Exercises CFT-course fall 2008, set 2.

Due on wednesday, october 15, 2008.

1. The conformal ward identity.

a. Carefully derive the conformal ward identity

$$\delta\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle = \frac{1}{2\pi i}\oint_C dz\varepsilon(z)\langle T(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle + c.c.\ .$$

b. Show that for primary fields, this gives rise to the equation

$$\langle T(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle = \sum_i \Big(\frac{h_i}{(z-z_i)^2} + \frac{\partial_{z_i}}{(z-z_i)}\Big)\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle\ .$$

2. The Schwartzian derivative.

a. Show that the Schwartzian derivative

$$\{w;z\} = \frac{w'''}{w'} - \frac{3}{2}\Big(\frac{w''}{w'}\Big)^2\ ,$$

(with $w' = \frac{dw}{dz}$ etc), is zero if and only if $w(z) = \frac{az+b}{cz+d}$. Start by solving the first order differential equation for (w''/w'), and proceed by solving for $v = w'$.

b. Show that for successive transformations $z \to w \to u$, one has the following composition rule

$$\{u;z\} = \{w;z\} + \Big(\frac{dw}{dz}\Big)^2\{u;w\}$$

3. The two point function on the cylinder.

a. Calculate the two point function of a primary operator $\phi(z,\bar{z})$ with scaling dimensions $(h,\bar{h})$ on the cylinder, by using the conformal map $w = \frac{L\log z}{2\pi}$. Result:

$$\langle\phi(w_1,\bar{w}_1)\phi(w_2,\bar{w}_2)\rangle = \Big(\frac{2\pi}{L}\Big)^{2h+2\bar{h}}\big(2\sinh\big(\pi(w_1-w_2)/L\big)\big)^{-2h}\big(2\sinh\big(\pi(\bar{w}_1-\bar{w}_2)/L\big)\big)^{-2\bar{h}}$$

b. Let $w = t + iu$. Analyze the result of a. in the limits $t_1 - t_2 \ll L$ and $t_1 - t_2 \gg L$. For simplicity, you may assume that $h = \bar{h}$.