Exercises CFT-course fall 2008, set 3.

Due on friday, october 31, 2008.

1. The massless free boson ϕ has the action

$$S = \frac{g}{2}\int dx^2 \partial_\mu\phi\partial^\mu\phi \ .$$

a. Determine the propagator of the free massless boson theory:

$$\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = \frac{-1}{4\pi g}\big(\ln(z-w)+\ln(\bar{z}-\bar{w}\big) + \text{const.}$$

b. Determine the form of the stress-energy tensor, namely $T(z) = -2\pi g :\partial\phi(z)\partial\phi(z):$.

c. Calculate the OPE $T(z)T(w)$.

d. Show that the vertex operator $V =: e^{i\alpha\phi} : (z)$ is a primary field of dimension $h_\alpha = \frac{\alpha^2}{8\pi g}$.

2. The massless free fermion.

The action of the massless free fermion can be written as

$$S = g\int dx^2(\overline{\psi}\partial\overline{\psi} + \psi\overline{\partial}\psi) \ ,$$

where ψ and $\overline{\psi}$ are the two components of the spinor Ψ. It is given that the equation of motion are $\partial\overline{\psi} = \overline{\partial}\psi = 0$, while the propagator takes the form $\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle = \frac{1}{2\pi g}\frac{1}{z-w}$. The stress-energy tensor can be written as $T(z) = -\pi g : \psi(z)\partial\psi(z):$.

a. Show that $\psi(z)$ is a primary field.

b. Calculate the OPE $T(z)T(w)$.

3. The Virasoro algebra.

Show that the Virasoro algebra has the following form

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \ ,$$

where $L_n = \oint \frac{dz}{2\pi i} z^{n+1}T(z)$.

4. Mode expansion of primary fields.

In analogy to the stress-energy tensor, we define a mode expansion of (holomorphic) primary fields $\phi(z)$ as

$$\phi(z) = \sum_{n\in\mathbb{Z}-h}\phi_n z^{-n-h} \qquad \phi_n = \oint\frac{dz}{2\pi i}z^{h+n-1}\phi(z) \ ,$$

where ϕ_{-n} has weight n.

a. Show that regularity requires $\phi_n|0\rangle = 0$ for $n \geq -h+1$.

b. Calculate the commutator $[L_n, \phi_m] = (n(h-1)-m)\phi_{m+n}$ by first calculating $[L_n, \phi(w)]$ and show that $L_0\phi_{-h}|0\rangle = h|h\rangle$, as expected.