Exercises CFT-course fall 2008, set 12.

1. Derive the Knizhnik-Zamolodchikov equation

$$(\partial_{z_i} - \frac{1}{k+g^\vee}\sum_{j\neq i}\frac{\sum_a t^a_{\lambda_i}\otimes t^a_{\lambda_j}}{z_i - z_j})\langle\phi_{\lambda_1}(z_1,\bar z_1)\cdots\phi_{\lambda_N}(z_N,\bar z_N)\rangle = 0$$

from the null vector $(L_{-1} - \frac{\sum_a J^a_{-1}t^a_\lambda}{k+g^\vee})\phi_\lambda = 0$.

2. Solving the Knizhnik-Zamolodchikov equation.

In this exercise, we will solve the KZ equation in the case of $su(2)_k$ ($su(n)_k$ is not much more complicated), for the primary fields ϕ_λ, where $\lambda = \omega_1$, i.e. the fundamental representation.

We define $z_{ij} = (z_i - z_j)$, and write the correlator as

$$\mathcal{G}(\{z_i\},\{\bar z_i\}) = \langle\phi_\lambda(z_1,\bar z_1)\phi^\lambda(z_2,\bar z_2)\phi^\lambda(z_3,\bar z_3)\phi_\lambda(z_4,\bar z_4)\rangle = (z_{14}z_{32}\bar z_{14}\bar z_{32})^{-2h}\mathcal{G}(x,\bar x)\ ,$$

where $h = h_{\omega_1} = \frac{(\omega_1,\omega_1+2\rho)}{2(k+2)} = \frac{3}{4(k+2)}$. The meaning of the notation ϕ^λ is that this field corresponds to the conjugate representation ω_1^* (which gives an additional sign when the t's act on it). We will use the definition $x = \frac{z_{12}z_{34}}{z_{14}z_{32}}$.

As usual, the correlator decomposes into holomorphic and anti-holomorphic parts, which we denote by $\mathcal{F}(x)$ and $\bar{\mathcal{F}}(\bar x)$. In addition, it decomposes into (in general) two $su(2)$ invariant pieces, so we have $\mathcal{F}(x) = I_1F_1(x) + I_2F_2(x)$, where $I_1 = \delta_{m_1,m_2}\delta_{m_3,m_4}$ and $I_2 = \delta_{m_1,m_3}\delta_{m_2,m_4}$. Here, the m_i refer to the components of the representation of the field at z_i.

Finally, after introducing this notation, the chiral part of the KZ equations read

$$\Big(\partial_{z_i} - \frac{1}{k+2}\sum_{j\neq i}\frac{\sum_a t^a_i\otimes t^a_j}{z_{ij}}\Big)(z_{14}z_{32})^{-2h}(I_1F_1(x) + I_2F_2(x)) = 0$$

a. Show that the equation for $i = 1$ reads

$$\Big[\Big(\frac{-2h}{z_{14}}\Big) + \Big(\frac{x}{z_{12}} - \frac{x}{z_{14}}\Big)\partial_x - \frac{1}{k+2}\Big(\frac{t^a_1\otimes t^a_2}{z_{12}} + \frac{t^a_1\otimes t^a_3}{z_{13}} + \frac{t^a_1\otimes t^a_4}{z_{14}}\Big)\Big](I_1F_1(x) + I_2F_2(x)) = 0$$

and find the three other equations as well.

We will now concentrate on the action of the t's on the I_1 and I_2. In the fundamental representation of $su(2)$, one has $\sum_a t^a_{ij}t^a_{kl} = \delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ij}\delta_{kl}$, from which the value of the quadratic Casimir follows $\sum_a (t^at^a)_{ij} = \frac{3}{2}\delta_{ij}$.

b. Calculate the relevant products $\sum_a t^a_1\otimes t^a_2 I_1$ etc. and express the result in I_1 and I_2 Remember that the t's act from the right, and with an additional sign, in the case of the conjugate representations, i.e. $\sum_a t^a_1\otimes t^a_2 I_1 = -\sum_{a,m_1',m2'} t^a_{m_1,m_1'}t^a_{m_2',m_2}\delta_{m_1',m_2'}\delta_{m_3,m_4} = -\sum_a (t^at^a)_{m_1,m_2}\delta_{m_3,m_4}$ etc.

c. Show, by taking a suitable linear combination of the equations, that one ends up with the following coupled equations for F_1 and F_2

$$\partial_x F_1 = \frac{-1}{2(k+2)}\Big(\frac{3F_1+2F_2}{x}+\frac{F_1}{1-x}\Big)$$
$$\partial_x F_2 = \frac{1}{2(k+2)}\Big(\frac{F_2}{x}+\frac{2F_1+3F_2}{1-x}\Big)$$

d. Decouple the equations, and show that they can be brought into the form of the hypergeometric equation $x(1-x)\partial_x^2 f(x)+(c-(a+b+1)x)\partial_x f(x)-abf(x)=0$. The resulting solutions are

$$F_1^{(0)}(x) = x^{-2h}(1-x)^{h_\theta-2h}f(\frac{1}{k+2},-\frac{1}{k+2},1-\frac{2}{k+2};x)$$
$$F_2^{(0)}(x) = \frac{1}{k}x^{1-2h}(1-x)^{h_\theta-2h}f(1+\frac{1}{k+2},1-\frac{1}{k+2},2-\frac{2}{k+2};x)$$

and

$$F_1^{(1)}(x) = x^{h_\theta-2h}(1-x)^{h_\theta-2h}f(\frac{1}{k+2},\frac{3}{k+2},1+\frac{2}{k+2};x)$$
$$F_2^{(1)}(x) = -2x^{h_\theta-2h}(1-x)^{h_\theta-2h}f(\frac{1}{k+2},\frac{3}{k+2},\frac{2}{k+2};x)$$

where $h_\theta = \frac{(\theta,\theta+2\rho)}{2(k+2)}$, and a convenient normalization is chosen.

e. Which of these solutions form the two different conformal blocks?