# Exercises CFT-course fall 2008, set 5.

Due on friday, november 7, 2008.

1. Unitarity of $SU(2)$ representations.

   Consider the $SU(2)$ algebra

   $$[J_+, J_-] = 2J_0 \qquad [J_0, J_\pm] = \pm J_\pm$$

   and let $|j\rangle$ be a highest weight, $J_+|j\rangle = 0$, with $J_0$ eigenvalue $j$. Show that one can only construct unitary highest weight representations for $j$ either a non-negative integer of a positive half-integer.

2. Non-unitarity of Virasoro representations with $0 < c < 1$.

   The vanishing curves of the 'Kac determinant' are (for instance) given by

   $$h_{r,s}(c) = \frac{1-c}{96}\left[\left((r+s) \pm (r-s)\sqrt{\frac{25-c}{1-c}}\right)^2 - 4\right] .$$

   Given that the Kac determinant only has positive eigenvalues when $c > 1$ and $h > 0$, argue that unitary representations are excluded in the region $0 < c < 1$ and $h > 0$, except for possibly those points on the vanishing curves.

3. Consequences of null-vectors.

a. Show that a necessary condition for the (chiral) correlator of primaries

   $$\langle \phi_{2,1}(z_1)\phi_{r,s}(z_2)\phi(z_3)\rangle$$

   to be non-zero is that the scaling dimension $h$ of $\phi(z_3)$ is equal to $h = h_{r\pm1,s}$, by making use of the level 2 null vector condition. Recall:

   $$h_{r,s} = \frac{(rp - sp') - (p - p')^2}{4pp'} ,$$

   where $p$ and $p'$ are relative prime and $0 < r < p'$ and $0 < s < p$.

   Note that this condition $h = h_{r\pm1,s}$ is not sufficient in general, as the three point function might still vanish. The necessary and sufficient condition is expressed in terms of the 'fusion rules', which for $\phi_{r,1}$ read

   $$\phi_{r,1} \times \phi_{r',s} = \sum_{\substack{k=|r-r'|+1 \\ k+r+r'=1 \bmod 2}}^{k_{\max}} \phi_{k,s}$$

   where $k_{\max} = \min(r + r' - 1, 2p' - r - r' - 1)$. These fusion rules express which three point functions are non-zero.

b. Ignoring the truncation in the upper limit $k_{\max}$, interpret the fusion rules for $\phi_{r,1}$ in terms of tensor products of $SU(2)$.