Last week:

* Time ordering on cylinder $\to$ radial ordering on plane.

* Mode expansion: $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, $L_n = \frac{1}{2\pi i}\oint_0 z^{n+1} T(z)\, dz$

* OPE: $T(z)T(w) = \cdots \Rightarrow [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$

Action of L_n on vacuum: $L_n |0\rangle = 0$ for $n \geq -1$

On a state $|\varphi_j\rangle = \varphi_j(0)|0\rangle$:
($\varphi_j(z)$: primary)

$$L_0|\varphi_j\rangle = \left[\frac{1}{2\pi i}\oint_C dz\, z\, T(z)\, \varphi_j(0)\right]|0\rangle$$

$$\underset{\substack{\uparrow \\ \text{OPE}}}{=} h_j \varphi_j(0)|0\rangle = h_j|\varphi_j\rangle$$

OPE: $T(z)\varphi_j(0) \sim \frac{h_j}{z^2}\varphi_j(0) + \frac{1}{z}\partial_{z'}\varphi_j(z')\Big|_{z'=0}$ + reg

For primary fields, the term $\frac{h_j \phi_j(0)}{z^2}$ is the most singular term in $T(z)\phi_j(0)$ by definition. Thus we have $L_n |\phi_j\rangle = 0$ for $n \geq 1$ if ϕ_j is primary.

The states $L_n |\phi_j\rangle$ with $n \leq -1$ correspond to descendants of ϕ_j

From the Virasoro algebra: $[L_0, L_{-n}] = nL_{-n}$, so we find

$$L_0 (L_{-n} |\phi_j\rangle) = (L_{-n}L_0 + nL_{-n}) |\phi_j\rangle = (h_j + n)(L_{-n} |\phi_j\rangle)$$, so the

L_n, w/ $n \leq 0$ raise the scaling dimension of $|\phi_j\rangle$ by $(-n)$

The L_n with $n > 0$ are lowering operators, w/ $L_n |\phi_j\rangle = 0$ for $n > 0$, so $|\phi_j\rangle$ is a lowest weight state.

The states $L_{-n_k} \cdots L_{-n_2} L_{-n_1} |\phi_j\rangle$, with $n_i \geq 0$ are nevertheless called descendants.

The descendant fields $\hat{L}_{-n}\phi(w)$ w/ $n > 0$ are obtained from the OPE $T(z)\,\phi(w)$:

$$T(z)\,\phi(w) = \sum_{n \geq 0} (z-w)^{n-2}\,\hat{L}_{-n}\,\phi(w) = \frac{\overbrace{\hat{L}_0\,\phi(w)}^{h\phi(w)}}{(z-w)^2} + \frac{\hat{L}_{-1}\,\phi(w)}{(z-w)} + \hat{L}_{-2}\,\phi(w) + (z-w)\,\hat{L}_{-3}\,\phi(w) + \cdots$$

We'll define: $$\phi^{(-n)}(w) := \hat{L}_{-n}\,\phi(w) = \oint \frac{dz}{2\pi i}\,\frac{1}{(z-w)^{n-1}}\,T(z)\,\phi(w)$$

Examples: $\phi^{(0)}(w) = h\,\phi(w)$;

$\phi^{(-1)}(w) = \partial\phi(w)$

$$\left(\hat{L}_{-2}\mathbb{1}\right)(w) = \oint \frac{dz}{(2\pi i)}\,\frac{1}{(z-w)}\,T(z)\,\mathbb{1} = T(w).$$

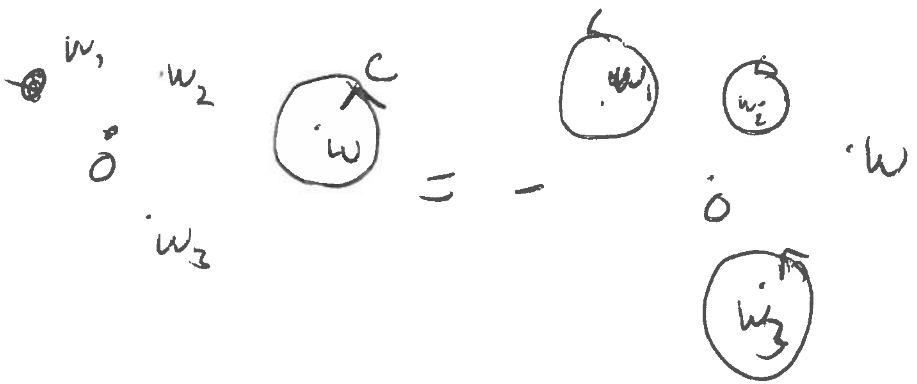
$T(z)$ is a descendent of the identity!

Using the Ward identity, we can calculate correlators of descendant fields.

Take primary fields $\phi_1(w_1) \ldots \phi_n(w_n)$, and $\phi^{(-k)}(w)$ a descendant:

$$\langle \phi^{(-k)}(w)\, \phi_1(w_1) \cdots \phi_n(w_n) \rangle = \oint_{C_w} \frac{dz}{2\pi i} (z-w)^{1-k} \langle T(z)\, \phi(w)\, \underbrace{\phi_1(w_1) \cdots \phi_n(w_n)}_{X(w_i)} \rangle$$

C_w: no w_i's inside



$$= -\sum_{j=1}^{n} \oint_{C_{w_j}} \frac{dz}{2\pi i} (z-w)^{1-k} \langle T(z)\, \phi(w)\, X(w_j) \rangle$$

$$= -\sum_{j=1}^{n} \oint_{C_{w_j}} \frac{dz}{2\pi i} \overset{(z-w)^{1-k}}{\vee} \left(\frac{h_j}{(z-w_j)^2} + \frac{\partial_{w_j}}{(z-w_j)} \right) \langle \phi(w)\, \phi_1(w_1) \cdots \phi_n(w_n) \rangle$$

OPE $T(z)\, \phi_j(w_j)$

$$\underset{\substack{\uparrow \\ \text{do the} \\ \text{z integral}}}{=} \underbrace{-\sum_{j=1}^{n}\left(\frac{(1-k)\,h_j}{(w_j-w)^k}+\frac{\partial_{w_j}}{(w_j-w)^{k-1}}\right)}_{:=\,\mathcal{L}_{-k}} \langle \phi(w)\,\phi_1(w_1)\cdots\phi_n(w_n)\rangle$$

So, $\langle \phi^{(-k)}(w)\,\phi_1(w_1)\cdots\phi_n(w_n)\rangle = \mathcal{L}_{-k}\langle\phi(w)\,\phi_1(w_1)\cdots\phi_n(w_n)\rangle$

In addition: $\mathcal{L}_{-1} = -\sum_j \partial_{w_j} = \partial_w$, because $\left(\partial_w + \sum_j \partial_{w_j}\right)\langle\phi(w)\phi_1(w_1)\cdots\phi_n(w_n)\rangle = 0$.

For one arbitrary descendant field: $\phi^{(-k_1,-k_2,\dots,-k_\ell)}(w) = \hat{L}_{-k_1}\cdots\hat{L}_{-k_\ell}\phi(w)$, we have: $\langle\phi^{(-k_1,\dots,-k_\ell)}(w)\,\phi_1(w_1)\cdots\phi_n(w_n)\rangle = \mathcal{L}_{-k_1}\mathcal{L}_{-k_2}\cdots\mathcal{L}_{-k_\ell}\langle\phi(w)\phi_1(w_1)\cdots\phi_n(w_n)\rangle$.

More than one descendant: more complicated!

Representations of the Virasoro algebra

To ~~test~~ construct a rep. of Vir, we take a lowest weight state $|\varphi\rangle$, w/ ~~wh~~ weight h, and act w/ raising operators L_{-n}. This gives an ∞ tower of states, called a Verma module.

Level	sc. dim	states	# states
0	h	$\|\varphi\rangle$	1
1	$h+1$	$L_{-1}\|\varphi\rangle$	1
2	$h+2$	$L_{-2}\|\varphi\rangle$; $(L_{-1})^2\|\varphi\rangle$	2
3	$h+3$	$L_{-3}\|\varphi\rangle$; $L_{-2}L_{-1}\|\varphi\rangle$; $(L_{-1})^3\|\varphi\rangle$	3
⋮			
N	$h+N$	....	$p(N)$

OBS: ~~there~~ can be null states (see next week)

$P(N)$: # of partitions of N into positive integers

Generating function for $p(N)$:

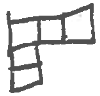
$$\sum_{N=0}^{\infty} p(N)\, q^N = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} =: \frac{1}{(q)_\infty}$$

$$\frac{1}{(q)_\infty} = \underbrace{(1+q+q^2+q^3+\cdots)}_{\#1's}\underbrace{(1+q^2+q^4+\cdots)}_{\#2's}\underbrace{(1+q^3+q^6+\cdots)}_{\#3's}\cdots\cdots$$

Young diagram

$q^5 = q^5 \cdot 1 \cdot 1 \cdots\cdots$: $1+1+1+1+1$ 

$q^5 = q^3 \cdot q^2 \cdot 1, \cdots\cdots$: $1+1+1+2$ 

$q^5 = q^2 \cdot 1 \cdot q^3 \cdot 1 \cdots\cdots$: $1+1+3$ 

$q^5 = q^1 \cdot q^4 \cdot 1 \cdots$: $1+2+2$  $\Rightarrow p(5)=6$

$q^5 = q^1 \cdot 1 \cdot 1 \cdot q^4$: $1+4$ 

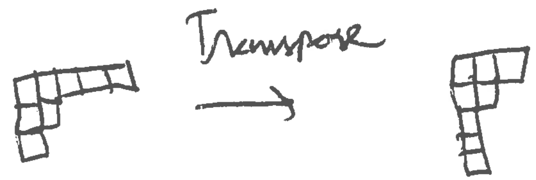
$q^5 = 1 \cdot 1 \cdot 1 \cdot 1 \cdot q^5 \cdot 1 \cdots$: 5 

Partitions play an important role in representation theory; ~~there's~~ they're fun to play with.

Let's define: $$\frac{1}{(q)_M} = \frac{1}{\prod_{n=1}^{M}(1-q^n)} = \sum_{N=0}^{\infty} P(N,M)\, q^N,$$ where

$P(N,M)$ is # of partitions of N into integers smaller or equal to M.

By transposing the Young diagram, we see that $P(N,M)$ is also the # of partitions of N into maximally M integers:



Durfee square: $$\frac{1}{(q)_\infty} = \sum_{M \geq 0} \frac{q^{M^2}}{(q)_M (q)_M}$$

M: size of max. square fitting in a Young tableau: