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Wavefunctions for topological quantum registers

E. Ardonne [a,b,*], K. Schoutens [c]

[a] Microsoft station Q, Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA
[b] Center for the Physics of Information, California Institute of Technology, Pasadena, CA 91125, USA
[c] Institute for Theoretical Physics, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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Abstract

We present explicit wavefunctions for quasi-hole excitations over a variety of non-abelian quantum Hall states: the Read–Rezayi states with $k \geqslant 3$ clustering properties and a paired spin-singlet quantum Hall state. Quasi-holes over these states constitute a topological quantum register, which can be addressed by braiding quasi-holes. We obtain the braid properties by direct inspection of the quasi-hole wavefunctions. We establish that the braid properties for the paired spin-singlet state are those of 'Fibonacci anyons', and thus suitable for universal quantum computation. Our derivations in this paper rely on explicit computations in the parafermionic conformal field theories that underly these particular quantum Hall states.

1. Introduction

The realization that quantum Hall systems may harness what is called non-abelian braid statistics has led to two exciting prospects. The first is that experiments can be set up where, for the first time, the existence of non-abelian statistics in nature can be established. The second prospect, following early ideas of Kitaev [1], is that systems exhibiting non-abelian braid statistics can give rise to quantum registers and may offer unique possibilities for what has come to be known as topological quantum computation or fault-tolerant quantum computation.

* Corresponding author. Fax: +1 805 8932431.
E-mail address: ardonne@kitp.ucsb.edu (E. Ardonne).

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Proposals for experimental detection of non-abelian statistics [1–4] have mostly focused on the so-called $\nu = 5/2$ quantum Hall state, which is believed to be described by the 'pfaffian' or Moore–Read state [5,6]. To this date, this is the single quantum Hall plateau where the indications for an underlying non-abelian state are rather firm. For the plateau at $\nu = 12/5$ there is a more speculative case for a connection with a non-abelian $k = 3$ Read–Rezayi state [7]. Proposals to test if the state observed at $\nu = 12/5$ is indeed non-abelian have been described in [8,9].

An interesting domain for the search for non-abelian quantum Hall states is that of multi-component quantum Hall states, which can be realized as double layer or as spin-singlet states. The double layer case in particular may offer the experimental flexibility needed for tuning into a regime where a non-abelian state takes the upper hand. Examples of candidates are the spin-singlet (or double layer) states of [10], exhibiting a separation of spin and charge, and the spin-singlet analogues of the Moore–Read and Read–Rezayi states that we introduced in [11] and studied further (with N. Read and E. Rezayi) in [12].

Ideas for topological quantum computation in quantum Hall systems boil down to the following. Having available a non-abelian quantum Hall state as the underlying medium, the injection of quasi-hole excitations is known to open up an internal space (the 'quantum register'), whose dimensionality grows with the number of quasi-holes. This quantum register can be addressed by performing adiabatic quasi-hole braidings, which give rise to matrices acting on the register.

It is well known that the non-abelian braiding in the Moore–Read state is not sufficiently rich to enable universal quantum gates on the quantum register (note that there are proposals to combine topological and non-topological operations to obtain universal gates in this state [13–15]). It is also known that 'higher' non-abelian quantum Hall states, such as those in the Read–Rezayi series with $k = 3, 5, 6, \ldots$, do offer a prospect of universal quantum computation [16,17], in the sense that universal quantum gates, such as the 2-qubit CNOT gate, can be approximated with arbitrary precision by a well-chosen sequence of braid matrices [18,19]. In this paper, we shall establish that the paired spin-singlet state (AS state) of [11] is also universal for quantum computation. It is the unique paired quantum Hall state with this property.

The simplest scenario for braid matrices with sufficient structure for universal topological quantum computation is offered by the so-called 'Fibonacci anyons'. *In abstracto*, these are particles of two types, '0' and '1', with fusion rules

$$0 \times 0 = 0, \quad 0 \times 1 = 1, \quad 1 \times 1 = 0 + 1. \tag{1.1}$$

Through a universal connection between, on the one hand, fusion rules and, on the other, braid matrices, it has been established that the simplest braid matrices for Fibonacci anyons are, in the notation of [20],

$$R = \begin{pmatrix} (-1)^{4/5} & 0 \\ 0 & (-1)^{-3/5} \end{pmatrix}, \quad F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}, \quad B = (-1)^{-4/5} \begin{pmatrix} \tau & (-1)^{-3/5}\sqrt{\tau} \\ (-1)^{-3/5}\sqrt{\tau} & (-1)^{-1/5}\tau \end{pmatrix}, \tag{1.2}$$

where $\tau = \frac{1}{2}(\sqrt{5} - 1)$.

Quasi-holes over non-abelian quantum Hall states cannot straightforwardly be identified with 'non-abelian anyons', but there are important parallels, in particular where the fundamental relations between fusion and braiding properties are concerned. These relations were first studied in the context of the algebraic approach rational conformal field theories, as developed by Moore and Seiberg in [21]. In the context of quantum Hall systems, these same relations have been exemplified in the work of Slingerland and Bais [22], who used an associated quantum group structure to obtain explicit results for braid matrices for the Read–Rezayi quantum Hall states for general k.

Before the work of [22], Nayak and Wilczek [23] had derived explicit wavefunctions for four quasi-hole excitations over the Moore–Read state. These wavefunctions provide detailed information on the internal state associated with four quasi-holes, as a function of the locations of these excitations. From these wavefunctions, braid properties are derived by direct inspection. The work of [23] was based on a 'coordinate' rather than an 'algebraic' approach to (rational) conformal field theory (CFT), and employed bosonization techniques for mastering the relevant CFT correlators.

It is the purpose of this paper to present explicit expressions for wavefunctions of quantum registers associated to non-abelian quantum Hall states that are sufficiently rich to enable universal topological quantum computation. We will in particular focus on two distinct quantum Hall states that both give rise to braid matrices of the type displayed in Eq. (1.2) (up to additional abelian phase factors). The first is the $k = 3$ Read–Rezayi state and the second is the paired ($k = 2$) AS spin-singlet quantum Hall state. We shall also write some of the wavefunctions for quasi-holes over the general k Read–Rezayi states. Note that throughout this paper, we will assume that the only effect of braiding comes from the explicit monodromy.

The appearance of the 'Fibonacci-type' braid matrices in the quantum Hall systems can be understood from a coarse graining of the fusion rules of the parafermionic CFTs underlying these states. For the example of the spin-singlet state, this takes the form

$$\text{`0'} = \{\mathbf{1}, \psi_1, \psi_2, \psi_3\}, \quad \text{`1'} = \{\sigma_1, \sigma_2, \sigma_3, \rho\} \tag{1.3}$$

with the ψ_i denoting the parafermion sectors and σ_i, ρ labeling the various parafermion spin fields in the CFT. The fusion rules of these fields (see Table B.1 in Section B.1) are such that the coarse graining into '0' and '1' respects the relations given in Eq. (1.1). We would like to stress that the 'Fibonacci anyon' aspect of the quantum Hall quasi-holes captures a limited fraction of their relevant properties: for example, the fundamental quasi-holes come with different sets of quantum numbers and the detailed fusion rules (and operator product expansions) of the fields in the parafermionic CFTs lead to detailed structure expressed in the wavefunctions that we derive in this paper.

The 'coordinate CFT' approach that we follow to derive the quasi-hole wavefunctions delves deep into the results for Wess–Zumino–Witten (WZW) and parafermion CFTs derived in the mid-1980s. We shall in particular rely on results of Knizhnik and Zamolodchikov (KZ) [24]. Their expression for four-point functions of particular primary fields in WZW models will form the cornerstone of the 'contraction arguments' that we employ to determine closed form expressions for the various quantum register states that we consider. The 'master formulas' that we develop enable an easy evaluation of correlators that have until now not appeared in the literature, and that are not easily computed with the methods of KZ. We present some explicit examples in Sections A.4 and B.4.

The presentation in this paper is organized as follows. In Section 2, we will explain the method we use to obtain the quasi-hole correlators by using the Moore–Read state as an example. In Section 3, we apply this method to the $k = 3$ Read–Rezayi states. We will provide a detailed derivation of the quasi-hole wavefunctions, and use these to calculate the braid behaviour of the quasi-holes. In Section 4, we apply our method to the paired spin-singlet states proposed by the authors. In Section 5, we compare the braid results of Sections 3 and 4 with the results obtained by using the associated quantum groups. In the two Appendices A and B, we give the details of the parafermion CFTs corresponding to $su(2)_k$ and $su(3)_2$, respectively. Namely, we provide the fusion rules, the details of the operator product expansions, including the various coefficients, the spin-field correlators used to derive the quasi-hole wavefunctions and the braid behaviour of the various hypergeometric functions which show up in the correlators. In addition, we give some new parafermionic correlators, which were used to obtain the various OPE coefficients. In Section A.6, we give the quasi-hole wavefunctions for the Read–Rezayi states for arbitrary k.

2. General form of quasi-hole wavefunctions

In our analysis of non-abelian quantum Hall states, we rely on the so-called qH-CFT connection where wavefunctions of a quantum Hall (qH) system are expressed as chiral correlators (conformal blocks) of an associated conformal field theory (CFT) [5]. The connection hinges on the identification in the CFT of an *electron operator* $\psi_{\mathrm{e}}(z)$, carrying charge $q = -1$. The quantum Hall wavefunction is then expressed as

$$\Psi_{\mathrm{qH}}(z_1,\ldots,z_N) = \lim_{z_\infty\to\infty}(z_\infty)^{\frac{N^2}{\nu}}\langle\psi_{\mathrm{e}}(z_1)\psi_{\mathrm{e}}(z_2)\cdots\psi_{\mathrm{e}}(z_N)Q_{\mathrm{bg}}(z_\infty)\rangle \tag{2.1}$$

with Q_{bg} representing a neutralizing background (ionic) charge and the factor $(z_\infty)^{\frac{N^2}{\nu}}$ is included to obtain a non-vanishing result. (The case of spin-full fermions has some additional structure.) Note that we drop the Gaussian factors throughout this paper.

The injection of a quasi-hole at position w is represented by the insertion in the CFT correlator of the *quasi-hole operator* $\phi_{\mathrm{qh}}(w)$. Mutual locality of the quasi-holes and the electrons in the quantum Hall condensate implies that the operator product expansion (OPE) between electron and quasi-hole operators is of the form

$$\phi_{\mathrm{qh}}(w)\psi_{\mathrm{e}}(z) = (w-z)^n\phi'_{\mathrm{qh}}(z) \tag{2.2}$$

with n a non-negative integer. Note that the mutual locality puts a constraint on the possible quasi-hole operators.

In all cases studied in this paper, the electron and quasi-hole operators are expressed in terms of free bosonic fields (representing charge and spin) and of a parafermionic CFT. The latter are closely related to Wess–Zumino–Witten (WZW) theories. For the order-k Read–Rezayi (RR) states the 'CFT-data' are: $SU(2)_k$ WZW theory and the associated $\mathbb{Z}_k$ parafermions, while the paired spin-singlet states are connected to the $SU(3)_2$ WZW theory and to the associated higher-rank parafermions. We refer to [25] for a review and further details.

For the Moore–Read (MR) state (which is the $k = 2$ member of the RR series), the parafermion theory reduces to a single real (Majorana) fermion $\psi(z)$. This allows a direct evaluation of the electron wavefunction using the Wick theorem, leading to a 'pfaffian' wavefunction. Quasi-holes over this pfaffian state can be characterized by pair braking in the pfaffian BCS factor [26]. The full dependence of a multi-quasi-hole wavefunction on all coordinates is set by a CFT correlator

$$\langle\sigma(w_1)\cdots\sigma(w_n)\psi(z_1)\cdots\psi(z_N)\rangle^{(0,1)} \tag{2.3}$$

with $\sigma(w)$ the spin fields for the Majorana fermion. In a 1996 paper, Nayak and Wilczek used bosonization techniques to derive an explicit expression for the full four quasi-hole wavefunction [23]. It has the general form

$$\begin{aligned}\Psi^{(0,1)}_{\mathrm{MR}}(w_1,w_2,w_3,w_4;z_1,z_2,\ldots,z_N) = {} & A^{(0,1)}(\{w\})\Psi_{12,34}(\{w\},\{z\}) \\ & + B^{(0,1)}(\{w\})\Psi_{13,24}(\{w\},\{z\}). \end{aligned} \tag{2.4}$$

In this expression, the factors $\Psi_{12,34}$ and $\Psi_{13,24}$, which are polynomial in all coordinates $(\{w\},\{z\})$, represent independent (in this case: two) ways in which four quasi-holes can break up pairs in the electron condensate. The prefactors $A^{(0,1)}$ and $B^{(0,1)}$ are given by (note that we are giving the result for the fermionic case $M = 1$)

$$\begin{aligned} A^{(p)}(\{w_i\}) &= \frac{(-1)^{-\frac{p}{2}}}{2}(w_{12}w_{34})^{\frac14}x^{\frac14}\left((-1)^p\sqrt{1-\sqrt{x}}+\sqrt{1+\sqrt{x}}\right), \\ B^{(p)}(\{w_i\}) &= -\frac{(-1)^{-\frac{p}{2}}}{2}(w_{12}w_{34})^{\frac14}x^{-\frac14}(1-x)^{\frac12}\left(-\sqrt{1-\sqrt{x}}+(-1)^p\sqrt{1+\sqrt{x}}\right), \end{aligned} \tag{2.5}$$

where $p = 0, 1$ and we used the anharmonic ratio $x = \frac{w_{12}w_{34}}{w_{14}w_{32}}$, (where $w_{ij} = (w_i - w_j)$), which is the same as the anharmonic ratio used in [24], but differs from the one used in [23]. The labels $(0, 1)$ refer to the fusion channel of the four-quasi-hole state; it is this index which acts as the qubit index in the context of topological quantum computation.

The formula (2.4) admits an elegant interpretation: it describes precisely how the fusion channel basis of the four-quasi-hole states decomposes over a basis set by patterns in quasi-hole induced pair-breaking in the condensate. This decomposition is given as a function of the quasi-hole coordinates w_i, meaning that it can be followed as quasi-particles move through the condensate. This in particular implies that quasi-hole braiding properties can be read off from these wavefunctions, as was done in [23]. Clearly, the information stored in these wavefunctions goes well beyond braiding properties; we expect that some of this additional structure will be relevant for the optimal design of experimental protocols aimed at demonstrating non-abelian statistics and at quantum computation.

In this paper we show that quasi-hole wavefunction for non-abelian quantum Hall states with potential for universal topological quantum computation can be cast in a form similar to (2.4). To achieve this goal, we rely, in a first step, on known expressions for the multi-parafermion correlators representing the quantum Hall condensate in the absence of excitations [7,27,12,28]. Analyzing the factors associated with the injection of quasi-holes (of various kinds) then leads to 'master formulas' not unlike (2.4). In a final step we consider various 'contractions' of this master formula and use those to relate the coefficients such as $A^{(0,1)}$ and $B^{(0,1)}$ to correlators having just four parafermion spin-fields. The latter can extracted from [24], where they have been expressed in terms of hypergeometric functions.

3. Quasi-hole wavefunctions for the $k = 3$ Read–Rezayi states

In this section, we use the approach outlined in the previous section to obtain the wavefunctions of quasi-holes over the $k = 3$ Read–Rezayi (RR) states, [7], which can be viewed as clustered analogues of the paired Moore–Read (MR) state [5]. The RR states have been studied in detail by various authors. Among the advances are explicit wavefunctions in terms of the electron coordinates, both with and without the presence of quasi-holes.

Even though the structure of the quasi-hole wavefunctions in terms of the electron coordinates is known, see [27,29], the full quasi-hole wave functions, which also exhibit the full dependence on the quasi-hole coordinates, have only been known for up to four quasi-holes in the MR state (see [23]) and for an arbitrary number of quasi-holes in the Laughlin states. In this paper, we fill in this gap, and calculate the full (four) quasi-hole wavefunctions for the RR states and their spin-singlet analogues. In this section, we provide the details of the $k=3$, $M=0$ case of the RR states. We give the general k, M results in Appendix A, where we also give the details of the fusion rules, OPEs, some general parafermion correlators, and the braid relations.

Before we turn to the quasi-hole wavefunctions, we first give the wavefunction of the $k=3$ RR states without quasi-holes. In this case the number of electrons (even though for $M=0$, the particles are bosons, we will refer to them as electrons) has to be a multiple of 3. It was shown in [27] that the following wavefunction is equivalent to the wavefunction presented in the original paper [7]. Divide the electrons into three groups S_a, $a=1,2,3$ of equal size. For each group, we write a Laughlin factor

$$\Psi^2_{S_a}(\{z\}) = \prod_{\substack{i<j \\ i,j\in S_a}} (z_i - z_j)^2. \tag{3.1}$$

To obtain the RR wavefunction, we sum over all inequivalent ways to divide the electrons into three groups of equal size

$$\Psi^{k=3}_{\mathrm{RR}}(\{z\}) = \frac{1}{\mathcal{N}} \sum_{S_1,S_2,S_3} \left[\Psi^2_{S_1} \Psi^2_{S_2} \Psi^2_{S_3} \right]. \tag{3.2}$$

The normalization is $\mathcal{N} = 3^{\frac{N}{2}}/3!$, chosen consistently with the operator product expansion of the parafermion fields. In effect, the sum amounts to symmetrization of all coordinates.

The $k=3$ clustering property is manifest from Eq. (3.2): from the wavefunction (3.2) it is clear that we can put three electrons at the same location, without obtaining zero, because there will always be a term in the summation for which the three electrons belong to different groups. Putting four or more electrons at the same location gives a vanishing wavefunction.

3.1. The CFT formulation

The $k=3$ RR wavefunctions for N electrons and n quasi-holes can be expressed in terms of a parafermionic correlator in the following way

$$\begin{aligned} \Psi_{\mathrm{RR}}(w_1,\ldots,w_n;z_1,\ldots,z_N) = {} & \langle \sigma_1(w_1)\cdots\sigma_1(w_n)\psi_1(z_1)\cdots\psi_1(z_N)\rangle^{(0,1)} \\ & \times \prod_{i<j}(z_i-z_j)^{\frac{2}{3}+M} \prod_{i,j}(w_i-z_j)^{\frac{1}{3}} \prod_{i<j}(w_i-w_j)^{\frac{1}{6}-\frac{M}{2(3M+2)}}. \end{aligned} \tag{3.3}$$

From the fusion rules, it follows that the number of electrons N and n have to satisfy the relation $2N+n=0 \bmod 3$ in order for the correlator to be non-zero. In addition, a state with only one quasi-hole is impossible.

3.2. The quasi-hole wavefunctions

We focus on the case of four quasi-holes. In this case, there are two fusion channels (labeled by (0) and (1)) for the parafermion correlator, namely $\sigma_1\sigma_1 \sim \psi_1$ for the (0) channel and $\sigma_1\sigma_1 \sim \sigma_2$ for the (1) channel.

Following [27] and specifying to $N = 3r + 1$ electrons, with r an integer, we can write the following ansatz for the wavefunction for four quasi-holes, (3.3),

$$\Psi_{\mathrm{RR}}^{(0,1)}(w_1, w_2, w_3, w_4; z_1, \ldots, z_N) = A^{(0,1)}(\{w\})\Psi_{12,34}(\{w\}, \{z\}) + B^{(0,1)}(\{w\})\Psi_{13,24}(\{w\}, \{z\}). \tag{3.4}$$

Throughout the paper, we will choose the phase of wavefunctions in such a way that the function $A^{(0)}(\{w\})$ has no phase.

To specify the functions $\Psi_{12,34}$ and $\Psi_{13,24}$, we divide the electrons in three groups in such a way that S_1 contains $(N-1)/3 + 1$ electrons and S_2 and S_3 contain $(N-1)/3$ electrons.

Splitting the four quasi-holes into two groups, we have

$$\begin{aligned}
\Psi_{12,34} &= \frac{1}{\mathcal{N}} \sum_{S_1,S_2,S_3} \left[\prod_{i\in S_2}(z_i - w_1)(z_i - w_2) \prod_{j\in S_3}(z_j - w_3)(z_j - w_4) \Psi_{S_1}^2 \Psi_{S_2}^2 \Psi_{S_3}^2 \right], \\
\Psi_{13,24} &= \frac{1}{\mathcal{N}} \sum_{S_1,S_2,S_3} \left[\prod_{i\in S_2}(z_i - w_1)(z_i - w_3) \prod_{j\in S_3}(z_j - w_2)(z_j - w_4) \Psi_{S_1}^2 \Psi_{S_2}^2 \Psi_{S_3}^2 \right]
\end{aligned} \tag{3.5}$$

with $\mathcal{N} = 3^{\frac{N}{2}}$. In the following, we will use the case of $N = 4$ electrons to determine the functions $A^{(0,1)}(\{w\})$ and $B^{(0,1)}(\{w\})$. Note that in the functions (3.5), the quasi-holes do not have a zero with electrons of the first group before symmetrization. Nevertheless, the maximum degree of each z is the same, because the number of electrons in the first group is one bigger in comparison to the other two groups.

There is a third possible way of dividing the four quasi-holes into two groups. However, this does not give an independent function, because we have the following relation

$$\Psi_{14,23} = x\Psi_{12,34} + (1 - x)\Psi_{13,24}. \tag{3.6}$$

Relations of this kind were first studied by Nayak and Wilczek in [23], in order to reduce the overcomplete basis to a linear independent one.

We should note that an (explicitly) independent basis for the Read–Rezayi states (for arbitrary k) was recently formulated by Read [29] (building on results presented in [31]). However, for our present purposes, namely deriving the full quasi-hole wavefunctions and studying the braid behaviour of the quasi-holes, it is more convenient to use the basis states (3.5), because the formulas and the transformation properties under the braiding of quasi-holes are simpler.

The strategy to obtain the functions $A^{(p)}$ and $B^{(p)}$ in (3.4) is as follows. Making use of OPEs, including the OPE coefficients, we reduce the correlator in the wavefunction (3.3) to correlators involving just four $\sigma_{1,2}$ fields. The latter can be extracted from the results in [24]. In particular we consider the following limits

$$\begin{array}{ll} \text{(I)} & z_2 \to z_1, \quad z_4 \to z_3, \quad z_3 \to w_4, \quad z_1 \to w_2, \\ \text{(II)} & z_2 \to z_1, \quad z_4 \to z_3, \quad z_3 \to w_4, \quad z_1 \to w_3. \end{array} \tag{3.7}$$

In both limits, we first fuse the two pairs of ψ_1 fields to two ψ_2 fields. Fusing these ψ_2 fields with σ_1 fields, we obtain σ_2 fields (see Section A.1 for more details about the fusion rules). Thus, we obtain the following correlators

$$\begin{aligned} &\lim_{\text{(I)}} \langle \sigma_1\sigma_1\sigma_1\sigma_1\psi_1\psi_1\psi_1\psi_1 \rangle^{(0,1)} \propto \langle \sigma_1\sigma_2\sigma_1\sigma_2 \rangle^{(0,1)}, \\ &\lim_{\text{(II)}} \langle \sigma_1\sigma_1\sigma_1\sigma_1\psi_1\psi_1\psi_1\psi_1 \rangle^{(0,1)} \propto \langle \sigma_1\sigma_1\sigma_2\sigma_2 \rangle^{(0,1)}. \end{aligned} \tag{3.8}$$

Keeping track of the various OPE coefficients and various phases is crucial in this procedure. Many of these coefficients for the $\mathbb{Z}_k$ parafermion CFT are given in [30]. The other coefficients we need were obtained from consistency relations on the four-point correlators. We give these coefficients in Appendix A.2.

With respect to the various phases which need to be taken into account we note the following. To be able to fuse the fields ψ_2 with the appropriate spin field, one has to move the field $\psi_2(z_1)$ in between the σ_1 fields. The resulting phase depends on the fusion channel, and can be obtained from the OPE of ψ_2 and the fusion channel under consideration. As an example, in the limits above we need to find the phase associated with $\sigma_1(w_3)\sigma_1(w_4)\psi_2(z_1) = (-1)^\alpha \psi_2(z_1)\sigma_1(w_3)\sigma_1(w_4)$. In the (0) channel, we have $\sigma_1\sigma_1 \sim \psi_1$, and the phase $\alpha = -\frac{4}{3}$ follows from

$$\psi_1(z)\psi_2(z') \sim (z-z')^{-\frac{4}{3}} \sim (-1)^{-\frac{4}{3}}(z'-z)^{-\frac{4}{3}} \sim (-1)^{-\frac{4}{3}}\psi_2(z')\psi_1(z),$$

while in the (1) channel we have $\sigma_1\sigma_1 \sim \sigma_2$ and the phase $\alpha = -\frac{1}{3}$ follows from

$$\sigma_2(z)\psi_2(z') \sim \varepsilon(z')(z-z')^{-\frac{1}{3}} \sim (-1)^{-\frac{1}{3}}\varepsilon(z')(z'-z)^{-\frac{1}{3}} \sim (-1)^{-\frac{1}{3}}\psi_2(z')\sigma_2(z).$$

Taking the phases and OPE coefficients into account, we obtain the limits (I) and (II) of the wavefunction $\Psi^{(p)}_{\text{RR}}$, with $p = 0, 1$, Eq. (3.3)

$$\begin{aligned} \lim_{\text{(I)}} \Psi^{(p)}_{\text{RR}} &= -(-1)^p \tfrac{4}{9}(w_{12}w_{34})^{\frac{5}{6}}(1-x)^{\frac{1}{6}} \langle \sigma_1(w_1)\sigma_2(w_2)\sigma_1(w_3)\sigma_2(w_4) \rangle^{(p)} \left(w_{42}^4 w_{14} w_{32}\right), \\ \lim_{\text{(II)}} \Psi^{(p)}_{\text{RR}} &= (-1)^p \tfrac{4}{9}(w_{12}w_{34})^{\frac{5}{6}} x^{-\frac{2}{3}}(1-x)^{\frac{5}{6}} \langle \sigma_1(w_1)\sigma_1(w_2)\sigma_2(w_3)\sigma_2(w_4) \rangle^{(p)} \left(w_{34}^4 w_{14} w_{32}\right), \end{aligned} \tag{3.9}$$

where we used the notation $w_{ij} = (w_i - w_j)$ and the following convention for the anharmonic ratio x

$$x = \frac{(w_{12})(w_{34})}{(w_{14})(w_{32})}, \quad 1-x = \frac{(w_{13})(w_{42})}{(w_{14})(w_{32})}, \quad \frac{x}{1-x} = \frac{(w_{12})(w_{34})}{(w_{13})(w_{42})}, \tag{3.10}$$

which is the same convention as used in [24], but differs from the one used in [30]. The explicit form of correlators $\langle \sigma_1\sigma_2\sigma_1\sigma_2 \rangle^{(0,1)}$ and $\langle \sigma_1\sigma_1\sigma_2\sigma_2 \rangle^{(0,1)}$ can be extracted from the results of [24]. In Appendix A.3 we present formulas expressing these correlators in terms of hypergeometric functions.

Because we can easily take the limits (I) and (II) of the functions $\Psi_{12,34}$ and $\Psi_{13,24}$ in the case of four electrons, namely

$$\begin{aligned}
&\lim_{(\mathrm{I})} \Psi_{12,34} = -\frac{4}{9} w_{14} w_{32} (w_{42})^4, \\
&\lim_{(\mathrm{II})} \Psi_{12,34} = 0, \\
&\lim_{(\mathrm{I})} \Psi_{13,24} = 0, \\
&\lim_{(\mathrm{II})} \Psi_{13,24} = \frac{4}{9} w_{14} w_{32} (w_{34})^4,
\end{aligned} \tag{3.11}$$

we finally obtain relations for the functions $A^{(0,1)}$ and $B^{(0,1)}$ which are easily solved to give

$$\begin{aligned}
A^{(0)} &= (w_{12}w_{34})^{\frac{7}{10}} x^{\frac{3}{10}} \mathcal{F}_1^{(0)}(x), \\
B^{(0)} &= -(w_{12}w_{34})^{\frac{7}{10}} x^{-\frac{7}{10}} (1-x) \mathcal{F}_2^{(0)}(x), \\
A^{(1)} &= -(-1)^{\frac{2}{5}} C (w_{12}w_{34})^{\frac{7}{10}} x^{\frac{3}{10}} \mathcal{F}_1^{(1)}(x), \\
B^{(1)} &= (-1)^{\frac{2}{5}} C (w_{12}w_{34})^{\frac{7}{10}} x^{-\frac{7}{10}} (1-x) \mathcal{F}_2^{(1)}(x).
\end{aligned} \tag{3.12}$$

Here, $C = \frac{1}{2}\sqrt{\frac{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}}$ and the functions $\mathcal{F}_i^{(p)}$, with $p = 0, 1$ and $i = 1, 2$ are given in (A.12).

With the explicit form of the functions $A^{(p)}$ and $B^{(p)}$, we have completely specified the wavefunction for the $k = 3$, $M = 0$ Read–Rezayi state in the presence of four quasi-holes. We will give the general k, M results in Appendix A. As an immediate application, we can study the behaviour under exchange of quasi-holes, by making use of transformation properties of the hypergeometric functions, which are also given in Appendix A.

3.3. Braid behaviour

To study the braid behaviour under the exchange of quasi-holes, we first note that the anharmonic ratio transforms as $x \mapsto \frac{-x}{1-x}$ for $(1 \leftrightarrow 2)$, $x \mapsto 1 - x$ for $(2 \leftrightarrow 3)$ and $x \mapsto \frac{1}{x}$ for $(1 \leftrightarrow 3)$. In addition, we find that, by making use of (3.6),

$$\begin{aligned}
\Psi_{12,34} &\overset{1\leftrightarrow 3}{\rightarrow} x\Psi_{12,34} + (1-x)\Psi_{13,24}, \\
\Psi_{13,24} &\overset{1\leftrightarrow 2}{\rightarrow} x\Psi_{12,34} + (1-x)\Psi_{13,24},
\end{aligned} \tag{3.13}$$

while the other transformations of the Ψ's are clear. The braid transformations of the functions $\mathcal{F}_i^{(p)}$ are given in Appendix A.5. Combining all the results, we find the following braid behaviour for general M

$$\begin{aligned}
&\Psi_{\mathrm{RR}}^{(p)} \overset{1\leftrightarrow 2}{\rightarrow} (U_{12})_q^p \Psi_{\mathrm{RR}}^{(q)}, \quad U_{12} = (-1)^{-\frac{1}{2}-\frac{M}{2(3M+2)}} \begin{pmatrix} (-1)^{-\frac{4}{5}} & 0 \\ 0 & (-1)^{\frac{3}{5}} \end{pmatrix}, \\
&\Psi_{\mathrm{RR}}^{(p)} \overset{2\leftrightarrow 3}{\rightarrow} (U_{23})_q^p \Psi_{\mathrm{RR}}^{(q)}, \quad U_{23} = (-1)^{-\frac{3}{10}-\frac{3M}{2(3M+2)}} \begin{pmatrix} \tau & (-1)^{-\frac{2}{5}}\sqrt{\tau} \\ (-1)^{\frac{2}{5}}\sqrt{\tau} & -\tau \end{pmatrix}, \\
&\Psi_{\mathrm{RR}}^{(p)} \overset{1\leftrightarrow 3}{\rightarrow} (U_{13})_q^p \Psi_{\mathrm{RR}}^{(q)}, \quad U_{13} = (-1)^{\frac{3}{10}-\frac{M}{2(3M+2)}} \begin{pmatrix} \tau & (-1)^{\frac{1}{5}}\sqrt{\tau} \\ -\sqrt{\tau} & (-1)^{\frac{1}{5}}\tau \end{pmatrix},
\end{aligned} \tag{3.14}$$

where we use τ to denote the inverse of the golden ratio, i.e. $\tau = \frac{\sqrt{5}-1}{2}$. These matrices are unitary, and satisfy $U_{13} = U_{23} \cdot U_{12} \cdot U_{23}^{-1}$. Note that if we evenly distribute the phases of the off-diagonal elements, which amounts to a 'gauge' transformation, the matrices U_{12}, U_{23} and U_{13} become proportional to the (inverses of the) R, F and B matrices of the 'Fibonacci' anyons, as displayed in Eq. (1.2).

4. Quasi-hole wavefunctions for a paired spin-singlet state

4.1. A paired spin-singlet quantum Hall state

In the search for a topological quantum liquid suited for universal quantum computation, the paired spin-singlet quantum Hall state of [11] imposes itself as a natural candidate. This state is the $k = 2$ member of a series of spin-singlet states introduced and studied in [11,12]. In many ways, these states are direct extensions of the Read–Reazayi states to spin-full (spin-1/2) fermions.

The simplest fermionic spin-singlet state (with $M = 1$) has filling fraction $\nu = 4/7$. At this particular filling spin-singlet quantum Hall states have been observed [32], but their precise nature has never been determined.

The braiding properties of the paired spin-singlet state are essentially more complicated than those of the paired spin-polarized (Moore–Read) state. In fact, we will show that the braiding in the paired ($k = 2$) spin-singlet state is similar to that in the $k = 3$ Read–Rezayi state, the similarity being due to what is known as level-rank duality between the affine Lie algebras $su(2)_3$ and $su(3)_2$. With this, the $k = 2$ state offers the perspective of universal topological quantum computation in a paired quantum Hall state. This being enough excitement for us now, we shall in this paper not address the $k > 2$ spin-singlet states. Their wavefunctions can be obtained using the methods described in this paper.

4.2. The CFT formulation

The various wavefunctions for the $k = 2$ spin-singlet state in the presence of quasi-holes are all expressed as correlators in a CFT. For $M = 0$ the CFT is precisely the (chiral) $SU(3)_2$ WZW model while for $M \neq 0$ we have a deformation thereof, with a modified compactification radius of the charge boson φ_c, see [12]. In all cases, the theory is conveniently represented as a product of the $su(3)_2$ parafermions, as introduced by Gepner in [33] and the CFT of spin and charge bosons φ_s and φ_c.

The fundamental quasi-holes over this quantum Hall state come in different types. One type has spin-1/2 and charge $1/(4M + 3)$; a second option is to have spin-less quasi-holes of charge $2/(4M + 3)$. For $M > 0$ the latter have the smaller scaling dimension and are thereby the most 'relevant' in the sense of scaling arguments. We expect that experimental protocols for the detection of non-abelian statistics and for quantum computation will be most easily implemented using the spin-less quasi-holes.

Using the quantum Hall—CFT connection, we can write the wavefunction for a state with the number of quasi-holes of the various types specified as $n_\uparrow$, $n_\downarrow$, n_3. These numbers satisfy

$$N_\uparrow + n_\uparrow = N_\downarrow + n_\downarrow, \tag{4.1}$$

in order for the state to be a spin-singlet and

$$3(N_\uparrow + N_\downarrow) + (n_\uparrow + n_\downarrow) + 2n_3 = 0 \bmod 4 \tag{4.2}$$

such that the fields can be fused to the identity sector. The case of only one quasi-hole is an exception, because a state with only one quasi-hole is impossible.

In full generality the quasi-hole wavefunction reads [12]

$$
\begin{aligned}
&\Psi^{M}_{\mathrm{AS}}(w_1^{\uparrow},\ldots,w_{n_\uparrow}^{\uparrow};w_1^{\downarrow},\ldots,w_{n_\downarrow}^{\downarrow};w_1,\ldots,w_{n_3};z_1^{\uparrow},\ldots,z_{N_\uparrow}^{\uparrow};z_1^{\downarrow},\ldots,z_{N_\downarrow}^{\downarrow})\\
&\quad=\langle\sigma_\uparrow(w_1^{\uparrow})\cdots\sigma_\uparrow(w_{n_\uparrow}^{\uparrow})\sigma_\downarrow(w_1^{\downarrow})\cdots\sigma_\downarrow(w_{n_\downarrow}^{\downarrow})\sigma_3(w_1)\cdots\sigma_3(w_{n_3})\psi_1(z_1^{\uparrow})\cdots\psi_1(z_{N_\uparrow}^{\uparrow})\psi_2(z_1^{\downarrow})\cdots\psi_2(z_{N_\downarrow}^{\downarrow})\rangle\\
&\qquad\times\left[\widetilde{\Psi}^{(2,2,1)}_{\mathrm{H}}(z_i^{\uparrow};z_j^{\downarrow})\right]^{1/2}\widetilde{\Psi}^{M}_{\mathrm{L}}(z_i^{\uparrow};z_j^{\downarrow})\prod_{i,j}(z_i^{\uparrow}-w_j^{\uparrow})^{\frac{1}{2}}\prod_{i,j}(z_i^{\downarrow}-w_j^{\downarrow})^{\frac{1}{2}}\prod_{i,j}(z_i^{\uparrow}-w_j)^{\frac{1}{2}}\prod_{i,j}(z_i^{\downarrow}-w_j)^{\frac{1}{2}}\\
&\qquad\times\prod_{i<j}(w_i^{\uparrow}-w_j^{\uparrow})^{\frac{1}{3}-\frac{M}{3(4M+3)}}\prod_{i,j}(w_i^{\uparrow}-w_j^{\downarrow})^{-\frac{1}{6}-\frac{M}{3(4M+3)}}\prod_{i<j}(w_i^{\downarrow}-w_j^{\downarrow})^{\frac{1}{3}-\frac{M}{3(4M+3)}}\\
&\qquad\times\prod_{i,j}(w_i^{\uparrow}-w_j)^{\frac{1}{6}-\frac{2M}{3(4M+3)}}\prod_{i,j}(w_i^{\downarrow}-w_j)^{\frac{1}{6}-\frac{2M}{3(4M+3)}}\prod_{i<j}(w_i-w_j)^{\frac{1}{3}-\frac{4M}{3(4M+3)}}.
\end{aligned}\tag{4.3}
$$

Below we present explicit formulas for these wavefunctions in the special cases of four quasi-holes with (i) $n_3=4$, (ii) $n_\uparrow=n_\downarrow=2$, (iii) $n_\uparrow=4$ and (iv) $n_\uparrow=3, n_\downarrow=1$. To avoid clutter, we write the expressions for $M=0$; the braid relations will be specified for general M.

4.3. Evaluating quasi-hole wavefunctions

We first give the wavefunction without any quasi-holes in the form given in [28], see also [27]. Assuming $N_\uparrow=N_\downarrow$ both even we have (the sum is over all independent ways of dividing the electrons in two groups, both containing $N_\uparrow/2$ spin-up electrons and $N_\downarrow/2$ spin-down electrons; in effect this amounts to symmetrization over the spin-up and spin-down electrons)

$$
\Psi^{M=0}_{\mathrm{AS}}=\frac{1}{\mathcal{N}}\sum_{S_1,S_2}\Psi^{221}_{S_1}(z_i^{\uparrow},z_{j'}^{\downarrow})\Psi^{221}_{S_2}(z_k^{\uparrow},z_{l'}^{\downarrow}),\tag{4.4}
$$

with $\mathcal{N}=2^{\frac{N_\uparrow+N_\downarrow}{2}}/2$ and with $\Psi^{221}_{S_a}$ denoting the 221 state restricted to $z_i^{\uparrow},z_{j'}^{\downarrow}\in S_a$

$$
\Psi^{221}_{S_a}(z_i^{\uparrow},z_{j'}^{\downarrow})=\prod_{\substack{i<j\\ i,j\in S_a}}(z_i^{\uparrow}-z_j^{\uparrow})^2\prod_{\substack{i'<j'\\ i',j'\in S_a}}(z_{i'}^{\downarrow}-z_{j'}^{\downarrow})^2\prod_{\substack{i,j'\\ i,j'\in S_a}}(z_i^{\uparrow}-z_{j'}^{\downarrow}).\tag{4.5}
$$

Note that from now on, we will put a prime on the index of spin-down particles. In some cases we will drop the arrows on the quasi-hole coordinates.

The validity of expression (4.4) is most easily understood from the characterization of the paired spin-singlet as the maximal-degree zero-energy eigenfunction of a specific three-body hamiltonian.

4.3.1. The case $n_3 = 4$

The wavefunction takes the form

$$
\begin{aligned}
&\Psi_{\mathrm{AS}}[3333](w_1,w_2,w_3,w_4;z_1^{\uparrow},\ldots,z_{N_\uparrow}^{\uparrow};z_{1'}^{\downarrow},\ldots,z_{N'_\downarrow}^{\downarrow})\\
&\quad=\langle\sigma_3(w_1)\sigma_3(w_2)\sigma_3(w_3)\sigma_3(w_4)\psi_1(z_1^{\uparrow})\cdots\psi_1(z_{N_\uparrow}^{\uparrow})\psi_2(z_{1'}^{\downarrow})\cdots\psi_2(z_{N'_\downarrow}^{\downarrow})\rangle\\
&\qquad\times\left[\widetilde{\Psi}^{(2,2,1)}_{\mathrm{H}}(z_i^{\uparrow};z_{j'}^{\downarrow})\right]^{1/2}\prod_{i,j}(z_i^{\uparrow}-w_j)^{\frac{1}{2}}\prod_{i',j}(z_{i'}^{\downarrow}-w_j)^{\frac{1}{2}}\prod_{i<j}(w_i-w_j)^{\frac{1}{3}}.
\end{aligned}\tag{4.6}
$$

Defining

$$\Psi_{ab,cd} = \frac{1}{2\mathcal{N}} \sum_{S_1,S_2} \left[\prod_{i,j' \in S_1} (z_i^{\uparrow} - w_a)(z_{j'}^{\downarrow} - w_a)(z_i^{\uparrow} - w_b)(z_{j'}^{\downarrow} - w_b) \right] \Psi_{S_1}^{221}(z_i^{\uparrow}, z_{j'}^{\downarrow})$$
$$\times \left[\prod_{k,l' \in S_2} (z_i^{\uparrow} - w_c)(z_{j'}^{\downarrow} - w_c)(z_i^{\uparrow} - w_d)(z_{j'}^{\downarrow} - w_d) \right] \Psi_{S_2}^{221}(z_k^{\uparrow}, z_{l'}^{\downarrow}). \tag{4.7}$$

we propose the following expression

$$\Psi_{\mathrm{AS}}^{(0,1)}[3333](w_1, w_2, w_3, w_4; z_1^{\uparrow}, \ldots, z_{N_\uparrow}^{\uparrow}; z_{1'}^{\downarrow}, \ldots, z_{N'_\downarrow}^{\downarrow})$$
$$= A^{(0,1)}[3333](\{w\})\Psi_{12,34}(\{w\},\{z\}) + B^{(0,1)}[3333](\{w\})\Psi_{13,24}(\{w\},\{z\}). \tag{4.8}$$

Following steps that are similar to those presented in Section 3.2, we can determine the coefficients in the master formula Eq. (4.8). The particular limits we employ are

$$\begin{aligned} &(\mathrm{I}) \quad z_1^{\uparrow} \to z_2^{\uparrow},\ z_{1'}^{\downarrow} \to z_{2'}^{\downarrow}, \\ &(\mathrm{II}) \quad z_{1'}^{\downarrow} \to z_{2'}^{\downarrow},\ z_1^{\uparrow} \to w_1,\ z_2^{\uparrow} \to w_2. \end{aligned} \tag{4.9}$$

They give

$$\begin{aligned} &\lim_{(\mathrm{I})} \langle \sigma_3\sigma_3\sigma_3\sigma_3\psi_1\psi_1\psi_2\psi_2 \rangle^{(0,1)} \propto \langle \sigma_3\sigma_3\sigma_3\sigma_3 \rangle^{(0,1)}, \\ &\lim_{(\mathrm{II})} \langle \sigma_3\sigma_3\sigma_3\sigma_3\psi_1\psi_1\psi_2\psi_2 \rangle^{(0,1)} \propto \langle \sigma_\uparrow\sigma_\uparrow\sigma_3\sigma_3 \rangle^{(0,1)}, \end{aligned} \tag{4.10}$$

leading to the result

$$\begin{aligned} A^{(0)}[3333] &= (w_{12}w_{34})^{\frac{4}{5}} x^{-\frac{2}{15}} (1-x)^{\frac{2}{3}} \mathcal{F}_2^{(0)}(x), \\ B^{(0)}[3333] &= (w_{12}w_{34})^{\frac{4}{5}} x^{-\frac{2}{15}} (1-x)^{\frac{2}{3}} \mathcal{F}_1^{(0)}(x), \\ A^{(1)}[3333] &= -(-1)^{\frac{3}{5}} C (w_{12}w_{34})^{\frac{4}{5}} x^{-\frac{2}{15}} (1-x)^{\frac{2}{3}} \mathcal{F}_2^{(1)}(x), \\ B^{(1)}[3333] &= -(-1)^{\frac{3}{5}} C (w_{12}w_{34})^{\frac{4}{5}} x^{-\frac{2}{15}} (1-x)^{\frac{2}{3}} \mathcal{F}_1^{(1)}(x). \end{aligned} \tag{4.11}$$

Here, $C = \frac{1}{3}\sqrt{\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}}$ and the functions $\mathcal{F}_i^{(p)}(x)$, with $p = 0, 1$ and $i = 1, 2$ are given in (B.11). Note that, while we use the same notation as in Section 3.2, the actual functions $\mathcal{F}_i^{(p)}(x)$ and the value of C differ between the two cases.

4.3.2. The case $n_\uparrow = n_\downarrow = 2$

The wavefunction reads

$$\Psi_{\mathrm{AS}}[\uparrow\uparrow\downarrow\downarrow](w_1^{\uparrow}, w_2^{\uparrow}; w_{3'}^{\downarrow}, w_{4'}^{\downarrow}; z_1^{\uparrow}, \ldots, z_{N_\uparrow}^{\uparrow}; z_1^{\downarrow}, \ldots, z_{N_\downarrow}^{\downarrow})$$
$$= \langle \sigma_\uparrow(w_1^{\uparrow})\sigma_\uparrow(w_2^{\uparrow})\sigma_\downarrow(w_{3'}^{\downarrow})\sigma_\downarrow(w_{4'}^{\downarrow})\psi_1(z_1^{\uparrow}) \cdots \psi_1(z_{N_\uparrow}^{\uparrow})\psi_2(z_1^{\downarrow}) \cdots \psi_2(z_{N_\downarrow}^{\downarrow}) \rangle$$
$$\times [\widetilde{\Psi}_{\mathrm{H}}^{(2,2,1)}(z_i^{\uparrow}; z_{j'}^{\downarrow})]^{1/2} \prod_{i,j} (z_i^{\uparrow} - w_j^{\uparrow})^{\frac{1}{2}} \prod_{i',j'} (z_{i'}^{\downarrow} - w_{j'}^{\downarrow})^{\frac{1}{2}}$$
$$\times \prod_{i<j} (w_i^{\uparrow} - w_j^{\uparrow})^{\frac{1}{3}} \prod_{i,j'} (w_i^{\uparrow} - w_{j'}^{\downarrow})^{-\frac{1}{6}} \prod_{i'<j'} (w_{i'}^{\downarrow} - w_{j'}^{\downarrow})^{\frac{1}{3}}. \tag{4.12}$$

The master formula now reads (from now on, we will label the quasi-holes consecutively, to avoid confusion when braiding quasi-holes; as a reminder, we will put primes on the labels of the spin-down quasi-holes)

$$
\begin{aligned}
&\Psi_{\mathrm{AS}}^{(0,1)}[\uparrow\uparrow\downarrow\downarrow](w_1^{\uparrow}, w_2^{\uparrow}; w_{3'}^{\downarrow}, w_{4'}^{\downarrow}; z_1^{\uparrow}, \ldots, z_{N_\uparrow}^{\uparrow}; z_{1'}^{\downarrow}, \ldots, z_{N'_\downarrow}^{\downarrow}) \\
&\quad = A^{(0,1)}[\uparrow\uparrow\downarrow\downarrow](\{w\})\Psi_{13',24'}(\{w\},\{z\}) + B^{(0,1)}[\uparrow\uparrow\downarrow\downarrow](\{w\})\Psi_{14',23'}(\{w\},\{z\}).
\end{aligned}
\tag{4.13}
$$

with

$$
\Psi_{13',24'} = \frac{1}{2\mathcal{N}} \sum_{S_1,S_2} \left[\prod_{i,j'\in S_1} (z_i^{\uparrow} - w_1^{\uparrow})(z_{j'}^{\downarrow} - w_{3'}^{\downarrow})\right] \Psi_{S_1}^{221}(z_i^{\uparrow}, z_{j'}^{\downarrow}) \left[\prod_{k,l'\in S_2} (z_k^{\uparrow} - w_2^{\uparrow})(z_{l'}^{\downarrow} - w_{4'}^{\downarrow})\right] \Psi_{S_2}^{221}(z_k^{\uparrow}, z_{l'}^{\downarrow}).
\tag{4.14}
$$

and similarly $\Psi_{14',23'}$. Note that in this case there is no natural third way to distribute the quasi-holes over S_1, S_2.

To determine $A^{(0,1)}[\uparrow\uparrow\downarrow\downarrow]$ and $B^{(0,1)}[\uparrow\uparrow\downarrow\downarrow]$ we put $N_\uparrow = N_\downarrow = 2$ and consider the limits

$$
\begin{aligned}
&\text{(I)} \quad z_1^{\uparrow} \to z_2^{\uparrow}, \quad z_{1'}^{\downarrow} \to z_{2'}^{\downarrow}, \\
&\text{(II)} \quad z_1^{\uparrow} \to w_1^{\uparrow}, \quad z_2^{\uparrow} \to w_2^{\uparrow}, \quad z_{1'}^{\downarrow} \to w_{3'}^{\downarrow}, \quad z_{2'}^{\downarrow} \to w_{4'}^{\downarrow}.
\end{aligned}
\tag{4.15}
$$

They give

$$
\begin{aligned}
&\lim_{\text{(I)}} \langle \sigma_\uparrow\sigma_\uparrow\sigma_\downarrow\sigma_\downarrow\psi_1\psi_1\psi_2\psi_2\rangle^{(0,1)} \propto \langle \sigma_\uparrow\sigma_\uparrow\sigma_\downarrow\sigma_\downarrow\rangle^{(0,1)} \\
&\lim_{\text{(II)}} \langle \sigma_\uparrow\sigma_\uparrow\sigma_\downarrow\sigma_\downarrow\psi_1\psi_1\psi_2\psi_2\rangle^{(0,1)} \propto \langle \sigma_3\sigma_3\sigma_3\sigma_3\rangle^{(0,1)}.
\end{aligned}
\tag{4.16}
$$

In the (0) channel this gives the equations

$$
\begin{aligned}
&\text{(I)} \quad (A+B)^{(0)}[\uparrow\uparrow\downarrow\downarrow] = (w_{12}w_{3'4'})^{-\frac{1}{5}} x^{\frac{13}{15}} (1-x)^{-\frac{1}{3}} \mathcal{F}_1^{(0)}(x), \\
&\text{(II)} \quad \left(\sqrt{1-x}\,A + \frac{1}{\sqrt{1-x}}\,B\right)^{(0)} [\uparrow\uparrow\downarrow\downarrow] = (w_{12}w_{3'4'})^{-\frac{1}{5}} x^{\frac{13}{15}} (1-x)^{\frac{1}{6}} [\mathcal{F}_1^{(0)}(x) + \mathcal{F}_2^{(0)}(x)].
\end{aligned}
\tag{4.17}
$$

The equations in channel (1) have a similar structure. The solutions are

$$
\begin{aligned}
A^{(0)}[\uparrow\uparrow\downarrow\downarrow] &= (w_{12}w_{3'4'})^{-\frac{1}{5}} x^{\frac{13}{15}} (1-x)^{-\frac{1}{3}} \left[\mathcal{F}_1^{(0)}(x) - \frac{1-x}{x}\mathcal{F}_2^{(0)}(x)\right], \\
B^{(0)}[\uparrow\uparrow\downarrow\downarrow] &= (w_{12}w_{3'4'})^{-\frac{1}{5}} x^{\frac{13}{15}} (1-x)^{-\frac{1}{3}} \left[\frac{1-x}{x}\mathcal{F}_2^{(0)}(x)\right], \\
A^{(1)}[\uparrow\uparrow\downarrow\downarrow] &= -(-1)^{\frac{3}{5}} C (w_{12}w_{3'4'})^{-\frac{1}{5}} x^{\frac{13}{15}} (1-x)^{-\frac{1}{3}} \left[\mathcal{F}_1^{(1)}(x) - \frac{1-x}{x}\mathcal{F}_2^{(1)}(x)\right], \\
B^{(1)}[\uparrow\uparrow\downarrow\downarrow] &= -(-1)^{\frac{3}{5}} C (w_{12}w_{3'4'})^{-\frac{1}{5}} x^{\frac{13}{15}} (1-x)^{-\frac{1}{3}} \left[\frac{1-x}{x}\mathcal{F}_2^{(1)}(x)\right].
\end{aligned}
\tag{4.18}
$$

Interchanging the positions of $\sigma_\uparrow(w_2)$ and $\sigma_\downarrow(w_{3'})$ gives a different basis for the four quasi-hole wavefunctions in this sector. With

$$\Psi_{\mathrm{AS}}^{(0,1)}[\uparrow\downarrow\uparrow\downarrow](w_1^\uparrow,w_{2'}^\downarrow;w_3^\uparrow,w_{4'}^\downarrow;z_1^\uparrow,\ldots,z_{N_\uparrow}^\uparrow;z_{1'}^\downarrow,\ldots,z_{N'_\downarrow}^\downarrow)$$
$$=A^{(0,1)}[\uparrow\downarrow\uparrow\downarrow](\{w\})\Psi_{12',34'}(\{w\},\{z\})+B^{(0,1)}[\uparrow\downarrow\uparrow\downarrow](\{w\})\Psi_{14',32'}(\{w\},\{z\}). \tag{4.19}$$

the coefficients are found to be

$$\begin{aligned}
A^{(0)}[\uparrow\downarrow\uparrow\downarrow] &= (-1)^{\frac{1}{2}}(w_{12'}w_{34'})^{-\frac{1}{5}}x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\Big[\mathcal{F}_2^{(0)}(x)-\frac{x}{1-x}\mathcal{F}_1^{(0)}(x)\Big],\\
B^{(0)}[\uparrow\downarrow\uparrow\downarrow] &= (-1)^{\frac{1}{2}}(w_{12'}w_{34'})^{-\frac{1}{5}}x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\Big[\frac{x}{1-x}\mathcal{F}_1^{(0)}(x)\Big],\\
A^{(1)}[\uparrow\downarrow\uparrow\downarrow] &= (-1)^{\frac{1}{10}}C(w_{12'}w_{34'})^{-\frac{1}{5}}x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\Big[\mathcal{F}_2^{(1)}(x)-\frac{x}{1-x}\mathcal{F}_1^{(1)}(x)\Big],\\
B^{(1)}[\uparrow\downarrow\uparrow\downarrow] &= (-1)^{\frac{1}{10}}C(w_{12'}w_{34'})^{-\frac{1}{5}}x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\Big[\frac{x}{1-x}\mathcal{F}_1^{(1)}(x)\Big].
\end{aligned} \tag{4.20}$$

4.3.3. The case $n_\uparrow = 4$

The wavefunction takes the form

$$\begin{aligned}
&\Psi_{\mathrm{AS}}[\uparrow\uparrow\uparrow\uparrow](w_1^\uparrow,w_2^\uparrow,w_3^\uparrow,w_4^\uparrow;z_1^\uparrow,\ldots,z_{N_\uparrow}^\uparrow;z_{1'}^\downarrow,\ldots,z_{N'_\downarrow}^\downarrow)\\
&\quad=\langle\sigma_\uparrow(w_1^\uparrow)\sigma_\uparrow(w_2^\uparrow)\sigma_\uparrow(w_3^\uparrow)\sigma_\uparrow(w_4^\uparrow)\psi_1(z_1^\uparrow)\cdots\psi_1(z_{N_\uparrow}^\uparrow)\psi_2(z_{1'}^\downarrow)\cdots\psi_2(z_{N'_\downarrow}^\downarrow)\rangle\\
&\qquad\times[\widetilde{\Psi}_{\mathrm{H}}^{(2,2,1)}(z_i^\uparrow;z_{j'}^\downarrow)]^{1/2}\prod_{i,j}(z_i^\uparrow-w_j^\uparrow)^{\frac{1}{2}}\prod_{i<j}(w_i^\uparrow-w_j^\uparrow)^{\frac{1}{3}}.
\end{aligned} \tag{4.21}$$

From now on, we will drop the up-arrow on the quasi-hole coordinates $\{w_i\}$. Defining

$$\Psi_{ab,cd}=\frac{1}{2\mathcal{N}}\sum_{S_1,S_2}\Bigg[\prod_{i\in S_1}(z_i^\uparrow-w_a)(z_i^\uparrow-w_b)\Bigg]\Psi_{S_1}^{221}(z_i^\uparrow,z_{j'}^\downarrow)\Bigg[\prod_{i\in S_2}(z_i^\uparrow-w_c)(z_i^\uparrow-w_d)\Bigg]\Psi_{S_2}^{221}(z_k^\uparrow,z_{l'}^\downarrow), \tag{4.22}$$

we propose the following expression

$$\Psi_{\mathrm{AS}}^{(0,1)}[\uparrow\uparrow\uparrow\uparrow](w_1,w_2,w_3,w_4;z_1^\uparrow,\ldots,z_{N_\uparrow}^\uparrow;z_{1'}^\downarrow,\ldots,z_{N'_\downarrow}^\downarrow)$$
$$=A^{(0,1)}[\uparrow\uparrow\uparrow\uparrow](\{w\})\Psi_{12,34}(\{w\},\{z\})+B^{(0,1)}[\uparrow\uparrow\uparrow\uparrow](\{w\})\Psi_{13,24}(\{w\},\{z\}). \tag{4.23}$$

To determine the coefficients in this master formula, we set $N_\uparrow=2$, $N_\downarrow=6$ and take the two limits

$$\begin{aligned}
&(\mathrm{I})\quad z_{1'}^\downarrow\to z_1^\uparrow,\ z_{2'}^\downarrow\to z_2^\uparrow,\ z_{4'}^\downarrow\to z_{3'}^\downarrow,\ z_{6'}^\downarrow\to z_{5'}^\downarrow,\ z_1^\uparrow\to w_2,\ z_2^\uparrow\to w_4,\\
&(\mathrm{II})\quad z_{1'}^\downarrow\to z_1^\uparrow,\ z_{2'}^\downarrow\to z_2^\uparrow,\ z_{4'}^\downarrow\to z_{3'}^\downarrow,\ z_{6'}^\downarrow\to z_{5'}^\downarrow,\ z_1^\uparrow\to w_3,\ z_2^\uparrow\to w_4,
\end{aligned} \tag{4.24}$$

which give

$$\begin{aligned}
&\lim_{(\mathrm{I})}\langle\sigma_\uparrow\sigma_\uparrow\sigma_\uparrow\sigma_\uparrow\psi_1\psi_1\psi_2\psi_2\psi_2\psi_2\psi_2\psi_2\rangle^{(0,1)}\propto\langle\sigma_\uparrow\sigma_\downarrow\sigma_\uparrow\sigma_\downarrow\rangle^{(0,1)},\\
&\lim_{(\mathrm{II})}\langle\sigma_\uparrow\sigma_\uparrow\sigma_\uparrow\sigma_\uparrow\psi_1\psi_1\psi_2\psi_2\psi_2\psi_2\psi_2\psi_2\rangle^{(0,1)}\propto\langle\sigma_\uparrow\sigma_\uparrow\sigma_\downarrow\sigma_\downarrow\rangle^{(0,1)}.
\end{aligned} \tag{4.25}$$

Using the spin-field correlators given in Eq. (B.10) we find

$$
\begin{aligned}
A^{(0)}[\uparrow\uparrow\uparrow\uparrow] &= (w_{12}w_{34})^{\frac{4}{5}}x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\mathcal{F}_2^{(0)}(x),\\
B^{(0)}[\uparrow\uparrow\uparrow\uparrow] &= (w_{12}w_{34})^{\frac{4}{5}}x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\mathcal{F}_1^{(0)}(x),\\
A^{(1)}[\uparrow\uparrow\uparrow\uparrow] &= -(-1)^{\frac{3}{5}}C(w_{12}w_{34})^{\frac{4}{5}}x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\mathcal{F}_2^{(1)}(x),\\
B^{(1)}[\uparrow\uparrow\uparrow\uparrow] &= -(-1)^{\frac{3}{5}}C(w_{12}w_{34})^{\frac{4}{5}}x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\mathcal{F}_1^{(1)}(x).
\end{aligned}
\tag{4.26}
$$

Because the spin-up quasi-holes have to satisfy exactly the same braid properties as the spin-less quasi-holes for $M = 0$, it is not surprising at all that the functional form of the functions (4.26) is exactly the same as in (4.11).

4.3.4. The case $n_\uparrow = 3$, $n_\downarrow = 1$

The wavefunction is

$$
\begin{aligned}
&\Psi_{\mathrm{AS}}[\uparrow\uparrow\uparrow\downarrow](w_1^\uparrow, w_2^\uparrow, w_3^\uparrow; w_{4'}^\downarrow; z_1^\uparrow, \ldots, z_{N_\uparrow}^\uparrow; z_{1'}^\downarrow, \ldots, z_{N'_\downarrow}^\downarrow)\\
&\quad = \langle \sigma_\uparrow(w_1^\uparrow)\sigma_\uparrow(w_2^\uparrow)\sigma_\uparrow(w_3^\uparrow)\sigma_\uparrow(w_{4'}^\downarrow)\psi_1(z_1^\uparrow)\cdots\psi_1(z_{N_\uparrow}^\uparrow)\psi_2(z_{1'}^\downarrow)\cdots\psi_2(z_{N'_\downarrow}^\downarrow)\rangle\\
&\qquad \times \left[\widetilde{\Psi}_{\mathrm{H}}^{(2,2,1)}(z_i^\uparrow; z_{j'}^\downarrow)\right]^{1/2} \prod_{i,j}(z_i^\uparrow - w_j^\uparrow)^{\frac{1}{2}} \prod_{i'}(z_{i'}^\downarrow - w_{4'}^\downarrow)^{\frac{1}{2}}\\
&\qquad \times \prod_{i<j}(w_i^\uparrow - w_j^\uparrow)^{\frac{1}{3}} \prod_i (w_i^\uparrow - w_{4'}^\downarrow)^{-\frac{1}{6}}.
\end{aligned}
\tag{4.27}
$$

In this case, we will in general have $N_\uparrow = 2r + 1$ spin-up electrons and $N_\downarrow = 2r + 3$ spin-down electrons, with r an integer. Thus, to define the functions $\Psi_{ab,cd}$, we will divide the electrons in two groups, where the first group S_1 contains $(N_\uparrow - 1)/2$ spin-up electrons and $(N_\downarrow - 1)/2 + 1$ spin-down electrons. The second group S_2 has the remaining $(N_\uparrow - 1)/2 + 1$ spin-up electrons, and the remaining $(N_\downarrow - 1)/2$ spin-down electrons. We can now define

$$
\Psi_{ab,cd'} = \frac{1}{\mathcal{N}'}\sum_{S_1,S_2}\left[\prod_{i\in S_1}(z_i^\uparrow - w_a^\uparrow)(z_i^\uparrow - w_b^\uparrow)\right]\Psi_{S_1}^{221}(z_i^\uparrow, z_{j'}^\downarrow)\left[\prod_{i,j'\in S_2}(z_i^\uparrow - w_c^\uparrow)(z_{j'}^\downarrow - w_d^\downarrow)\right]\Psi_{S_2}^{221}(z_k^\uparrow, z_{l'}^\downarrow),
\tag{4.28}
$$

with the normalization $\mathcal{N}' = 2^{\frac{N_\uparrow+1}{2}}2^{\frac{N_\downarrow+1}{2}}$, where the sum is over all ways of dividing the electrons into the two groups. Note that we again have three different ways of splitting the quasi-holes. However, the relation between them differs form the 'usual' relation, because we now have

$$
\Psi_{23,14'} = x\frac{w_{14'}}{w_{34'}}\Psi_{12,34'} - (1-x)\frac{w_{14'}}{w_{4'2}}\Psi_{13,24'}.
\tag{4.29}
$$

The master formula now reads

$$
\begin{aligned}
&\Psi_{\mathrm{AS}}^{(0,1)}[\uparrow\uparrow\uparrow\downarrow]\left(w_1^\uparrow, w_2^\uparrow, w_3^\uparrow; w_{4'}^\downarrow; z_1^\uparrow, \ldots, z_{N_\uparrow}^\uparrow; z_{1'}^\downarrow, \ldots, z_{N'_\downarrow}^\downarrow\right)\\
&\quad = A^{(0,1)}[\uparrow\uparrow\uparrow\downarrow](\{w\})\Psi_{12,34'}(\{w\},\{z\}) + B^{(0,1)}[\uparrow\uparrow\uparrow\downarrow](\{w\})\Psi_{13,24'}(\{w\},\{z\}).
\end{aligned}
\tag{4.30}
$$

Specifying $N_\uparrow = 1$ and $N_\downarrow = 3$, and taking the limits

$$
\begin{aligned}
&(\mathrm{I}) \quad z_{1'}^\downarrow \to z_1^\uparrow,\ z_{3'}^\downarrow \to z_{2'}^\downarrow,\ z_1^\uparrow \to w_2^\uparrow,\\
&(\mathrm{II}) \quad z_{1'}^\downarrow \to z_1^\uparrow,\ z_{3'}^\downarrow \to z_{2'}^\downarrow,\ z_1^\uparrow \to w_3^\uparrow,
\end{aligned}
\tag{4.31}
$$

we obtain the following result

$$
\begin{aligned}
A^{(0)}[\uparrow\uparrow\uparrow\downarrow] &= \frac{(w_{12}w_{34'})^{\frac{4}{5}}}{w_{34'}} x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\mathcal{F}_2^{(0)}(x), \\
B^{(0)}[\uparrow\uparrow\uparrow\downarrow] &= -\frac{(w_{12}w_{34'})^{\frac{4}{5}}}{w_{4'2}} x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\mathcal{F}_1^{(0)}(x), \\
A^{(1)}[\uparrow\uparrow\uparrow\downarrow] &= -(-1)^{\frac{3}{5}}C\frac{(w_{12}w_{34'})^{\frac{4}{5}}}{w_{34'}} x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\mathcal{F}_2^{(1)}(x), \\
B^{(1)}[\uparrow\uparrow\uparrow\downarrow] &= (-1)^{\frac{3}{5}}C\frac{(w_{12}w_{34'})^{\frac{4}{5}}}{w_{4'2}} x^{-\frac{2}{15}}(1-x)^{\frac{2}{3}}\mathcal{F}_1^{(1)}(x).
\end{aligned}
\tag{4.32}
$$

Note that in calculating the braid relations, the 'extra' factors of $w_{34'}$ and $w_{4'2}$ are precisely 'compensated' by the additional factors in the relation between the $\Psi_{ab,cd'}$'s in Eq. (4.29).

4.4. Braiding relations

The braid properties of the various four quasi-hole wavefunctions are easily evaluated using the transformation properties of the functions $\mathcal{F}_i^{(p)}(x)$ as specified in Section B.5. We display them here for general M.

For braiding the neutral quasi-holes of charge $2/(4M+3)$, denoted by the label '3', we obtain the following results

$$
\begin{aligned}
&\Psi_{\mathrm{AS}}^{(p)}[3333] \overset{1\leftrightarrow 2}{\rightarrow} (U_{12})_q^p \Psi_{\mathrm{AS}}^{(q)}[3333], \quad U_{12} = (-1)^{-\frac{2}{3}-\frac{4M}{3(4M+3)}} \begin{pmatrix} (-1)^{\frac{4}{5}} & 0 \\ 0 & (-1)^{-\frac{3}{5}} \end{pmatrix}, \\
&\Psi_{\mathrm{AS}}^{(p)}[3333] \overset{2\leftrightarrow 3}{\rightarrow} (U_{23})_q^p \Psi_{\mathrm{AS}}^{(q)}[3333], \quad U_{23} = (-1)^{\frac{4}{5}-\frac{4M}{4M+3}} \begin{pmatrix} \tau & (-1)^{-\frac{3}{5}}\sqrt{\tau} \\ (-1)^{\frac{3}{5}}\sqrt{\tau} & -\tau \end{pmatrix}, \\
&\Psi_{\mathrm{AS}}^{(p)}[3333] \overset{1\leftrightarrow 3}{\rightarrow} (U_{13})_q^p \Psi_{\mathrm{AS}}^{(q)}[3333], \quad U_{13} = (-1)^{\frac{8}{15}-\frac{4M}{3(4M+3)}} \begin{pmatrix} \tau & -(-1)^{-\frac{1}{5}}\sqrt{\tau} \\ \sqrt{\tau} & (-1)^{-\frac{1}{5}}\tau \end{pmatrix}.
\end{aligned}
\tag{4.33}
$$

For braiding spin-full quasi-holes, of charge $1/(4M+3)$, the corresponding matrices are

$$
\begin{aligned}
U_{12} &= (-1)^{-\frac{2}{3}-\frac{M}{3(4M+3)}} \begin{pmatrix} (-1)^{\frac{4}{5}} & 0 \\ 0 & (-1)^{-\frac{3}{5}} \end{pmatrix}, \\
U_{23} &= (-1)^{\frac{4}{5}-\frac{M}{4M+3}} \begin{pmatrix} \tau & (-1)^{-\frac{3}{5}}\sqrt{\tau} \\ (-1)^{\frac{3}{5}}\sqrt{\tau} & -\tau \end{pmatrix}, \\
U_{13} &= (-1)^{\frac{8}{15}-\frac{M}{3(4M+3)}} \begin{pmatrix} \tau & -(-1)^{-\frac{1}{5}}\sqrt{\tau} \\ \sqrt{\tau} & (-1)^{-\frac{1}{5}}\tau \end{pmatrix}.
\end{aligned}
\tag{4.34}
$$

These matrices are found by explicit inspection of the wavefunctions for the cases ($n_\uparrow = 3$, $n_\downarrow = 1$) and $n_\uparrow = 4$. For ($n_\uparrow = 2$, $n_\downarrow = 2$) equivalent matrices are found for situations where the braiding does not mix the spin-labels, such as $1 \leftrightarrow 2$ for $\Psi^{(p)}_{\mathrm{AS}}[\uparrow\uparrow\downarrow\downarrow]$ and $1 \leftrightarrow 3$ for $\Psi^{(p)}_{\mathrm{AS}}[\uparrow\downarrow\uparrow\downarrow]$.

The different M-dependence between Eqs. (4.33) and (4.34) reflects the fact that for $M \neq 0$ the conformal dimension of σ_3 differs from that of $\sigma_\uparrow$ and $\sigma_\downarrow$. For $M = 0$ the three quasi-hole types are connected by an $su(3)$ symmetry and the braiding properties are necessarily the same.

By a simple change of basis ('gauge transformation') the matrices U_{12}, U_{23} and U_{13} acquire a form which can be identified with that of the R, F and B matrices for the Fibonacci anyons, Eq. (1.2), up to overall M-dependent phase factors. This establishes the suitability of this particular quantum register for universal topological quantum computation.

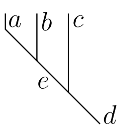
5. Quantum group approach

In this section, we will use the CFT-quantum group connection to calculate the braid properties of the quasi-holes, and confirm that the results from this approach is indeed consistent with the results obtained here from the explicit wavefunctions for the quasi-hole states.

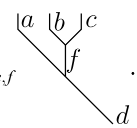
The approach used here follows the lines of Slingerland and Bais, [22], to which we refer for details and more references. More details on the CFT-quantum group connection can be found in [34,35]. For the $su(2)_k$ case in particular, see [36]. More details about the quantum groups themselves can be found in, for instance, [38].

At a basic level, the connection between conformal field theory and quantum groups states that the braid properties of fields in the conformal field theory are the same as the braid properties of particles carrying a quantum group representation. Because the latter are specified by the R-matrix of the quantum group, one can calculate the braid properties of the quasi-holes in an algebraic way. In addition to the R-matrix, one will also need to know the 6j-symbols, because to describe general braidings, on needs to know how to change between the different bases of the tensor product of three representations. This information is encoded in the (q-deformed) 6j-symbols, or the F-matrices, see for instance, [20] for a nice review.

To be more explicit, the F-matrices describe the basis transformation between the two different ways in which one can take the tensor product, or fusion, of three representations, or particle types. The first way is to first fuse a particle of type a with a particle of type b, which gives, say, a particle of type e. Finally, one fuses this particle e which the third particle c, with particle d as outcome. The other way of fusing particles a, b and c is to first fuse b and c into f, which is fused with a to give d. Pictorially, we can describe the relation between these two bases as

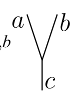


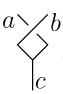
$$= \sum (F_d^{a,b,c})_{e,f}$$



. (5.1)

The exchange of two particles a and b in a definite fusion channel c is described by the R-matrix. Pictorially, we have

$R_c^{a,b}$ 

$=$ 

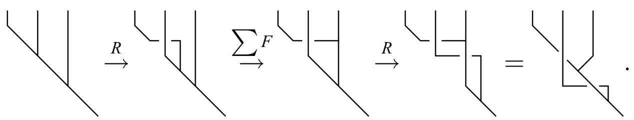
. (5.2)

In order that the F and R matrices describe consistent braiding, they have to satisfy consistency conditions, which go under the name of the pentagon and hexagon equations [21].

The F and R matrices obtained from the quantum groups automatically satisfy these equations.

We can now express the braiding of quasi-holes in terms of the R and F-matrices. The braiding of the quasi-holes 1 and 2 (using the notation of Sections 3 and 4), is simply given by the elements of the R matrix, and depends on the fusion channel.

The exchange of particles 2 and 3 can be done in more than one way. In our case, it turns out that we first need to exchange particles 1 and 2, followed by acting with the F-matrix, and finally, exchanging the third particle with the 'intermediate' particle. Pictorially, we have (for clarity, we dropped the labels a, b, etc.)



This corresponds to the following form of the braid matrix U_{23}

$$(U_{23})_{e,f} = R_d^{b,f} (F_d^{a,b,c})_f^e R_e^{a,b}. \tag{5.3}$$

Note that we do not sum over repeated indices. Similarly, we have the following expression for U_{13}

$$(U_{13})_{e,f} = R_d^{a,f} (R_f^{b,c})^{-1} (F_d^{a,b,c})_f^e. \tag{5.4}$$

We will use the expressions (5.3) and (5.4) in the following subsections to compare the braid results of the previous sections with the results obtained using the quantum groups. Note that by changing the relative phase of the different blocks, we can change the relative phase between the off diagonal elements of the matrices U_{13} and U_{23}. Thus, only the sum of the off-diagonal phases is a physical quantity. Nevertheless, we reproduce the matrices exactly.

We will not explain in detail how to obtain the F and R matrices from the quantum groups. For this, we refer to [22] and a forthcoming paper [37], in which details will be given on the calculation of the F and R matrices in the case of $U_q(su(3))$. Here, we will be brief in our description, and quote the results we need from the literature.

A quantum group associated to a Lie algebra is a q-deformation of its universal enveloping algebra. At generic values of q, the representations of the quantum group are similar to the representations of the Lie algebra. However, if q equals specific roots of unity (which we will specify below in the cases of our interest), the representation theory is rather different. Concentration on the case $U_q(su(2))$ (see, for instance, [22], where the quantum group picture was used to calculate the braid behaviour of the quasi-holes of the Read–Rezayi states), we find that for $q = e^{\frac{2\pi i}{k+2}}$, there are $k+1$ unitary highest weight representations. In addition, the tensor product of such representations gets truncated in comparison to the tensor product of $su(2)$. In fact, the truncated tensor products are equivalent to the fusion rules of $su(2)_k$. The braid properties of the quasi-holes are in fact described by the F and R matrices of the quantum group at these special values of q.

The calculation of the F matrices can be done in a similar fashion as in the case of ordinary groups, by first calculating the Clebsch–Gordan coefficients, and from those the $6j$-coefficients. Of course, in the quantum group case, one has to use the q-deformed raising and lowering operators.

Generically, the F-matrices can be expressed in terms of the so-called q numbers $\lfloor n \rfloor$, which are defined as

$$\lfloor n \rfloor = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{5.5}$$

or, equivalently, $\lfloor n \rfloor = \sum_{i=1}^{n} q^{\frac{n+1}{2}-i}$. Note that for $q = 1$, we have $\lfloor n \rfloor = n$.

For the Read–Rezayi states with $k = 3$ and the paired spin-singlet states proposed by the authors, the corresponding value of q is $q = e^{\frac{2\pi i}{5}}$, for which we have $\lfloor 0 \rfloor = 0$, $\lfloor 1 \rfloor = \lfloor 4 \rfloor = 1$, $\lfloor 2 \rfloor = \lfloor 3 \rfloor = \frac{1+\sqrt{5}}{2}$ and $\lfloor n+5 \rfloor = -\lfloor n \rfloor$ for $n \in \mathbb{N}$.

5.1. The case $su(2)_k$

To calculate the braid properties of the quasi-holes over the Read–Rezayi states, we need to specify to which $su(2)_k$ representation they correspond, and the possible fusion channels as well. Because all the braid properties are encoded by the braid properties of four quasi-holes and because we know the four quasi-hole correlators, we will focus on that case. The corresponding quantum group is $U_q(su(2))$, with $q = e^{\frac{2\pi i}{k+2}}$. The quasi-holes correspond to the representation $l = 1$ (i.e. spin $\frac{1}{2}$), and the two fusion channels to the representations $l = 0$ and $l = 2$.

Let us focus on the matrix U_{12} first. This matrix describes the braiding of the first two quasi-holes, which depends on the fusion channel. For $k = 3$, we have $R_0^{1,1} = (-1)^{\frac{7}{10}}$ and $R_2^{1,1} = (-1)^{\frac{1}{10}}$. This is in agreement with the matrix U_{12} of Eq. (3.14).

For general k, we have $R_0^{1,1} = -q^{-\frac{3}{4}} = (-1)^{\frac{2k+1}{2(k+2)}}$ and $R_2^{1,1} = q^{\frac{1}{4}} = (-1)^{\frac{1}{2(k+2)}}$, which confirms U_{12} of Eq. (A.42). Note that the M dependence easily follows from the general form of the wavefunction, Eq. (A.33).

In addition, we have for $k = 3$

$$F_1^{1,1,1} = \begin{pmatrix} -\tau & \sqrt{\tau} \\ \sqrt{\tau} & \tau \end{pmatrix}. \tag{5.6}$$

For general k, we obtain the following result

$$F_1^{1,1,1} = \frac{1}{\lfloor 2 \rfloor} \begin{pmatrix} -1 & \sqrt{\lfloor 3 \rfloor} \\ \sqrt{\lfloor 3 \rfloor} & 1 \end{pmatrix} = \frac{1}{d_k} \begin{pmatrix} -1 & \sqrt{d_k^2 - 1} \\ \sqrt{d_k^2 - 1} & 1 \end{pmatrix}, \tag{5.7}$$

where $d_k = 2\cos(\frac{\pi}{k+2})$ is the quantum dimension of the fundamental representation of $su(2)_k$.

Now, the matrix U_{23} corresponds to

$$(U_{23})_{e,f} = R_d^{b,f} (F_d^{a,b,c})_f^e R_e^{a,b}, \tag{5.8}$$

with $a = b = c = d = 1$ and both e and f can take the values $0, 2$. Note that we do not sum over repeated indices. Similarly, U_{13} corresponds to

$$(U_{13})_{e,f} = R_d^{a,f} (R_f^{b,c})^{-1} (F_d^{a,b,c})_f^e, \tag{5.9}$$

also with $a = b = c = d = 1$ and $e, f = 0, 2$.

Using these results for the R and F matrices, we easily see that the $k = 3$ results of Eq. (3.14) and the general results of Eq. (A.42) are reproduced exactly.

We would like to note that the $k = 3$ braid matrices are, up to an overall factor, directly related to the braid matrices of the Fibonacci theory. Note that they are in fact related to the 'mirror-image' of the matrices (1.2). This is related to the fact that the F matrix of the spin-1/2 particles in the $su(2)_3$ theory is given by Eq. (5.6) instead of Eq. (1.2).

5.2. The case $su(3)_2$

We will now compare the results of braiding in the spin-singlet case to the results which can be obtained by using a quantum group picture. To do this, we will have to know the 6*j*-symbols (or F-symbols) of the quantum group of $su(3)$. The results of a direct calculation of these 6*j*-symbols for general q will be presented elsewhere [37], together with the R-matrix 'eigenvalues' and a detailed analysis of the 6*j*-symbols related to cosets. Here, we will merely quote a small number of F-symbols and braid factors.

For the comparison of the braid matrices with the results obtained from the quantum group picture, it is easiest to work with a set of parafermion fields which stay as close as possible to the representations used in the KZ equation, see Appendix B.1. Hence, we will work with the representatives $a = \mathbf{3}$, $b = \mathbf{3}$, $c = \mathbf{3}$, and $d = \mathbf{8}$. Note that we cannot take $\mathbf{3}$ as the last representative as well. The reason is that after fusing the first two fields, we find that the possible channels correspond to $\mathbf{6}$ or $\bar{\mathbf{3}}$. Fusing these intermediate channels automatically gives us the $\mathbf{8}$. It follows that we need the following data, which can be obtained from the quantum group of $su(3)$ [37]

$$F_{\mathbf{8}}^{\mathbf{3},\mathbf{3},\mathbf{3}} = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}, \tag{5.10}$$

where the intermediate fusion channels are $\mathbf{6}$ and $\bar{\mathbf{3}}$ (in that order) and

$$R_{\mathbf{6}}^{\mathbf{3},\mathbf{3}} = (-1)^{\frac{2}{15}} \quad R_{\bar{\mathbf{3}}}^{\mathbf{3},\mathbf{3}} = (-1)^{\frac{11}{15}} \quad R_{\mathbf{8}}^{\mathbf{3},\mathbf{6}} = (-1)^{\frac{2}{3}} \quad R_{\mathbf{8}}^{\mathbf{3},\bar{\mathbf{3}}} = (-1)^{\frac{1}{15}}. \tag{5.11}$$

The symbols $R_{\mathbf{6}}^{\mathbf{3},\mathbf{3}}$ and $R_{\bar{\mathbf{3}}}^{\mathbf{3},\mathbf{3}}$ correspond to the diagonal elements of U_{12} as they should. In addition, upon using Eqs. (5.3) and (5.4), we obtain the matrices of Eq. (4.34).

5.3. $su(3)_2$ parafermion correlators

To verify the braid behaviour of the various $su(3)_2$ parafermion correlators, we will use the following data

$$F_{\mathbf{8}}^{\mathbf{8},\mathbf{8},\mathbf{8}} = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}, \tag{5.12}$$

where the intermediate fusion channels are $\mathbf{1}$ and $\mathbf{8}$ and

$$R_{\mathbf{8}}^{\mathbf{8},\mathbf{1}} = R_{\mathbf{8}}^{\mathbf{1},\mathbf{8}} = 1 \quad R_{\mathbf{1}}^{\mathbf{8},\mathbf{8}} = (-1)^{-\frac{6}{5}} \quad R_{\mathbf{8}}^{\mathbf{8},\mathbf{8}} = (-1)^{-\frac{3}{5}}. \tag{5.13}$$

With this data, we exactly obtain the braid behaviour which can be derived from the correlator of four $\bar{\rho}_1$ fields in Eq. (B.13). In addition, we obtain the braid behaviour of four σ_3 fields of Eq. (B.10) up to an overall sign. To explain the origin of this sign, we note that if one expresses the $SU(3)_2$ WZW primary in the adjoint representation in terms of the parafermion field σ_3, additional $u(1)$ factors are needed, see Eq. (B.4). These $u(1)$ factors give rise to the additional sign. Note that in the ρ sector, these $u(1)$ factors are absent.

Acknowledgments

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Appendix A. Detailed structure of the $su(2)_k$ parafermion theory

In this appendix, we will review the structure of the $\mathbb{Z}_k$ parafermion conformal field theory, as introduced in [30]. In fact, we will use the general setting of [33], so we can use the results to explain the details of the parafermion theory associated to $su(3)_2$ in the next appendix as well.

In the case of the $su(3)_2$ parafermions, which we will describe in the next appendix, it will be important to make a distinction between the chiral sectors of the parafermion theory, and the Virasoro primary fields. Each chiral sector contains an infinite number of Virasoro primary fields. However, we will often use the same notation for the chiral sector and the leading Virasoro primary field. For the chiral sectors we consider in this appendix, there will always be only one leading virasoro primary field. However, in the case of the $su(3)_2$ parafermions, there will be one chiral sector containing two leading Virasoro primary fields.

The fusion rules (which will be specified for the $su(2)_k$ parafermions below) describe the 'merging' of two parafermion sectors. To calculate the quasi-hole wavefunctions, we need to know the full details of what happens when two primary fields are brought to the same location inside correlators. This information is contained in the operator product expansions (OPEs), which will be given in (A.2) in the case of the parafermions associated to $su(2)_k$.

A.1. Fusion rules of the $su(2)_3$ parafermions

The sectors of the $su(2)_k/u(1)$ parafermion CFT (which we will sometimes denote by $\mathbb{Z}_k$) are labeled by two labels, an $su(2)_k$ label $l = 0, 1, \ldots, k$ and the $u(1)$ charge m, which is defined modulo $2k$, because the $u(1)$ theory is compactified. Thus, we write the sector (and the leading parafermion primary fields) as Φ^l_m.

The branching rules state that the only labels allowed are those which satisfy $l - m = 0 \bmod 2$. In addition, we need to identify the sectors [39] $\Phi^l_m \equiv \Phi^{k-l}_{m+k}$. It follows that there are $\frac{1}{2}k(k+1)$ parafermion primary fields, and for each field we can choose labels Φ^l_m with $l = 0, 1, \ldots, k$ and $m \in \{-l+2, -l+4, \ldots, l\}$. With labels chosen in this way, the dimensions of the fields are given by

$$h_{l,m} = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k}. \tag{A.1}$$

Note that if m is chosen outside of the range $m \in \{-l+2, -l+4, \ldots, l\}$, the scaling dimension will be given by $h_{l,m} + n^l_m$, where n^l_m is a positive integer.

Thus, for the parafermions $\psi_i = \Phi^0_{2i} \equiv \Phi^k_{2i-k}$, with $i = 0, 1, \ldots, k-1$ we find $h_{\psi_i} = \frac{i(k-i)}{k}$, while for the spin fields $\sigma_i = \Phi^i_i$ we find $h_{\sigma_i} = \frac{i(k-i)}{2k(k+2)}$. The 'neutral' fields $\varepsilon_i = \Phi^{2i}_0$ with $2i \in \{0, 1, \ldots, k\}$ have scaling dimension $h_{\varepsilon_i} = \frac{i(i+1)}{k+2}$.

Table A.1
Fusion rules of the parafermion and spin fields associated to the parafermion theory su(2)$_3$/u(1)

$\times$	σ_1	σ_2	ε	ψ_1	ψ_2
σ_1	$\psi_1 + \sigma_2$				
σ_2	$\mathbf{1} + \varepsilon$	$\psi_2 + \sigma_1$			
ε	$\psi_2 + \sigma_1$	$\psi_1 + \sigma_2$	$\mathbf{1} + \varepsilon$		
ψ_1	ε	σ_1	σ_2	ψ_2	
ψ_2	σ_2	ε	σ_1	$\mathbf{1}$	ψ_1

The fusion rules of the $\mathbb{Z}_k$ parafermion theory can be obtained from the fusion rules of the $SU(2)_k$ WZW conformal field theory. Explicitly, we have

$$\Phi^{l}_{m} \times \Phi^{l'}_{m'} = \oplus_{l'' \in l \times l'} \Phi^{l''}_{m+m'}, \tag{A.2}$$

where the sum is over the range $l'' = |l - l'|, |l - l'| + 2, \ldots, \min(l + l', 2k - l - l')$.

Specializing to the $\mathbb{Z}_3$ parafermions, we will use the following standard notation

$$\mathbf{1} = \Phi^0_0 \quad \psi_1 = \Phi^0_2 \quad \psi_2 = \Phi^0_4 \quad \sigma_1 = \Phi^1_1 \quad \sigma_2 = \Phi^2_2 \quad \varepsilon = \Phi^2_0. \tag{A.3}$$

The scaling dimensions are $h_{\psi_i} = \frac{2}{3}$, $h_{\sigma_i} = \frac{1}{15}$ and $h_\varepsilon = \frac{2}{5}$.

With this notation, we find the fusion rules as given in Table A.1.

Note that the structure of the fusion rules becomes simpler if we adopt the following notation $\psi_0 = \mathbf{1}$, $\psi_1 = \psi_1$, $\psi_2 = \psi_2$, $\tau_0 = \varepsilon$, $\tau_1 = \sigma_2$ and $\tau_2 = \sigma_1$. The labels of the fields are defined modulo 3. In this notation, the fusion rules are simply

$$\begin{aligned} &\psi_i \times \psi_j = \psi_{i+j}, \\ &\psi_i \times \tau_j = \tau_{i+j}, \\ &\tau_i \times \tau_j = \psi_{i+j} + \tau_{i+j}. \end{aligned} \tag{A.4}$$

A.2. OPEs

In this section, we give the OPEs of the leading Virasoro primary fields[1] (including the numerical coefficients). We will use schematic notation, writing

$$A(z)B(w) = (z - w)^{h_C - h_A - h_B} c^C_{A,B} C(w) + \cdots \tag{A.5}$$

as

$$AB = c^C_{A,B} C \tag{A.6}$$

and restricting ourselves to the leading field in each of the fusion channels.

We then have the following OPEs

$$\psi_1\psi_1 = \frac{2}{\sqrt{3}}\psi_2, \quad \psi_1\psi_2 = \mathbf{1}, \quad \psi_2\psi_2 = \frac{2}{\sqrt{3}}\psi_1, \tag{A.7}$$

[1] To be completely rigorous, we should view the fields as chiral vertex operators, or intertwiners. Then, the OPE coefficients would carry two additional labels, indicating the sectors the fields acts between. In this paper, we will not need this level of detail.

in accordance with the results in [30] for all fields ψ_i and general k. The OPEs between parafermions and spin fields are

$$\begin{aligned} &\psi_1\sigma_1 = \sqrt{\frac{2}{3}}\varepsilon, \quad \psi_1\sigma_2 = \frac{1}{\sqrt{3}}\sigma_1, \quad \psi_1\varepsilon = \frac{2}{\sqrt{3}}\sigma_2, \\ &\psi_2\sigma_1 = \frac{1}{\sqrt{3}}\sigma_2, \quad \psi_2\sigma_2 = \sqrt{\frac{2}{3}}\varepsilon, \quad \psi_2\varepsilon = \frac{2}{\sqrt{3}}\sigma_1, \end{aligned} \tag{A.8}$$

while the other OPEs of the spin fields are given by, with $C = \frac{1}{2}\sqrt{\frac{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}}$

$$\begin{aligned} &\sigma_1\sigma_1 = \frac{1}{\sqrt{3}}\psi_1 + \sqrt{2C}\sigma_2, \quad \sigma_1\sigma_2 = \mathbf{1} + \sqrt{C}\varepsilon, \quad \sigma_2\sigma_2 = \frac{1}{\sqrt{3}}\psi_2 + \sqrt{2C}\sigma_1, \\ &\sigma_1\varepsilon = \sqrt{\frac{2}{3}}\psi_2 + \sqrt{C}\sigma_1, \quad \sigma_2\varepsilon = \sqrt{\frac{2}{3}}\psi_1 + \sqrt{C}\sigma_2. \end{aligned} \tag{A.9}$$

The most interesting OPE turns out to be

$$\varepsilon\varepsilon = \mathbf{1} + 0\varepsilon + \sqrt{\frac{12C}{7}}\varepsilon', \tag{A.10}$$

that is, the first field appearing in the 'ε' channel is *not* the field ε, but a different Virasoro primary field ε', which has scaling dimension $h_{\varepsilon'} = \frac{7}{5}$. This result will be derived in Section A.4.

A.3. Spin-field correlators

In this appendix, we will explain how the four-point correlators of four 'spin fields' σ can be obtained from the four-point correlators of the WZW CFT as given in [24]. We will do this explicitly in the case of $su(2)_3$, and merely state the outcome in the other cases.

The correlators which are calculated in [24] are four-point correlators of WZW primary fields, transforming in the fundamental representation. These fields g can be written in terms of the spin fields σ_1 and σ_2, combined with a vertex operator. Explicitly, we have $g_1 = \sigma_1 e^{\frac{i\varphi}{6}}$ and $g_2 = \sigma_2 e^{-\frac{i\varphi}{6}}$. Because correlators of the form $\langle g_1g_2g_1g_2\rangle^{(p)}$ and $\langle g_1g_1g_2g_2\rangle^{(p)}$ are given in [24] and four-point correlators of the vertex operators $e^{\pm\frac{i\varphi}{6}}$ are easily calculated, we can derive the explicit form of the correlators $\langle\sigma_1\sigma_2\sigma_1\sigma_2\rangle^{(p)}$ and $\langle\sigma_1\sigma_1\sigma_2\sigma_2{}^{(p)}$, namely

$$\begin{aligned} &\langle\sigma_1(w_1)\sigma_2(w_2)\sigma_1(w_3)\sigma_2(w_4)\rangle^{(0)} = (w_{12}w_{34})^{-\frac{2}{15}}x^{\frac{3}{10}}(1-x)^{-\frac{1}{6}}\mathcal{F}_1^{(0)}(x), \\ &\langle\sigma_1(w_1)\sigma_2(w_2)\sigma_1(w_3)\sigma_2(w_4)\rangle^{(1)} = (-1)^{\frac{2}{5}}(w_{12}w_{34})^{-\frac{2}{15}}x^{\frac{3}{10}}(1-x)^{-\frac{1}{6}}C\mathcal{F}_1^{(1)}(x), \\ &\langle\sigma_1(w_1)\sigma_1(w_2)\sigma_2(w_3)\sigma_2(w_4)\rangle^{(0)} = (-1)^{\frac{2}{3}}(w_{12}w_{34})^{-\frac{2}{15}}x^{-\frac{1}{30}}(1-x)^{\frac{1}{6}}\mathcal{F}_2^{(0)}(x), \\ &\langle\sigma_1(w_1)\sigma_1(w_2)\sigma_2(w_3)\sigma_2(w_4)\rangle^{(1)} = -(-1)^{\frac{1}{15}}(w_{12}w_{34})^{-\frac{2}{15}}x^{-\frac{1}{30}}(1-x)^{\frac{1}{6}}C\mathcal{F}_2^{(1)}(x). \end{aligned} \tag{A.11}$$

The constant C is related to h used in [24] via $h = C^2$. The phases make sure that the correlators have the correct behaviour when $x \to 0$. In this limit, the four-point correlators reduce to normalized two-point functions. We remind the reader of the notation $w_{ij} = w_i - w_j$. The functions $\mathcal{F}_i^{(p)}(x)$ can be expressed in terms of the hypergeometric functions $F(a,b,c;x)$ in the following way

$$\begin{aligned}
\mathcal{F}_1^{(0)}(x) &= x^{-\frac{3}{10}}(1-x)^{\frac{1}{10}}F\left(\frac{1}{5},-\frac{1}{5},\frac{3}{5};x\right),\\
\mathcal{F}_2^{(0)}(x) &= \frac{1}{3}x^{\frac{7}{10}}(1-x)^{\frac{1}{10}}F\left(\frac{6}{5},\frac{4}{5},\frac{8}{5};x\right),\\
\mathcal{F}_1^{(1)}(x) &= x^{\frac{1}{10}}(1-x)^{\frac{1}{10}}F\left(\frac{1}{5},\frac{3}{5},\frac{7}{5};x\right),\\
\mathcal{F}_2^{(1)}(x) &= -2x^{\frac{1}{10}}(1-x)^{\frac{1}{10}}F\left(\frac{1}{5},\frac{3}{5},\frac{2}{5};x\right).
\end{aligned} \tag{A.12}$$

A.4. Further correlators

The master formula (3.4) can be used to obtain other correlation functions of parafermion fields, which are hard to obtain from the KZ equation, because they correspond to other SU(2) representations than the fundamental representations.

These correlators are obtained by taking various limits of the master formula (3.4). In fact, for $k = 3$, we can obtain all the possible four-point correlation functions of the spin fields σ_1, σ_2 and ε, because we can fuse the spin field σ_1 with an arbitrary number of fields ψ_i, which gives us all the possible spin fields. For arbitrary k, these methods gives us a way of calculating all the four-point correlators of the fields of the form $\Phi_m^1 \equiv \Phi_{m+k}^{k-1}$, in combination with an arbitrary number (which has to be allowed by fusion) of parafermion fields ψ_i.

We will now use the master formula (3.4) to find the correlator $\langle\sigma_1(w_1)\sigma_2(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(0,1)}$, by taking the limit $(z_1 \to w_2, z_2 \to w_2, z_3 \to w_3, z_4 \to w_4)$. This results in

$$\begin{aligned}
\langle\sigma_1(w_1)\sigma_2(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(0)} &= \frac{1}{2}w_{12}^{-\frac{2}{15}}w_{34}^{-\frac{4}{5}}x^{\frac{3}{10}}(1-x)^{\frac{1}{2}}\left[\frac{2-x}{1-x}\mathcal{F}_1^{(0)}(x)+\mathcal{F}_2^{(0)}(x)\right],\\
\langle\sigma_1(w_1)\sigma_2(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(1)} &= (-1)^{\frac{7}{5}}\frac{C}{2}w_{12}^{-\frac{2}{15}}w_{34}^{-\frac{4}{5}}x^{\frac{3}{10}}(1-x)^{\frac{1}{2}}\left[\frac{2-x}{1-x}\mathcal{F}_1^{(1)}(x)+\mathcal{F}_2^{(1)}(x)\right].
\end{aligned} \tag{A.13}$$

Here, we used that $C^{\varepsilon}_{\psi_1,\sigma_1} = \sqrt{\frac{2}{3}}$. We would like to note that this result is equivalent to the result obtained in [30], namely

$$\begin{aligned}
\langle\sigma_1(w_1)\sigma_2(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(0)} &= w_{12}^{-\frac{2}{15}}w_{34}^{-\frac{4}{5}}(1-x)^{-\frac{2}{5}}F\left(-\frac{1}{5},-\frac{4}{5},-\frac{2}{5};x\right)\\
\langle\sigma_1(w_1)\sigma_2(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(1)} &= (-1)^{\frac{7}{5}}w_{12}^{-\frac{2}{15}}w_{34}^{-\frac{4}{5}}x^{\frac{7}{5}}(1-x)^{-\frac{2}{5}}\sqrt{\rho}F\left(\frac{6}{5},\frac{3}{5},\frac{12}{5};x\right).
\end{aligned} \tag{A.14}$$

The equivalence of the two results follows from the fact that $\sqrt{\rho}/C = \frac{2}{7}$, with ρ given by

$$\rho = 4\frac{\Gamma^3(\frac{3}{5})\Gamma^2(\frac{6}{5})\Gamma(\frac{4}{5})}{\Gamma^2(\frac{12}{5})\Gamma^2(-\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{1}{5})} \tag{A.15}$$

and the following relations between hypergeometric functions

$$
\begin{aligned}
(1-x)^{-\frac{2}{5}}F\left(-\frac{1}{5},-\frac{4}{5},-\frac{2}{5};x\right) &= \frac{1}{2}x^{\frac{3}{10}}(1-x)^{\frac{1}{2}}\left[\frac{2-x}{1-x}\mathcal{F}_1^{(0)}(x)+\mathcal{F}_2^{(0)}(x)\right],\\
x^{\frac{7}{5}}(1-x)^{-\frac{2}{5}}F\left(\frac{6}{5},\frac{3}{5},\frac{12}{5};x\right) &= \frac{7}{4}x^{\frac{3}{10}}(1-x)^{\frac{1}{2}}\left[\frac{2-x}{1-x}\mathcal{F}_1^{(1)}(x)+\mathcal{F}_2^{(1)}(x)\right].
\end{aligned}
\tag{A.16}
$$

To obtain the OPE coefficient $C^{\varepsilon}_{\varepsilon,\varepsilon}$, we expand the correlators around $x=0$, with the result

$$
\begin{aligned}
&w_{12}^{\frac{2}{15}}w_{34}^{\frac{4}{5}}\langle\sigma_1(w_1)\sigma_2(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(0)} = 1+\frac{x^2}{15}+\frac{x^3}{15}+o(x^4),\\
&w_{12}^{\frac{2}{15}}w_{34}^{\frac{4}{5}}\frac{(-1)^{-\frac{7}{5}}7}{2\sqrt{h}}\langle\sigma_1(w_1)\sigma_2(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(1)} = x^{\frac{7}{5}}+\frac{7x^{\frac{12}{5}}}{10}+\frac{236\,x^{\frac{17}{5}}}{425}+o(x^{\frac{22}{5}}).
\end{aligned}
\tag{A.17}
$$

We find that the 1 (or σ_0) channel starts as $x^{\frac{7}{5}}$, as was observed in [30]. As a consequence, we find that the OPE coefficient $C^{\varepsilon}_{\varepsilon,\varepsilon}=0$.

Finally, we also reduced the master formula (3.4) to obtain the correlator $\langle\varepsilon(w_1)\varepsilon(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(0,1)}$, by taking the limit ($z_1\to w_2, z_2\to w_1, z_3\to w_3, z_4\to w_4$). This results in

$$
\begin{aligned}
\langle\varepsilon(w_1)\varepsilon(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(0)} &= \tfrac{1}{2}(w_{12}w_{34})^{-\frac{4}{5}}x^{\frac{3}{10}}(1-x)^{-\frac{1}{2}}[(2-x)\mathcal{F}_1^{(0)}(x)+(1+x)\mathcal{F}_2^{(0)}(x)],\\
\langle\varepsilon(w_1)\varepsilon(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(1)} &= \frac{-(-1)^{\frac{7}{5}}C}{2(w_{12}w_{34})^{\frac{4}{5}}}x^{\frac{3}{10}}(1-x)^{-\frac{1}{2}}[(2-x)\mathcal{F}_1^{(1)}(x)+(1+x)\mathcal{F}_2^{(1)}(x)].
\end{aligned}
\tag{A.18}
$$

Again, we find that the small x behaviour for the (1) or ε channel goes like $x^{\frac{7}{5}}$, explicitly

$$
\begin{aligned}
&(w_{12}w_{34})^{\frac{4}{5}}\langle\varepsilon(w_1)\varepsilon(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(0)} = 1+\frac{2x^2}{5}+\frac{2x^3}{5}+o(x^4),\\
&(w_{12}w_{34})^{\frac{4}{5}}\frac{(-1)^{-\frac{7}{5}}7}{12C}\langle\varepsilon(w_1)\varepsilon(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(1)} = x^{\frac{7}{5}}+\frac{7x^{\frac{12}{5}}}{10}+\frac{261x^{\frac{17}{5}}}{425}+o(x^{\frac{22}{5}}).
\end{aligned}
\tag{A.19}
$$

Using the braid properties of the functions $\mathcal{F}_{1,2}^{(0,1)}$, we find the following braid matrices for the correlator $\langle\varepsilon(w_1)\varepsilon(w_2)\varepsilon(w_3)\varepsilon(w_4)\rangle^{(0,1)}$

$$
\begin{aligned}
&U_{12}=\begin{pmatrix}(-1)^{-\frac{4}{5}} & 0\\ 0 & (-1)^{\frac{3}{5}}\end{pmatrix},\quad U_{23}=(-1)^{-\frac{4}{5}}\begin{pmatrix}\tau & (-1)^{-\frac{7}{5}}\sqrt{\tau}\\ (-1)^{\frac{7}{5}}\sqrt{\tau} & -\tau\end{pmatrix},\\
&U_{13}=(-1)^{\frac{4}{5}}\begin{pmatrix}\tau & (-1)^{-\frac{4}{5}}\sqrt{\tau}\\ \sqrt{\tau} & (-1)^{\frac{1}{5}}\tau\end{pmatrix}.
\end{aligned}
\tag{A.20}
$$

Again, we would like to see if we can obtain the same result using the quantum group picture. This time, we need the R and F-matrices corresponding to $l=2$ or spin-1, because the ε particles are represented by $\varepsilon=\Phi_0^2$ in the coset construction. The data we need is $R_0^{2,2}=(-1)^{-\frac{4}{5}}$, $R_0^{2,2}=(-1)^{\frac{3}{5}}$ and

$$
F_2^{2,2,2}=\begin{pmatrix}\tau & -\sqrt{\tau}\\ -\sqrt{\tau} & -\tau\end{pmatrix}.
\tag{A.21}
$$

With this data, we can calculate the braid matrices using (5.3) and (5.4), with the following result

$$
\begin{aligned}
&U_{12}=\begin{pmatrix}(-1)^{-\frac{4}{5}} & 0\\ 0 & (-1)^{\frac{3}{5}}\end{pmatrix},\quad U_{23}=(-1)^{-\frac{4}{5}}\begin{pmatrix}\tau & (-1)^{-\frac{2}{5}}\sqrt{\tau}\\ (-1)^{\frac{2}{5}}\sqrt{\tau} & -\tau\end{pmatrix},\\
&U_{13}=(-1)^{\frac{4}{5}}\begin{pmatrix}\tau & (-1)^{\frac{1}{5}}\sqrt{\tau}\\ -\sqrt{\tau} & (-1)^{\frac{1}{5}}\tau\end{pmatrix}
\end{aligned}
\tag{A.22}
$$

We see that we almost exactly reproduced the braid behaviour of Eq. (A.20). The only difference is the additional minus signs in the off-diagonal elements of the matrices. These additional minus signs are a result of the fact that the conformal block in the (1) or (ε) channel starts at a degree higher than naively expected from the fusion rules. However, these additional signs can be 'gauged' away, by redefining the block $\langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle^{(1)}$ with an additional sign. Thus, we conclude that the vanishing of the coefficient $C^{\varepsilon}_{\varepsilon,\varepsilon}$ does show up in the braid matrices, but its effect can be gauged away. Note that is was to be expected that the additional power in the correlator would not change the braid behaviour, because there is no (or hardly any) freedom for change without violating the pentagon and hexagon equations.

A.5. Braiding relations

From [24] we know that

$$
\mathcal{F}^{(p)}_{1,2}(1-x)=\sum_q C^p_q\mathcal{F}^{(q)}_{2,1}(x),
\tag{A.23}
$$

where, in this case, we have

$$
\begin{aligned}
&C^0_0=-C^1_1=\frac{\Gamma(\frac{2}{5})\Gamma(\frac{3}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{4}{5})}=\frac{\sqrt{5}-1}{2}=\tau,\quad C^1_0=-2\frac{\Gamma^2(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{3}{5})},\\
&C^0_1=\frac{1+C^0_0C^1_1}{C^1_0}=-\frac{1}{2}\frac{\Gamma^2(\frac{3}{5})}{\Gamma(\frac{2}{5})\Gamma(\frac{4}{5})}
\end{aligned}
\tag{A.24}
$$

In addition, we have the following expression for C, which is related to h used in [24]

$$
h=C^2=\frac{C^0_1}{C^1_0}=\frac{1}{4}\frac{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})},\quad \frac{C^0_1}{C}=C^1_0C=-\sqrt{\tau}.
\tag{A.25}
$$

For general k, these relations become

$$
\begin{aligned}
&C^0_0=-C^1_1=\frac{\Gamma(\frac{2}{k+2})\Gamma(\frac{k}{k+2})}{\Gamma(\frac{1}{k+2})\Gamma(\frac{k+1}{k+2})}=\frac{1}{2\cos(\frac{\pi}{k+2})},\quad C^1_0=-2\frac{\Gamma^2(\frac{2}{k+2})}{\Gamma(\frac{1}{k+2})\Gamma(\frac{3}{k+2})},\\
&C^0_1=\frac{1+C^0_0C^1_1}{C^1_0}=-\frac{1}{2}\frac{\Gamma^2(\frac{k}{k+2})}{\Gamma(\frac{k-1}{k+2})\Gamma(\frac{k+1}{k+2})}
\end{aligned}
\tag{A.26}
$$

In addition, we have the following expression for h, see [24]

$$
h=C^2=\frac{C^0_1}{C^1_0}=\frac{1}{4}\frac{\Gamma(\frac{1}{k+2})\Gamma(\frac{3}{k+2})\Gamma^2(\frac{k}{k+2})}{\Gamma(\frac{k-1}{k+2})\Gamma(\frac{k+1}{k+2})\Gamma^2(\frac{2}{k+2})},\quad \frac{C^0_1}{C}=C^1_0C=-\sqrt{1+C^0_0C^1_1}.
\tag{A.27}
$$

The transformation behaviour of the $\mathcal{F}^{(p)}_i(x)$ under $x\mapsto\frac{-x}{1-x}$ is as follows

$$\begin{aligned}
\mathcal{F}_1^{(0)}\left(\frac{-x}{1-x}\right) &= (-1)^{-\frac{3}{2(k+2)}}(1-x)^{\frac{3}{2(k+2)}}\left[\mathcal{F}_1^{(0)}(x)+\mathcal{F}_2^{(0)}(x)\right],\\
\mathcal{F}_2^{(0)}\left(\frac{-x}{1-x}\right) &= -(-1)^{-\frac{3}{2(k+2)}}(1-x)^{\frac{3}{2(k+2)}}\mathcal{F}_2^{(0)}(x),\\
\mathcal{F}_1^{(1)}\left(\frac{-x}{1-x}\right) &= -(-1)^{\frac{1}{2(k+2)}}(1-x)^{\frac{3}{2(k+2)}}\left[\mathcal{F}_1^{(1)}(x)+\mathcal{F}_2^{(1)}(x)\right],\\
\mathcal{F}_2^{(1)}\left(\frac{-x}{1-x}\right) &= (-1)^{\frac{1}{2(k+2)}}(1-x)^{\frac{3}{2(k+2)}}\mathcal{F}_2^{(1)}(x).
\end{aligned} \tag{A.28}$$

For $x\mapsto\frac{1}{x}$ we have

$$\begin{aligned}
\mathcal{F}_1^{(0)}\left(\frac{1}{x}\right) &= (-1)^{\frac{3}{2(k+2)}}x^{\frac{3}{2(k+2)}}\left[C_0^0\mathcal{F}_1^{(0)}(x)+(-1)^{\frac{k}{k+2}}C_1^0\mathcal{F}_1^{(1)}(x)\right],\\
\mathcal{F}_2^{(0)}\left(\frac{1}{x}\right) &= -(-1)^{\frac{3}{2(k+2)}}x^{\frac{3}{2(k+2)}}\left[C_0^0(\mathcal{F}_1^{(0)}(x)+\mathcal{F}_2^{(0)}(x))+(-1)^{\frac{k}{k+2}}C_1^0(\mathcal{F}_1^{(1)}(x)+\mathcal{F}_2^{(1)}(x))\right],\\
\mathcal{F}_1^{(1)}\left(\frac{1}{x}\right) &= -(-1)^{-\frac{1}{2(k+2)}}x^{\frac{3}{2(k+2)}}\left[C_0^1\mathcal{F}_1^{(0)}(x)+(-1)^{\frac{k}{k+2}}C_1^1\mathcal{F}_1^{(1)}(x)\right],\\
\mathcal{F}_2^{(1)}\left(\frac{1}{x}\right) &= (-1)^{-\frac{1}{2(k+2)}}x^{\frac{3}{2(k+2)}}\left[C_0^1(\mathcal{F}_1^{(0)}(x)+\mathcal{F}_2^{(0)}(x))+(-1)^{\frac{k}{k+2}}C_1^1(\mathcal{F}_1^{(1)}(x)+\mathcal{F}_2^{(1)}(x))\right].
\end{aligned} \tag{A.29}$$

A.6. More general k results

From [24], we can extract the following correlators for general k

$$\begin{aligned}
\langle\sigma_1(w_1)\sigma_{k-1}(w_2)\sigma_1(w_3)\sigma_{k-1}(w_4)\rangle^{(0)} &= (w_{12}w_{34})^{-\frac{k-1}{k(k+2)}}x^{\frac{3}{2(k+2)}}(1-x)^{-\frac{1}{2k}}\mathcal{F}_1^{(0)}(x),\\
\langle\sigma_1(w_1)\sigma_{k-1}(w_2)\sigma_1(w_3)\sigma_{k-1}(w_4)\rangle^{(1)} &= (-1)^{\frac{2}{k+2}}(w_{12}w_{34})^{-\frac{k-1}{k(k+2)}}x^{\frac{3}{2(k+2)}}(1-x)^{-\frac{1}{2k}}C\mathcal{F}_1^{(1)}(x),\\
\langle\sigma_1(w_1)\sigma_1(w_2)\sigma_{k-1}(w_3)\sigma_{k-1}(w_4)\rangle^{(0)} &= (-1)^{\frac{k-1}{k}}(w_{12}w_{34})^{-\frac{k-1}{k(k+2)}}x^{\frac{k-4}{2k(k+2)}}(1-x)^{\frac{1}{2k}}\mathcal{F}_2^{(0)}(x),\\
\langle\sigma_1(w_1)\sigma_1(w_2)\sigma_{k-1}(w_3)\sigma_{k-1}(w_4)\rangle^{(1)} &= -(-1)^{\frac{k-2}{k(k+2)}}(w_{12}w_{34})^{-\frac{k-1}{k(k+2)}}x^{\frac{k-4}{2k(k+2)}}(1-x)^{\frac{1}{2k}}C\mathcal{F}_2^{(1)}(x),
\end{aligned} \tag{A.30}$$

where the $\mathcal{F}_i^{(p)}$ are now k dependent

$$\begin{aligned}
\mathcal{F}_1^{(0)}(x) &= x^{-\frac{3}{2(k+2)}}(1-x)^{\frac{1}{2(k+2)}}F\left(\frac{1}{k+2},-\frac{1}{k+2},\frac{k}{k+2};x\right),\\
\mathcal{F}_2^{(0)}(x) &= \frac{1}{k}x^{\frac{2k+1}{2(k+2)}}(1-x)^{\frac{1}{2(k+2)}}F\left(\frac{k+3}{k+2},\frac{k+1}{k+2},\frac{2k+2}{k+2};x\right),\\
\mathcal{F}_1^{(1)}(x) &= x^{\frac{1}{2(k+2)}}(1-x)^{\frac{1}{2(k+2)}}F\left(\frac{1}{k+2},\frac{3}{k+2},\frac{k+4}{k+2};x\right),\\
\mathcal{F}_2^{(1)}(x) &= -2x^{\frac{1}{2(k+2)}}(1-x)^{\frac{1}{2(k+2)}}F\left(\frac{1}{k+2},\frac{3}{k+2},\frac{2}{k+2};x\right).
\end{aligned} \tag{A.31}$$

From this, we find the following OPE coefficients

$$C_{\sigma_1,\sigma_1}^{\psi_1} = C_{\sigma_1,\psi_{k-1}}^{\sigma_{k-1}} = \frac{1}{\sqrt{k}}. \tag{A.32}$$

The wavefunction of four quasi-holes and $N = (r + 1)k - 2$ electrons (with r a positive integer) can be written in the following form

$$
\begin{aligned}
&\Psi^{(0,1)}(w_1, w_2, w_3, w_4; z_1, \ldots, z_N) \\
&\quad = \langle \sigma_1(w_1)\sigma_1(w_2)\sigma_1(w_3)\sigma_1(w_4)\psi_1(z_1)\cdots\psi_1(z_N)\rangle^{(0,1)} \\
&\qquad \times (w_{12}w_{34})^{\frac{3}{k(kM+2)}} x^{-\frac{2}{k(kM+2)}} (1-x)^{\frac{1}{k(kM+2)}} \prod_{i,j}(w_i - z_j)^{\frac{1}{k}} \prod_{i<j}(z_i - z_j)^{\frac{kM+2}{k}}.
\end{aligned}
\tag{A.33}
$$

Again, we have the following master formula

$$
\begin{aligned}
\Psi^{(0,1)}(w_1, w_2, w_3, w_4; z_1, \ldots, z_N) &= A^{(0,1)}(\{w\})\Psi_{(12)(34)}(\{w\}, \{z\}) \\
&\quad + B^{(0,1)}(\{w\})\Psi_{(13)(24)}(\{w\}, \{z\}).
\end{aligned}
\tag{A.34}
$$

To specify the functions $\Psi_{(12)(34)}$ and $\Psi_{(13)(24)}$, we divide the electrons into k groups, namely $S_i = \{i, i + k, \ldots, N - (k - 2) + i\}$ for $i = 1, \ldots, k - 2$ and $S_j = \{j, j + k, \ldots, N - (2k - 2) + j\}$ for $j = k - 1, k$. Setting $M = 0$ for simplicity, we have

$$
\begin{aligned}
\Psi_{(12)(34)} &= \frac{1}{\mathcal{N}} \sum_{\{S_i\}} \left[\prod_{j \in S_{k-1}} (z_j - w_1)(z_j - w_2) \prod_{j' \in S_k} (z_{j'} - w_3)(z_{j'} - w_4) \prod_{i=1}^{k} \Psi_{S_i}^2 \right], \\
\Psi_{(13)(24)} &= \frac{1}{\mathcal{N}} \sum_{\{S_i\}} \left[\prod_{j \in S_{k-1}} (z_j - w_1)(z_j - w_3) \prod_{j' \in S_k} (z_{j'} - w_2)(z_{j'} - w_4) \prod_{i=1}^{k} \Psi_{S_i}^2 \right],
\end{aligned}
\tag{A.35}
$$

where the sum is over all in-equivalent ways of dividing the electrons in k groups and $\mathcal{N} = k^{\frac{N}{2}}/(k-2)!$.

We will consider the following two limits in the case of $N = 2k - 2$ electrons

$$
\begin{aligned}
&(\mathrm{I}) \quad \begin{cases} z_i \longrightarrow z_1, & i = 2, \ldots, k-1, \\ z_1 \longrightarrow w_2, & \\ z_j \longrightarrow z_k, & j = k+1, \ldots, 2k-2, \\ z_k \longrightarrow w_4, & \end{cases} \\
&(\mathrm{II}) \quad \begin{cases} z_i \longrightarrow z_1, & i = 2, \ldots, k-1, \\ z_1 \longrightarrow w_3, & \\ z_j \longrightarrow z_k, & j = k+1, \ldots, 2k-2. \\ z_k \longrightarrow w_4, & \end{cases}
\end{aligned}
\tag{A.36}
$$

On the one hand, the master formula reduces to a form containing the following two correlators $\langle \sigma_1\sigma_{k-1}\sigma_1\sigma_{k-1}\rangle$ and $\langle \sigma_1\sigma_1\sigma_{k-1}\sigma_{k-1}\rangle$ in the limits (I) and (II), respectively. On the other hand, we find

$$
\begin{aligned}
&\lim_{(\mathrm{I})} \Psi_{(12)(34)} = -\frac{((k-1)!)^2}{k^{k-1}} w_{14}w_{32}w_{42}^{2k-2}, \quad \lim_{(\mathrm{II})} \Psi_{(12)(34)} = 0, \\
&\lim_{(\mathrm{I})} \Psi_{(13)(24)} = 0, \quad \lim_{(\mathrm{II})} \Psi_{(13)(24)} = \frac{((k-1)!)^2}{k^{k-1}} w_{14}w_{32}w_{34}^{2k-2}.
\end{aligned}
\tag{A.37}
$$

The functions $A^{(p)}$ and $B^{(p)}$ again follow

$$
\begin{aligned}
A^{(0)} &= (w_{12}w_{34})^{\alpha}x^{\beta}(1-x)^{\gamma}\mathcal{F}_1^{(0)}(x),\\
B^{(0)} &= -(w_{12}w_{34})^{\alpha}x^{\beta-1}(1-x)^{1+\gamma}\mathcal{F}_2^{(0)}(x),\\
A^{(1)} &= -(-1)^{\frac{2}{k+2}}C(w_{12}w_{34})^{\alpha}x^{\beta}(1-x)^{\gamma}\mathcal{F}_1^{(1)}(x),\\
B^{(1)} &= (-1)^{\frac{2}{k+2}}C(w_{12}w_{34})^{\alpha}x^{\beta-1}(1-x)^{1+\gamma}\mathcal{F}_2^{(1)}(x),
\end{aligned}
\tag{A.38}
$$

where we introduced the following notation

$$
\begin{aligned}
\alpha &= \frac{2k+1}{2(k+2)} - \frac{3M}{2(kM+2)}, \quad \beta = \frac{3}{2(k+2)} + \frac{2M}{2(kM+2)},\\
\gamma &= -\frac{M}{2(kM+2)}.
\end{aligned}
\tag{A.39}
$$

In this derivation, we used that the OPE coefficients for the parafermion fields ψ_l are given by [30]

$$
C^2_{\psi_l,\psi_{l'}} = \frac{\Gamma(l+l'+1)\Gamma(k-l+1)\Gamma(k-l'+1)}{\Gamma(l+1)\Gamma(l'+1)\Gamma(k-l-l'+1)\Gamma(k+1)},
\tag{A.40}
$$

from which it follows that

$$
\prod_{l=1}^{k-2} C^2_{\psi_1,\psi_l} = \frac{((k-1)!)^2}{k^{k-2}}.
\tag{A.41}
$$

In addition, we find the following braid matrices

$$
\begin{aligned}
U_{12} &= (-1)^{-\frac{M}{2(kM+2)}}\begin{pmatrix} (-1)^{\frac{2k+1}{2(k+2)}} & 0\\ 0 & (-1)^{\frac{1}{2(k+2)}}\end{pmatrix},\\
U_{23} &= \frac{(-1)^{-\frac{3}{2(k+2)}-\frac{3M}{2(kM+2)}}}{d_k}\begin{pmatrix} 1 & (-1)^{-\frac{2}{k+2}}\sqrt{d_k^2-1}\\ (-1)^{\frac{2}{k+2}}\sqrt{d_k^2-1} & -1\end{pmatrix},\\
U_{13} &= \frac{(-1)^{\frac{3}{2(k+2)}-\frac{M}{2(kM+2)}}}{d_k}\begin{pmatrix} 1 & (-1)^{\frac{k-2}{k+2}}\sqrt{d_k^2-1}\\ -\sqrt{d_k^2-1} & (-1)^{\frac{k-2}{k+2}}\end{pmatrix},
\end{aligned}
\tag{A.42}
$$

where $d_k = (C_0^0)^{-1}$.

Appendix B. Detailed structure of the $su(3)_2$ parafermion theory

We now turn the $su(3)_2$ parafermions, as introduced by Gepner in [33]. This theory arises upon factoring two free fields from an $SU(3)_2$ WZW theory, namely, it is the $su(3)_2/[u(1)]^2$ coset CFT. In this appendix we present fusion rules, OPEs, four-point correlation functions and braiding properties.

B.1. Fusion rules

The $su(3)_2$ parafermion theory has 8 chiral sectors, which we label as

$$\{\mathbf{1}, \psi_1, \psi_2, \psi_{12}, \sigma_\uparrow, \sigma_\downarrow, \sigma_3, \rho\}.$$

It is important to realize that each of these sectors contains an infinite number of Virasoro primary fields. We sometimes use the same notation for the parafermion sector (say ψ_1) and for the leading Virasoro primary field ($\psi_1(z)$). We shall see that in the sector denoted as ρ there are two independent leading Virasoro primaries, which we shall write as $\rho_s(z)$ and $\rho_c(z)$.

At the level of parafermion sectors, the merging of two sectors is expressed via *fusion rules*, which we present in this section. Our computations in Section 4 require more detailed information at the level of (primary) fields. The latter is contained in the OPEs that we present in the next section of this appendix.

The fusion rules of the eight parafermion sectors can be derived from the coset description of this theory. For our purposes, we will use the defining coset $su(3)_2/[u(1)]^2$, as given in [33]. The parafermion sectors are labeled by an $su(3)$ label $\Lambda = (\Lambda_1, \Lambda_2)$ (when labeling the representations in terms of the dimensions, we would have $\mathbf{1} = (0,0)$, $\mathbf{3} = (1,0)$, $\bar{\mathbf{3}} = (0,1)$, $\mathbf{6} = (2,0)$, $\bar{\mathbf{6}} = (0,2)$ and $\mathbf{8} = (1,1)$) and two $u(1)$ labels $\lambda = (\lambda_1, \lambda_2)$.

There are various restrictions on the labels. First of all, we have the branching condition $\Lambda_1 + 2\Lambda_2 = (\lambda_1 + 2\lambda_2) \bmod 3$. The label λ is only defined up to 2 times (in general, k times) the root lattice of $su(3)$, which means the following sectors are identified

$$\Phi^{\Lambda}_{\lambda_1,\lambda_2} \equiv \Phi^{\Lambda}_{\lambda_1+4,\lambda_2-2}, \quad \Phi^{\Lambda}_{\lambda_1,\lambda_2} \equiv \Phi^{\Lambda}_{\lambda_1-2,\lambda_2+4}. \tag{B.1}$$

In addition, there are other identifications, which follow from the structure of the affine Lie algebra $su(3)_2$, see [39]

$$\Phi^{(\Lambda_1,\Lambda_2)}_{(\lambda_1,\lambda_2)} \equiv \Phi^{(2-\Lambda_1-\Lambda_2,\Lambda_1)}_{(\lambda_1+2,\lambda_2)} \equiv \Phi^{(\Lambda_2,2-\Lambda_1-\Lambda_2)}_{(\lambda_1,\lambda_2+2)}. \tag{B.2}$$

From these rules, it follows that there are indeed eight different sectors, or ‘parafermion fields’, as mentioned above. The fusion rules follow from the general rule

$$\Phi^{\Lambda}_{\lambda} \times \Phi^{\Lambda'}_{\lambda'} = \sum_{\Lambda'' \in \Lambda \times \Lambda'} \Phi^{\Lambda''}_{\lambda+\lambda'}. \tag{B.3}$$

The fusion rules can now easily be derived from the following set of ‘representations’ of the eight parafermion sectors, which close under fusion

$$\begin{aligned} &\mathbf{1} = \Phi^{(0,0)}_{(0,0)}, \quad \psi_1 = \Phi^{(0,0)}_{(2,-1)}, \quad \psi_2 = \Phi^{(0,0)}_{(1,-2)}, \quad \psi_{12} = \Phi^{(0,0)}_{(1,1)}, \\ &\sigma_\uparrow = \Phi^{(1,1)}_{(-1,2)}, \quad \sigma_\downarrow = \Phi^{(1,1)}_{(2,-1)}, \quad \sigma_3 = \Phi^{(1,1)}_{(1,1)}, \quad \rho = \Phi^{(1,1)}_{(0,0)}. \end{aligned} \tag{B.4}$$

Using the only non-trivial fusion rule of the $su(3)_2$ fields $(0,0)$ and $(1,1)$ (corresponding to the one and eight dimensional representation, respectively), namely, $(1,1) \times (1,1) = (0,0) + (1,1)$ and the identifications (B.1), we find the fusion rules as given in Table B.1.

Of course, these fusion rules can also be derived from the S-matrix and the Verlinde formula.

B.2. OPEs

To our knowledge, the OPEs of the leading Virasoro primary fields in the various sectors of the $su(3)_2$ parafermion theory have not been presented in the literature.

We have determined the leading terms in the OPEs of these Virasoro primaries. We observed a $\mathbb{Z}_3$ symmetry relating ($\sigma_\downarrow$,$\sigma_\uparrow$, σ_3) and ($\psi_1, \psi_2, \sqrt{2}\psi_{12}$). To streamline notations,

Table B.1
Fusion rules of the parafermion and spin sectors associated to the parafermion theory $su(3)_2/[u(1)]^2$ introduced by Gepner [33]

$\times$	$\sigma_\uparrow$	$\sigma_\downarrow$	σ_3	ρ	ψ_1	ψ_2	ψ_{12}
$\sigma_\uparrow$	$\mathbf{1}+\rho$						
$\sigma_\downarrow$	$\psi_{12}+\sigma_3$	$\mathbf{1}+\rho$					
σ_3	$\psi_1+\sigma_\downarrow$	$\psi_2+\sigma_\uparrow$	$\mathbf{1}+\rho$				
ρ	$\psi_2+\sigma_\uparrow$	$\psi_1+\sigma_\downarrow$	$\psi_{12}+\sigma_3$	$\mathbf{1}+\rho$			
ψ_1	σ_3	ρ	$\sigma_\uparrow$	$\sigma_\downarrow$	$\mathbf{1}$		
ψ_2	ρ	σ_3	$\sigma_\downarrow$	$\sigma_\uparrow$	ψ_{12}	$\mathbf{1}$	
ψ_{12}	$\sigma_\downarrow$	$\sigma_\uparrow$	ρ	σ_3	ψ_2	ψ_1	$\mathbf{1}$

we therefore write $\sigma_1=\sigma_\downarrow$, $\sigma_2=\sigma_\uparrow$, and $\psi_3=\sqrt{2}\psi_{12}$. We also employ Virasoro primaries ρ_i and $\bar{\rho}_i$, $i=1, 2, 3$. These fields are all linear combinations (specified below) of the fields $\rho_{\rm s}$ and $\rho_{\rm c}$. The scaling dimensions of the leading fields are

$$h_\psi=\frac{1}{2},\quad h_\sigma=\frac{1}{10},\quad h_\rho=\frac{3}{5}. \tag{B.5}$$

For fixing the details of the OPEs, especially those involving the fields $\rho_{\rm s}$, $\rho_{\rm c}$, we have relied on various contractions of the master formulas such as eq. (4.8), (4.13), etc., presented in Section 4.3. These formulas provide expressions for correlators of four spin fields σ_i plus an arbitrary number of parafermions ψ_j. By fusing some of the σ_i with the ψ_j, one produces various combinations of the fields $\rho_{s,c}$; in the end this gives enough information to uniquely fix the OPEs. [We remark that the logical structure of our reasoning has been quite delicate: we have relied on the general 'master formula' structure of correlators and on the 'seed' provided by the explicit correlators provided by the KZ paper and by some of the simplest OPEs, set by the parafermion fusion rules. Combined these turn out to be strong enough to fix both the coefficients in the master formulas and the OPEs of all fields involved.] In Section B.4 below we explicitly mention some of the contractions we used and we provide some additional correlation functions of the parafermion theory.

Employing the same schematic notation as in Section A.2, Eqs. (A.5), (A.6), we have the following OPEs

$$\begin{aligned}
\psi_i\psi_j &= \mathbf{1} && \text{for } i=j\\
&= \tfrac{1}{\sqrt{2}}\psi_k && \text{for } i\neq j,\ i\neq k,\ j\neq k,\\
\psi_i\sigma_j &= \bar{\rho}_i && \text{for } i=j\\
&= \tfrac{1}{\sqrt{2}}\sigma_k && \text{for } i\neq j,\ i\neq k,\ j\neq k,\\
\sigma_i\sigma_j &= \mathbf{1}+\sqrt{2C}\rho_i && \text{for } i=j\\
&= \tfrac{1}{\sqrt{2}}\psi_k+\sqrt{-3C}\ \sigma_k && \text{for } i\neq j,\ i\neq k,\ j\neq k,\\
\bar{\rho}_i\psi_j &= \sigma_j && \text{for } i=j\\
&= -\tfrac{1}{2}\sigma_j && \text{for } i\neq j,\\
(\bar{\rho}_i\sigma_j)^{(0)} &= \psi_j+\cdots && \text{for } i=j\\
&= -\tfrac{1}{2}\psi_j+\cdots && \text{for } i\neq j,\\
(\rho_i\sigma_j)^{(1)} &= \sqrt{2C}\sigma_j+\cdots && \text{for } i=j\\
&= -\tfrac{1}{2}\sqrt{2C}\sigma_j+\cdots && \text{for } i\neq j.
\end{aligned} \tag{B.6}$$

Note that we have written separate equations for fusing fields from the ρ and σ sectors into the (0) channel or the (1) channel.

The $\bar{\rho}_i$ and ρ_i can all be expressed in two independent Virasoro primaries ρ_s and ρ_c according to

$$\begin{aligned}
\rho_1 &= \tfrac{1}{2}(\rho_c - \sqrt{3}\rho_s), \quad \rho_2 = \tfrac{1}{2}(\rho_c + \sqrt{3}\rho_s), \quad \rho_3 = -\rho_c, \\
\bar{\rho}_1 &= \tfrac{1}{2}(\sqrt{3}\rho_c + \rho_s), \quad \bar{\rho}_2 = \tfrac{1}{2}(-\sqrt{3}\rho_c + \rho_s), \quad \bar{\rho}_3 = -\rho_s,
\end{aligned} \tag{B.7}$$

The list of basic OPEs is then completed by

$$\rho_s\rho_s = \mathbf{1} - \sqrt{2C}\rho_c, \quad \rho_s\rho_c = -\sqrt{2C}\rho_s \quad \rho_c\rho_c = \mathbf{1} + \sqrt{2C}\rho_c \tag{B.8}$$

giving

$$\begin{aligned}
\rho_i\rho_j &= \mathbf{1} - \sqrt{2C}\rho_i \qquad \text{for } i = j \\
&= -\tfrac{1}{2}\mathbf{1} - \sqrt{2C}\rho_k \quad \text{for } i \neq j,\ i \neq k,\ j \neq k, \\
\bar{\rho}_i\bar{\rho}_j &= \mathbf{1} + \sqrt{2C}\rho_i \qquad \text{for } i = j \\
&= -\tfrac{1}{2}\mathbf{1} + \sqrt{2C}\rho_k \quad \text{for } i \neq j,\ i \neq k,\ j \neq k.
\end{aligned} \tag{B.9}$$

In all these expressions, the constant C has the value given below Eq. (4.11).

B.3. Spin-field correlators

We now present the four-point correlation functions for the spin fields in the parafermion theory. They are obtained by factoring the four-point functions of fundamental fields in the $SU(3)_2$ WZW model, as given by [24], by factors associated to the spin and charge bosons

$$\begin{aligned}
\langle\sigma_\uparrow(w_1)\sigma_\uparrow(w_2)\sigma_\downarrow(w_3)\sigma_\downarrow(w_4)\rangle^{(0)} &= (w_{12}w_{34})^{-\frac{1}{5}}x^{\frac{8}{15}}(1-x)^{-\frac{1}{6}}\mathcal{F}_1^{(0)}(x), \\
\langle\sigma_\uparrow(w_1)\sigma_\uparrow(w_2)\sigma_\downarrow(w_3)\sigma_\downarrow(w_4)\rangle^{(1)} &= -(-1)^{\frac{3}{5}}C(w_{12}w_{34})^{-\frac{1}{5}}x^{\frac{8}{15}}(1-x)^{-\frac{1}{6}}\mathcal{F}_1^{(1)}(x), \\
\langle\sigma_\uparrow(w_1)\sigma_\downarrow(w_2)\sigma_\uparrow(w_3)\sigma_\downarrow(w_4)\rangle^{(0)} &= (-1)^{\frac{1}{2}}(w_{12}w_{34})^{-\frac{1}{5}}x^{\frac{1}{30}}(1-x)^{\frac{1}{3}}\mathcal{F}_2^{(0)}(x), \\
\langle\sigma_\uparrow(w_1)\sigma_\downarrow(w_2)\sigma_\uparrow(w_3)\sigma_\downarrow(w_4)\rangle^{(1)} &= (-1)^{\frac{1}{10}}C(w_{12}w_{34})^{-\frac{1}{5}}x^{\frac{1}{30}}(1-x)^{\frac{1}{3}}\mathcal{F}_2^{(1)}(x), \\
\langle\sigma_3(w_1)\sigma_3(w_2)\sigma_3(w_3)\sigma_3(w_4)\rangle^{(0)} &= (w_{12}w_{34})^{-\frac{1}{5}}x^{\frac{8}{15}}(1-x)^{\frac{1}{3}}\left[\mathcal{F}_1^{(0)}(x) + \mathcal{F}_2^{(0)}(x)\right], \\
\langle\sigma_3(w_1)\sigma_3(w_2)\sigma_3(w_3)\sigma_3(w_4)\rangle^{(1)} &= -(-1)^{\frac{3}{5}}C(w_{12}w_{34})^{-\frac{1}{5}}x^{\frac{8}{15}}(1-x)^{\frac{1}{3}}\left[\mathcal{F}_1^{(1)}(x) + \mathcal{F}_2^{(1)}(x)\right],
\end{aligned} \tag{B.10}$$

where x is as usual $x = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$. The functions $\mathcal{F}_i^{(p)}(x)$ are now given by

$$\begin{aligned}
\mathcal{F}_1^{(0)}(x) &= x^{-\frac{8}{15}}(1-x)^{\frac{1}{15}}F\left(\frac{1}{5},-\frac{1}{5},\frac{2}{5};x\right), \\
\mathcal{F}_2^{(0)}(x) &= \frac{1}{2}x^{\frac{7}{15}}(1-x)^{\frac{1}{15}}F\left(\frac{6}{5},\frac{4}{5},\frac{7}{5};x\right), \\
\mathcal{F}_1^{(1)}(x) &= x^{\frac{1}{15}}(1-x)^{\frac{1}{15}}F\left(\frac{2}{5},\frac{4}{5},\frac{8}{5};x\right), \\
\mathcal{F}_2^{(1)}(x) &= -3x^{\frac{1}{15}}(1-x)^{\frac{1}{15}}F\left(\frac{2}{5},\frac{4}{5},\frac{3}{5};x\right).
\end{aligned} \tag{B.11}$$

B.4. Further correlators

The master formulas developed in Section 4.3 can be used to gain insight into the OPEs and correlation functions involving the fields in the ρ sector of the $su(3)_2$ parafermion theory. In Section B.2 we already displayed some of the OPEs satisfied by ρ_s and ρ_c. In deriving these we proceeded as follows. Having defined the combinations ρ_i and $\bar{\rho}_i$ through the OPEs $(\sigma_i\sigma_i)^{(1)} = \sqrt{2C}\rho_i$ and $(\psi_i\sigma_i)^{(1)} = \bar{\rho}_i$, we observed that a specific contraction of the master formula Eq. (4.13) shows that $\langle\sigma_i\sigma_i\bar{\rho}_i\rangle$ vanishes. This implies that ρ_i and $\bar{\rho}_i$ are orthogonal (have a vanishing two-point function). Exploiting the symmetry among the $i = 1,2,3$ labels leads to the parametrization Eq. (B.7). All this leaves some freedom in the self-couplings of the ρ_c and ρ_s. To fix these, we observed that yet another contraction of the master formula implies that the three-point function $\langle\bar{\rho}_1\bar{\rho}_2\bar{\rho}_3\rangle$ vanishes. This contraction arises from the formula

$$\begin{aligned}&\langle\sigma_1(w_1)\sigma_2(w_2)\sigma_3(w_3)\psi_1(z_1)\psi_2(z_2)\psi_3(z_3)\rangle\\&\quad=(-1)^{-\frac{9}{10}}\frac{1}{2}\sqrt{\frac{3C}{2}}(z_{12}z_{23}z_{31})^{-\frac{1}{2}}(w_{12}w_{23}w_{31})^{-\frac{1}{10}}\\&\qquad\times\left[\frac{(z_1-w_2)(z_2-w_3)(z_3-w_1)+(z_1-w_3)(z_2-w_1)(z_3-w_2)}{[(z_1-w_2)(z_2-w_3)(z_3-w_1)(z_1-w_3)(z_2-w_1)(z_3-w_2)]^{\frac{1}{2}}}\right]\end{aligned}\tag{B.12}$$

in the limit where $z_i \to w_i$ for $i = 1,2,3$.

The vanishing of $\langle\bar{\rho}_1\bar{\rho}_2\bar{\rho}_3\rangle$ implies that the combination of ρ_c and ρ_s featuring in the (1) channel of the fusion product of $\bar{\rho}_1$ and $\bar{\rho}_2$ is orthogonal to $\bar{\rho}_3$ and thereby proportional to ρ_3. A final contraction yielding $\langle\sigma_2\sigma_2\bar{\rho}_1\bar{\rho}_1\rangle$ is then used to fix the normalization, giving Eq. (B.8).

Correlation functions involving one or more fields in the ρ sector are easily generated as suitable contractions of the various master formulas. Examples are the following four-point functions

$$\begin{aligned}&\langle\bar{\rho}_1(w_1)\bar{\rho}_1(w_2)\bar{\rho}_1(w_3)\bar{\rho}_1(w_4)\rangle^{(0)}=(w_{12}w_{34})^{-\frac{6}{5}}x^{\frac{8}{15}}(1-x)^{-\frac{2}{3}}(1-x+x^2)[\mathcal{F}_1^{(0)}(x)+\mathcal{F}_2^{(0)}(x)],\\&\langle\bar{\rho}_1(w_1)\bar{\rho}_1(w_2)\bar{\rho}_1(w_3)\bar{\rho}_1(w_4)\rangle^{(1)}=-(-1)^{\frac{3}{5}}C(w_{12}w_{34})^{-\frac{6}{5}}x^{\frac{8}{15}}(1-x)^{-\frac{2}{3}}(1-x+x^2)[\mathcal{F}_1^{(1)}(x)+\mathcal{F}_2^{(1)}(x)],\\&\langle\bar{\rho}_1(w_1)\bar{\rho}_1(w_2)\bar{\rho}_2(w_3)\bar{\rho}_2(w_4)\rangle^{(0)}=\frac{1}{4}(w_{12}w_{34})^{-\frac{6}{5}}x^{\frac{8}{15}}(1-x)^{\frac{4}{3}}\left[\left(1+\frac{3}{(1-x)^2}\right)\mathcal{F}_1^{(0)}(x)+\frac{x^2}{(1-x)^2}\mathcal{F}_2^{(0)}(x)\right],\\&\langle\bar{\rho}_{1(w_1)}\bar{\rho}_{1(w_2)}\bar{\rho}_{2(w_3)}\bar{\rho}_{2(w_4)}\rangle^{(1)}=-(-1)^{\frac{3}{5}}\frac{C}{4}(w_{12}w_{34})^{-\frac{6}{5}}x^{\frac{8}{15}}(1-x)^{\frac{4}{3}}\left[\left(1+\frac{3}{(1-x)^2}\right)\mathcal{F}_1^{(1)}(x)+\frac{x^2}{(1-x)^2}\mathcal{F}_2^{(1)}(x)\right].\end{aligned}\tag{B.13}$$

B.5. Braiding relations

We list the transformation properties of the functions $\mathcal{F}_{1,2}^{(0,1)}(x)$, as specified in Eq. (B.11) under transformations (i) $x \to 1-x$, (ii) $x \to \frac{-x}{1-x}$, (iii) $x \to \frac{1}{x}$.

For $x \to 1-x$ we have [24]

$$\mathcal{F}_{1,2}^{(p)}(1-x) = \sum_q C_q^p \mathcal{F}_{2,1}^{(q)}(x)\tag{B.14}$$

with

$$
\begin{aligned}
C_0^0 &= -C_1^1 = 3\frac{\Gamma(\frac{3}{5})\Gamma(-\frac{3}{5})}{\Gamma(\frac{1}{5})\Gamma(-\frac{1}{5})} = \frac{1}{2}(\sqrt{5}-1) = \tau, \quad C_0^1 = -3\frac{\Gamma^2(\frac{3}{5})}{\Gamma(\frac{2}{5})\Gamma(\frac{4}{5})}, \\
C_1^0 &= \frac{1+C_0^0C_1^1}{C_0^1} = -\frac{1}{3}\,\frac{\Gamma^2(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma(\frac{3}{5})}.
\end{aligned} \tag{B.15}
$$

We observe that

$$
C^2 = C_1^0/C_0^1 = \frac{1}{9}\,\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}, \quad C_1^0/C = C_0^1 C = -\sqrt{\tau}. \tag{B.16}
$$

For $x \to \frac{-x}{1-x}$ we have

$$
\begin{aligned}
\mathcal{F}_1^{(0)}\left(\frac{-x}{1-x}\right) &= (-1)^{-\frac{8}{15}}(1-x)^{\frac{1}{5}}\mathcal{F}_1^{(0)}(x), \\
\mathcal{F}_2^{(0)}\left(\frac{-x}{1-x}\right) &= (-1)^{-\frac{8}{15}}(1-x)^{\frac{1}{5}}\left[-x\mathcal{F}_1^{(0)}(x) + (1-x)\mathcal{F}_2^{(0)}(x)\right], \\
\mathcal{F}_1^{(1)}\left(\frac{-x}{1-x}\right) &= (-1)^{\frac{1}{15}}(1-x)^{\frac{1}{5}}\mathcal{F}_1^{(1)}(x), \\
\mathcal{F}_2^{(1)}\left(\frac{-x}{1-x}\right) &= (-1)^{\frac{1}{15}}(1-x)^{\frac{1}{5}}\left[-x\mathcal{F}_1^{(1)}(x) + (1-x)\mathcal{F}_2^{(1)}(x)\right].
\end{aligned} \tag{B.17}
$$

Finally for $x \to \frac{1}{x}$

$$
\begin{aligned}
\mathcal{F}_2^{(0)}\left(\frac{1}{x}\right) &= (-1)^{-\frac{2}{15}}x^{\frac{1}{5}}\left[C_0^0\mathcal{F}_2^{(0)}(x) - (-1)^{\frac{2}{5}}C_1^0\mathcal{F}_2^{(1)}(x)\right], \\
\mathcal{F}_2^{(1)}\left(\frac{1}{x}\right) &= (-1)^{-\frac{2}{15}}x^{\frac{1}{5}}\left[-(-1)^{\frac{2}{5}}C_0^1\mathcal{F}_2^{(0)}(x) - (-1)^{-1/5}C_1^1\mathcal{F}_2^{(1)}(x)\right], \\
\mathcal{F}_1^{(0)}\left(\frac{1}{x}\right) &= (-1)^{-\frac{2}{15}}x^{\frac{1}{5}}\left[C_0^0(x\mathcal{F}_1^{(0)}(x) - (1-x)\mathcal{F}_2^{(0)}(x)) - (-1)^{\frac{2}{5}}C_1^0(x\mathcal{F}_1^{(1)}(x) - (1-x)\mathcal{F}_2^{(1)}(x))\right], \\
\mathcal{F}_1^{(1)}\left(\frac{1}{x}\right) &= (-1)^{-\frac{2}{15}}x^{\frac{1}{5}}\left[-(-1)^{\frac{2}{5}}C_0^1(x\mathcal{F}_1^{(0)}(x) - (1-x)\mathcal{F}_2^{(0)}(x)) - (-1)^{-\frac{1}{5}}C_1^1(x\mathcal{F}_1^{(1)}(x) - (1-x)\mathcal{F}_2^{(1)}(x))\right].
\end{aligned} \tag{B.18}
$$

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