Exercises CFT-course fall 2008, set 6.

Due on wednesday, november 19, 2008.

1. The hypergeometric differential equation reads

$$\big(z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab\big)f(z) = 0$$

a. Find a solution around $z=0$ by substituting $f(z) = \sum_{n\geq 0} f_n z^n$, and express the result in terms of $(a)_n$, $(b)_n$ and $(c)_n$, where $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$, i.e. $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.
Answer: $f(z) = F(a,b,c;z) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n n!} z^n$. Note: the other solution around $z=0$ reads $z^{1-c}F(a-c+1, b-c+1, 2-c; z)$.

b. When are polynomial solutions of the hypergeometric differential equation possible?

2. In this exercise, we will derive an ordinary differential equation for the four point function of $\phi_{1,2}$ of the minimal models:

$$G(\{w_i\}) = \langle \phi_{1,2}(w_1)\phi_{1,2}(w_2)\phi_{1,2}(w_3)\phi_{1,2}(w_4)\rangle = (w_1-w_2)^{-2h}(w_3-w_4)^{-2h}F(x) \ ,$$

where $x = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$, and h is the conformal dimension of $\phi_{1,2}$.
The level-2 null-vector condition translates into the following partial differential equations:

$$\left(\frac{3}{2(2h+1)}\partial_{w_i}^2 - \sum_{j\neq i}\Big(\frac{\partial_{w_j}}{(w_i-w_j)} + \frac{h}{(w_i-w_j)^2}\Big)\right)G(\{w_i\}) = 0 \ .$$

a. Show, by considering the special values $z_1 = 0, z_2 = x, z_3 = \infty, z_4 = 1$, that $F(x)$ satisfies

$$\left(x(1-x)\partial_x^2 + \Big(\frac{2(1-4h)}{3} + \frac{4(h-1)}{3}x\Big)\partial_x - \frac{2h(2h+1)}{3}\frac{x}{1-x}\right)F(x) = 0$$

b. Show that the differential equation obtained in a. can be brought into the form of the hypergeometric differential equation, by defining a function $H(x) = f(x)F(x)$, such that $H(x)$ satisfies

$$\left(x(1-x)\partial_x^2 + \Big(\frac{2(1-4h)}{3} + \frac{4(4h-1)}{3}x\Big)\partial_x + \frac{4h(1-4h)}{3}\right)H(x) = 0 \qquad (1)$$

3. Show that in the case of the Ising model, two independent solutions of the differential equation (1) (with $h = \frac{1}{16}$) take the form

$$H^{\pm}(x) = \sqrt{\frac{1\pm\sqrt{1-x}}{2}} \ ,$$

by making a suitable (trigonometric) transformation.

Thus, the (chiral) four-point σ-correlator of the Ising model takes the form

$$\langle \sigma(w_1)\sigma(w_2)\sigma(w_3)\sigma(w_4)\rangle^{\pm} = (w_1 - w_2)^{-1/8}(w_3 - w_4)^{-1/8}(1-x)^{-1/8}\sqrt{\frac{1 \pm \sqrt{1-x}}{2}} \ .$$