Exercises CFT-course fall 2008, set 11.

1. The Sugawara construction.

   In this exercise, we will verify that the Sugawara constuction

   $$T(z) = \frac{1}{\beta} \sum_a \; : J^a J^a : (z)$$

   indeed gives a stress energy tensor. Here, the normal ordering symbols are defined to pick out the constant term, i.e. for two general fields $A(z)$ and $B(z)$ we have

   $$: AB : (w) = \frac{1}{2\pi i} \oint_w \frac{dz}{(z-w)} A(z) B(w)$$

   The contraction $\underline{A(z)B(w)}$ contains all the singular terms, so we have

   $$: AB : (w) = \lim_{z \to w} \Big(A(z)B(w) - \underline{A(z)B(w)} \Big)$$

   a. To calculate the OPE of T(z) with J^a, we will need contractions of the type $\underline{A(z) : BC : (w)}$. Argue that the integral

   $$\underline{A(z) : BC : (w)} = \frac{1}{2\pi i} \oint_w \frac{dx}{x-w} \Big(\underline{A(z)B(x)}C(w) + B(x)\underline{A(z)C(w)} \Big)$$

   indeed gives all the singular terms in this contraction, by considering the various poles in the integrant. Which terms do contribute to the remaining operator products?

   b. Use the result of a. to calculate first $\sum_b J^a(z) : \underline{J^b J^b} : (w)$ and from that the contraction $\underline{T(z)J^a(w)}$. Determine β, assuming that J^a is a Virasoro primary field.

   c. Calculate the singular terms of the OPE $T(z)T(w)$, and determine the central charge of the a general WZW cft.

2. Affine Lie algebra primary fields.

   Show that an affine Lie algebra primary field is also a Virasoro primary field, and determine the conformal dimension of the affine Lie algebra primary fields.