Note about the conformal Ward identity

Primary fields transform under arbitrary conformal transformations $z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$ with $f(z), \bar{f}(\bar{z})$ a holomorphic (anti-holomorphic) function, as

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \Big(\frac{\partial f(z)}{\partial z}\Big)^h \Big(\frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}}\Big)^{\bar{h}} \phi(f(z_1), \bar{f}(\bar{z_1})) \tag{1}$$

Quasi-primary fields transform in the same way, if the conformal transformation is a projective transformation (i.e., those which are invertible). The correlation function of primary fields transform covariantly, namely

$$\big\langle \phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\big\rangle = \prod_{j=1}^{n}\Big(\frac{\partial f(z_j)}{\partial z_j}\Big)^h \Big(\frac{\partial \bar{f}(\bar{z}_j)}{\partial \bar{z}_j}\Big)^{\bar{h}} \big\langle \phi_1(f(z_1),\bar{f}(\bar{z}_1))\cdots\phi_n(f(z_n),\bar{f}(\bar{z}_n))\big\rangle \tag{2}$$

For infinitesimal transformations $z \to z + \epsilon(z)$, $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$, we introduce the notation

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \phi'(z,\bar{z}) - \phi(z,\bar{z}) \tag{3}$$

Explicitly, we find that to first order

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \Big(h(\partial_z\epsilon) + \epsilon\partial_z + \bar{h}(\partial_{\bar{z}}\bar{\epsilon}) + \bar{\epsilon}\partial_{\bar{z}}\Big)\phi(z,\bar{z}) \tag{4}$$

Let us introduce the notation

$$\delta_{\epsilon,\bar{\epsilon}}\big\langle \phi(z_1,\bar{z}_1)\cdots\phi(z_n,\bar{z}_n)\big\rangle = \sum_{j=1}^{n}\big\langle \phi(z_1,\bar{z}_1)\cdots\delta_{\epsilon,\bar{\epsilon}}\phi(z_j,\bar{z}_j)\cdots\phi(z_n,\bar{z}_n)\big\rangle \tag{5}$$

The infinitesimal form of eq. (2), for projective transformations is given by

$$\delta_{\epsilon,\bar{\epsilon}}\big\langle \phi(z_1,\bar{z}_1)\cdots\phi(z_n,\bar{z}_n)\big\rangle = 0 \tag{6}$$

Here, we used that the action is invariant under projective conformal transformations.

For arbitrary infinitesimal conformal transformations, which are restricted to some region D_1, with boundary C, invariance of the action under projective conformal transformations gives, with $f(z) = z + \epsilon(z)$, etc.

$$\big\langle \phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\big\rangle_S = \prod_{j=1}^{n}\Big(\frac{\partial f(z_j)}{\partial z_j}\Big)^h \Big(\frac{\partial \bar{f}(\bar{z}_j)}{\partial \bar{z}_j}\Big)^{\bar{h}} \big\langle \phi_1(f(z_1),\bar{f}(\bar{z}_1))\cdots\phi_n(f(z_n),\bar{f}(\bar{z}_n))\big\rangle_{S+\delta S} \tag{7}$$

where the subscript on the correlation function denotes the action to be used to calculate the correlation function. In addition, the change in action is given by

$$\delta S = \frac{1}{2\pi i}\oint_C \epsilon(z)T(z)dz + \text{c.c.} \tag{8}$$

Collecting the first order terms, we obtain the conformal Ward identity

$$\delta_{\epsilon,\bar{\epsilon}}\big\langle \phi(z_1,\bar{z}_1)\cdots\phi(z_n,\bar{z}_n)\big\rangle = \frac{1}{2\pi i}\oint_C \epsilon(z)\big\langle T(z)\phi(z_1,\bar{z}_1)\cdots\phi(z_n,\bar{z}_n)\big\rangle dz + \text{c.c.} \tag{9}$$

where the correlation functions are to be calculated with respect to S.