Exercises CFT-course fall 2008, set 9.

Due on wednesday, January 7th, 2009.

1. Obtaining the weights of representations. We use the following definition of the Cartan matrix: $A_{i,j} = \frac{2(\alpha_i,\alpha_j)}{|\alpha_j|^2}$.

   Consider the exceptional algebra g_2, whose Cartan matrix is given by $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

a. The highest root is given by $\theta = (1,0)$. Express θ in terms of the root and co-roots, and give the dual Coxeter number.

b. Obtain all the roots and draw the root diagram. What is the dimension of g_2?

c. Consider the representation $(0,1)$. Obtain the weights, and the dimension of this representation.

2. Consider the $sp(4)$, whose Cartan matrix is given by $A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$.

a. The highest root is given by $\theta = (2,0)$. Express θ in terms of the root and co-roots, and give the dual Coxeter number.

b. Obtain all the roots and draw the root diagram. What is the dimension of $sp(4)$?

c. Consider the representations $(0,1)$ and $(1,0)$. Obtain their weights, and the dimension of these representations. Draw the weight diagrams.

3. The Weyl group $\mathcal{W}$. The simple Weyl reflections $s_i := s_{\alpha_i}$ (where α_i is a simple root) map weights to other weight via $s_{\alpha_i}\lambda = \lambda - \alpha_i(\lambda, \alpha_i^\vee)$. Arbitrary elements $w \in \mathcal{W}$ of the Weyl group can be written in terms of the simple Weyl reflections as $w = s_{i_k} \cdots s_{i_2} s_{i_1}$. The simple Weyl reflections satisfy $s_i^2 = 1$ and $(s_i s_j)^m_{ij} = 1$, where $m_{ij} = 2, 3, 4, 6$ for $A_{ij}A_{ji} = 0, 1, 2, 3$. Weights of a highest weight representation which are mapped onto each other by the action of the Weyl group have the same dimension.

a. Write the action of the simple Weyl reflections on simple roots in terms of the Cartan matrix, and show that the simple Weyl reflections map positive roots onto positive roots. The exception is $s_i\alpha_i = -\alpha_i$.

b. Show that the inner product is invariant under the action of the simple Weyl reflections, and use this to argue that the Weyl vector, which is defined as $\rho = \sum_{\alpha>0} \alpha$ can be written as $\rho = \sum_{i=1}^{r} \omega_i$, where the ω_i are the fundamental weights.