Exercises CFT-course fall 2008, set 10.

1. Simple roots and fundamental weights of affine Lie algebras.

   Arbitrary weights $\hat{\lambda}$ are characterized by their finite part λ, their level k_λ and their grade n_λ: $\hat{\lambda} = (\lambda; k_\lambda; n_\lambda)$. The inner product between two arbitrary weights is given by $(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda$, where (λ, μ) is the inner product in the corresponding finite dimensional Lie algebra.

   The simple roots are given by $\hat{\alpha}_i = (\alpha_i; 0; 0)$ for $i = 1, \ldots, r$, and $\hat{\alpha}_0 = (-\theta; 0; 1)$, where the highest root is given in terms of the marks and co-marks as $\theta = \sum_{i=1}^r a_i \alpha_i = \sum_{i=1}^r a_i^\vee \alpha_i^\vee$, with $\alpha_i^\vee = \frac{2}{|\alpha_i|^2}\alpha_i$ and $a_i^\vee = \frac{|\alpha_i|^2}{2} a_i$.

a. The fundamental weights $\hat{\omega}_i$ are defined to be dual to the simple co-roots $\hat{\alpha}_i^\vee = (\alpha_i^\vee; 0; 0)$, i.e. $(\hat{\omega}_i, \hat{\alpha}_i^\vee) = \delta_{i,j}$, for $i, j = 0, 1, \ldots, r$. Determine the form of the fundamental weights.

b. Determine the level k of an arbitrary weight $\hat{\lambda} = \sum_{i=0}^r \lambda_i \hat{\omega}_i + l\delta$ in terms of the dynkin labels λ_i. Express $\hat{\lambda}$ in terms of $\hat{\omega}_0$, the finite fundamental weights and δ.

c. Express λ_0 in terms of finite Lie algebra data and k.

2. Highest weight representations.

   Let $\hat{\lambda} = \sum_{i=0}^r \lambda_i \hat{\omega}_i$ be the highest weight of a highest weight representation, which completely characterizes the representation.

a. Argue that, for the highest weight, all the λ_i, $i = 0, 1, \ldots, r$ are non-negative.

b. Show that for a fixed, finite k, the number of highest weight representations of an affine Lie algebra is finite.

c. For the affine Lie algebra $\hat{g}_2$, based on the finite Lie algebra g_2, the co-marks are given by $(a_0^\vee, a_1^\vee, a_2^\vee) = (1, 2, 1)$. Give the highest weights of the possible highest weight representations of $\hat{g}_2$ at levels $k = 1, 2, 3$ in terms of their dynkin labels.

d. For the affine Lie algebra $\widehat{su}(r+1)$, which has rank r, one has that all the marks and co-marks are unity, $a_i = a_i^\vee = 1$. What is the number of highest weight representations for $\widehat{su}(2)$ and $\widehat{su}(3)$ in terms of k? What about $su(r+1)_k$?

3. The weight space of representations of affine Lie algebras.

   Consider the algebra $su(2)_k$, whose Cartan matrix is given by $\widehat{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Obtaining the weights of highest weight representations is done completely analogously as for finite dimensional Lie algebras, because the elements of the rows of the Cartan matrix are the Dynkin labels of the simple roots: $\hat{\alpha}_0 = 2\hat{\omega}_0 - 2\hat{\omega}_1 + \delta$, and $\hat{\alpha}_1 = -2\hat{\omega}_0 + 2\hat{\omega}_1$. Note, however, that the root $\hat{\alpha}_0$ has a term δ as well, so subtracting $\hat{\alpha}_0$ from an arbitrary weight decreases the grade by one. The only difference with the finite dimensional case is that here, one can subtract simple roots ad infinitum.

a. Obtain the weights of the $su(2)_k$ highest weight representation $\hat{\lambda} = [0, 1] = \hat{\omega}_1$, for the first seven grades. I.e., construct the analogous diagram as in figure 14.4 of 'the big yellow book'. Note that you do not have to calculate the weight multiplicities.

b. Repeat exercise 3a. for the highest weight representation $\hat{\lambda} = [2, 0] = 2\hat{\omega}_0$, and compare with figure 14.4.