Exercises CFT-course fall 2008, set 7.

Due on wednesday, december 3rd, 2008.

1. Majorana fermions with periodic and anti-periodic boundary conditions.

   The mode expansion for free (Majorana) fermions reads $\psi(z) = \sum_n \psi_n z^{-n-1/2}$, or $\psi_n = \oint \frac{dz}{2\pi i} z^{n-1/2} \psi(z)$.

a. Show that the modes obey $\{\psi_n, \psi_m\} = \delta_{n+m,0}$.

   We will now consider periodic and anti-periodic boundary conditions for the fermion $\psi(z)$ when z is moved around the origin: $\psi(e^{2\pi i} z) = \pm\psi(z)$. The modes n are half integer $n \in \mathbb{Z} + \frac{1}{2}$ in the periodic (P) case, and integer $n \in \mathbb{Z}$ in the anti-periodic (A) case.

b. Use the explicit mode expansions to show that

$$\langle \psi(z)\psi(w)\rangle_{\mathrm{P}} = \frac{1}{z-w}$$
$$\langle \psi(z)\psi(w)\rangle_{\mathrm{A}} = \frac{1}{2}\frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{z-w}$$

   It is given that

$$\langle \sigma(w_1)\psi(z_1)\psi(z_2)\sigma(w_2)\rangle = \frac{1}{2}(w_1-w_2)^{-1/8} \frac{\left(\frac{(z_1-w_1)(z_2-w_2)}{(z_1-w_2)(z_2-w_1)}\right)^{1/2} + \left(\frac{(z_1-w_2)(z_2-w_1)}{(z_1-w_1)(z_2-w_2)}\right)^{1/2}}{(z_1-z_2)}$$

c. Consider $\langle\sigma(\infty)|\psi(z_1)\psi(z_2)|\sigma(0)\rangle$, where $\langle\sigma(\infty)| = \lim_{w\to\infty}\langle 0|\sigma(w) w^{2h}$, and argue that σ 'changes the boundary conditions on ψ'.

2. Character formula's for the Ising model.

a. Calculate the character of the vacuum and ψ sector by calculating the partition function of states of the form

$$\psi_{-(n-1)/2-p_n} \cdots \psi_{-3/2-p_2}\psi_{-1/2-p_1}|0\rangle \ ,$$

   where $0 \le p_1 \le p_2 \le \cdots \le p_n$.

b. Repeat exercise a. for the σ sector, by considering

$$\psi_{-(n-1)-p_n} \cdots \psi_{-1-p_2}\psi_{-p_1}|\sigma\rangle \ ,$$

   again with $0 \le p_1 \le p_2 \le \cdots \le p_n$.

   Answer:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} = \prod_{n=1}^{\infty}(1+q^n) = \sum_{n\ge 0 \text{ even}} \frac{q^{n(n-1)/2}}{(q)_n} = \sum_{n \ge 1 \text{ odd}} \frac{q^{n(n-1)/2}}{(q)_n} \ . \qquad (1)$$

c. Obtain the last two equalities in (1), by making use of an identity due to Cauchy (which you don't have to prove):

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} x^n}{(q)_n} = \prod_{n=0}^{\infty} (1 + x q^n) \ ,$$

and considering the role of the zero mode ψ_0.

3. Modular transformation properties of the Ising characters.

   The following definitions and identities are given ($q = e^{2\pi i \tau}$):

$$\theta_2(\tau) = \sum_{n=\in\mathbb{Z}} q^{(n+1/2)^2/2} = 2q^{1/8} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2 \qquad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

$$\theta_3(\tau) = \sum_{n=\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-1/2})^2 \qquad \eta(\tau)^3 = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)$$

$$\theta_4(\tau) = \sum_{n=\in\mathbb{Z}} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-1/2})^2$$

a. Show, by making use of the results of exercise 2., that the characters of the Ising model are given by

$$\chi_0 = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \chi_{1/2} = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \chi_{1/16} = \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}$$

b. Calculate the transformation properties of the Ising characters under the transformation $\tau \to -1/\tau$, by making use of the Poisson resummation formula:

$$\sum_{n\in\mathbb{Z}} e^{-\pi a n^2 + bn} = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} e^{(b+2\pi i k)^2/(4\pi a)}$$

4. Constraints on c and h_i from the fusion rules.

   Different conformal field theories can have the same fusion rules. However, for a given set of fusion rules, the possible values of c and h_i are restricted by modular invariance.

   We will consider a theory with three fields $\mathbf{1}$, σ and ψ, with the fusion rules given by $\sigma \times \sigma = \mathbf{1} + \psi$, $\sigma \times \psi = \sigma$ and $\psi \times \psi = \mathbf{1}$. Thus the fusion matrices are (the fields are ordered as $\mathbf{1}, \sigma, \psi$)

$$N_{\mathbf{1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad N_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad N_\psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

These fusion rules are diagonalized by

$$S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \ .$$

Show that the general relation $(ST)^3 = C^2$, where $C = S^2$ is the conjugation matrix (which satisfies $C^2 = 1$), and $T_{i,j} = e^{2\pi i(h_i - c/8)}\delta_{i,j}$ gives rise to the constraints

$$h_\psi = \frac{1}{2} \bmod 1 \qquad\qquad h_\sigma - c/8 = 0 \bmod 1 \ .$$