Exercises CFT-course fall 2008, set 8.

Due on wednesday, december 17th, 2008.

1. The quadratic Casimir operator.

   The Cartan-Weyl basis reads

$$
\begin{aligned}
[H^i, H^j] &= 0 \\
[H^i, E^\alpha] &= \alpha^i E^\alpha \\
[E^\alpha, E^\beta] &= N_{\alpha,\beta} E^{\alpha+\beta} \qquad \alpha+\beta \in \Delta \\
&= \frac{2}{|\alpha|^2} \alpha \cdot H \qquad \alpha = -\beta \\
&= 0 \qquad \text{otherwise}
\end{aligned}
$$

   The $N_{\alpha,\beta}$ are constants and Δ is the set of all roots.

   Show that the Casimir operator

$$
\mathcal{C} = \sum_i H^i H^i + \sum_{\alpha>0} \frac{|\alpha|^2}{2} \big(E^\alpha E^{-\alpha} + E^{-\alpha} E^\alpha \big) \tag{1}
$$

   commutes with all the generators of the Lie algebra.
   Hint: use the invariance of the Killing form to show that $|\alpha|^2 N_{\alpha,\beta} = |\alpha+\beta|^2 N_{\beta,-(\alpha+\beta)}$

2. The Freudenthal recursion formula.

   In this exercise, we will prove the Freudenthal recursion formula, which gives the multiplicities of the weights λ' in a highest weight representation λ, namely $\text{mult}_\lambda(\lambda')$ in terms of the weights above λ':

$$
\big(|\lambda+\rho|^2 - |\lambda'+\rho|^2\big)\text{mult}_\lambda(\lambda') = 2\sum_{\alpha>0}\sum_{k=1}^{\infty}(\lambda'+k\alpha,\alpha)\text{mult}_\lambda(\lambda'+k\alpha) \ ,
$$

   where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$.

a. Show that for each weight state $|\lambda', i\rangle$, where $i = 1, \ldots, n_{\lambda'} = \text{mult}_\lambda(\lambda')$, in the highest weight representation $|\lambda\rangle$, one has $\mathcal{C}|\lambda', i\rangle = (\lambda, \lambda+2\rho)|\lambda', i\rangle$. and argue that in the subspace $|\lambda'\rangle$, one has $\text{Tr}_{\lambda'}\,\mathcal{C} = n_{\lambda'}(\lambda, \lambda+2\rho)$.

   We will now calculate $\text{Tr}_{\lambda'}\,\mathcal{C}$ differently.

b. First, calculate $\text{Tr}_{\lambda'}\sum_i H^i H^i$.

   To calculate the remainder, we will make use of the fact that all the states $|\lambda', i\rangle$ can also be considered as weights in a representation of the $su(2)$ subalgebra ($E^\alpha, E^{-\alpha}, \frac{2\alpha\cdot H}{|\alpha|^2} = \sqrt{2}H$) (where H is in the Cartan-Weyl basis).

c. Consider the quadratic Casimir $\mathcal{C}'$ of this $su(2)$ algebra. Show that $\mathcal{C}' = \frac{1}{2}H^\alpha H^\alpha + E^\alpha E^{-\alpha} + E^{-\alpha}E^\alpha$, and $\mathcal{C}'|\lambda', i\rangle = \frac{1}{2}(a(a+2))|\lambda', i\rangle$, where (the integer) a is the highest weight of the $su(2)$ representation under consideration.

d. Suppose that the highest weight is in fact $\lambda' + k\alpha$, where $k \geq 0$. Show that $a = \frac{2(\alpha, \lambda' + k\alpha)}{|\alpha|^2}$ and deduce

$$\frac{|\alpha|^2}{2}\big(E^\alpha E^{-\alpha} + E^{-\alpha}E^\alpha\big)|\lambda'\rangle = (k(k+1)(\alpha,\alpha) + (2k+1)(\lambda',\alpha))|\lambda'\rangle$$

All the weights $|\lambda', i\rangle$ in the ($n_{\lambda'}$-dimensional) weight space $|\lambda'\rangle$ have a corresponding value of k, which can be the same for different weights.

e. Argue that

$$\mathrm{Tr}_{\lambda'}\,\frac{|\alpha|^2}{2}\big(E^\alpha E^{-\alpha} + E^{-\alpha}E^\alpha\big) = \sum_{k\geq 0}(n_{\lambda'+k\alpha} - n_{\lambda'+(k+1)\alpha})(k(k+1)(\alpha,\alpha) + (2k+1)(\lambda',\alpha))$$

f. Finish the proof of the Freudenthal recursion formula.