the excited state.

The braid matrices for n quasi-hole excitations were obtained by Nayak and Wilczek [**19**], who showed a direct connection with the rotation matrices of the group $SO(2n)$. We refer to [**23**] for more general results on braiding matrices.

4.2. Quasi-hole counting formulas. The CFT approach to the excited state wave functions and their braid properties is highly efficient. One would like however, to 'keep both feet on the ground' and understand the fundamental degeneracies that characterize the non-abelian statistics in a more direct way. This can be done by selecting a (ultra-local) hamiltonian that has the qH state as its ground state, and then (numerically) studying the spectrum of excited states.

These numerical computations are most easily performed by studying a small number of particles in a spherical geometry. By tuning the number of flux quanta to the value $N_\phi = \frac{1}{\nu}N - \mathcal{S}$, where $\mathcal{S}$ is the so-called shift [**26**], one realizes the qH state as the unique ground state. Cranking up the number N_ϕ and performing a numerical diagonalization, one obtains characteristic degeneracies for quasi-hole excitations.

Following [**20**], we first explain the counting of degeneracies for the case of the pfaffain state. To understand the degeneracies of quasi-hole excitations on the sphere, two effects should be taken into account. The first is a choice of fusion path or, equivalently, a choice of numbers F and $m_1,\ldots,m_F$ in the formula (4.2). The second effect is the so-called orbital degeneracy: the quasi-holes are not localized on the sphere, but can occupy one of a finite number of available orbitals, each of which is characterized by a definite angular momentum L_z. These orbital degeneracies are well-known from the analysis of integer and abelian qH states.

For the pfaffain, the orbital degeneracy factor depends on the number F of unpaired electrons. Fixing this number F, we have $d_n^{(F)}$ different choices for the quasi-hole wave function. To each of those we can associate [**20**] an orbital degeneracy factor equal to $\left(\begin{array}{c}\frac{N-F}{2}+n\\ n\end{array}\right)$. Putting it all together, we have the following total degeneracy

$$\#(N,n)=\sum_F \left(\begin{array}{c}\frac{n}{2}\\ F\end{array}\right)\left(\begin{array}{c}\frac{N-F}{2}+n\\ n\end{array}\right)\ , \tag{4.4}$$

in agreement with numerical results [**20**]. The degeneracies d_n relevant for a situation where n quasi-holes are at fixed positions are recovered by suppressing the orbital factors, $d_n=\sum_F' d_n^{(F)}=2^{n/2-1}$, where the sum is over even (odd) F for N even (odd). This number is in agreement with a direct count of the number of fusion paths of the n Ising spin fields [**19**].

For the more general clustered qH states, the degeneracies have basically the same form: an orbital part and an intrinsic part, stemming from the non-trivial fusion rules. The difference is however, that we can not rely on explicit wave functions to break up the intrinsic degeneracy. One can work around this by extracting from the parafermion CFT the relevant combinatorial factors, using the methods put forward in [**22, 6**].

For the Read-Rezayi states, the counting has been worked out in [**11**], with the result

$$\#(N,n;k)=\sum_F \left\{\begin{array}{c} n\\ F\end{array}\right\}_k \left(\begin{array}{c}\frac{N-F}{k}+n\\ n\end{array}\right)\ , \tag{4.5}$$

with n the number of quasi-holes, $n=k\Delta N_\phi$. The symbols $\{{n \atop F}\}_k$ represent the degeneracies due to the fusion rules. In [**6, 11**], these were described in terms of recursion relations; explicit formulas (based on binomials) for general k can be found in [**4**]. The sum $d_n=\sum_F\{{n \atop F}\}_k$, which equals the total number of fusion paths for the spin fields contained in n quasi-hole operators, sets the dimension of the braid matrices for braiding 2 out of the n quasi-particles.

For the non-abelian spin-singlet states of [**3**], the counting goes along the same lines, with the additional complication that we have to deal with two spin components, with are combined in a non-trivial way. This is reflected in the counting formulas by a doubling of the number of binomial factors. By inserting an amount ΔN_ϕ of extra flux, one creates quasi-holes, which can have either spin. The total number of quasi-holes is fixed, $n^\uparrow+n^\downarrow=2k\Delta N_\phi$. The symbols $\{\}_k$ now depend on four parameters $\{{n^\uparrow \atop F_1}{n^\downarrow \atop F_2}\}_k$ and we have $d_{n^\uparrow,n^\downarrow}=\sum_{F_1,F_2}\{{n^\uparrow \atop F_1}{n^\downarrow \atop F_2}\}_k$. The case $k=2$ is worked out in detail in [**2**], where the results are checked against numerical data. Explicit results for the symbols $\{\}_k$ can be found in [**4**].

The numbers d_n (for both spin-polarized and spin-singlet states) are easily extracted from the known fusion rules of the $\widehat{su}(2)_k$ and $\widehat{su}(3)_k$ CFTs. For both $\widehat{su}(2)_3$ and $\widehat{su}(3)_2$ the numbers d_n are Fibonacci numbers. The asymptotic behavior for $n \to \infty$ is found to be

$$d_n \sim [2\cos\frac{\pi}{k+2}]^n \tag{4.6}$$

for the spin-polarized clustered states, and

$$d_p \sim [1+2\cos\frac{2\pi}{k+3}]^p \tag{4.7}$$

for the spin-singlet non-abelian states, where $p = n^\uparrow + n^\downarrow$.

4.3. Exclusion statistics and K matrix structure. In [**1**] a proposal was made for a K-matrix structure of the paired and clustered qH states discussed in this paper. It was established that the exclusion statistics of edge excitations over these states (in a suitable basis) can be captured by a statistics matrix of the form (2.4), supplemented by a prescription that some of the particles described by this matrix be viewed as pseudo-particles. We refer to the first paper of [**1**] for a physical picture underlying these K-matrices, and to the second paper of [**1**] for mathematical details.

It is a pleasure to thank P. Bouwknegt, R. van Elburg, S. Guruswamy, N. Read and E. Rezayi for collaborations on this subject. This research is supported in part by the Foundation FOM of the Netherlands.

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