The conformal group in d dimensions:

Transformations that leave all angles invariant; are local scale transformations

So, we consider the line element: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \mu, \nu = 1, \dots, d$

For arbitrary transformations $x^\mu \to x'^\mu$, we have $g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$

For conformal invariant transformations, we demand: $g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$

For these transformations, $\frac{v \cdot w}{(v^2 w^2)^{\frac{1}{2}}}$ is constant, so angles are conserved

[$v \cdot w = g_{\mu\nu} v^\mu w^\nu$; summation convention]

Simple examples:
- translations
- rotations
- dilatations

Let's consider infinitesimal transformations:

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \quad , \text{w/} \quad \varepsilon_{\mu} = g_{\mu\nu}\varepsilon^{\nu} \; ; \; (g_{\mu\nu})^{-1} = g^{\mu\nu}$$

The line element ~~ds~~ transforms as:

$$ds^2 \to ds^2 - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)\, dx^\mu dx^\nu ;$$ so, for conformal transformations, we have

$$* \quad \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x)\, g_{\mu\nu}.$$

~~~~ We take the original metric to be orthogonal: $g_{\mu\nu} = \delta_{\mu\nu}$

To find the constraints on $f(x)$, we take the trace: $g^{\mu\nu} g_{\mu\nu} = d$

$$\Rightarrow g^{\mu\nu}(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) = 2(\partial \cdot \varepsilon) = f(x)\, d$$

Act w/ $-\partial_\rho$ on (*): $\quad -\partial_\rho \partial_\mu \varepsilon_\nu - \partial_\rho \partial_\nu \varepsilon_\mu = -(\partial_\rho f)\, \delta_{\mu\nu}$

$(\nu \leftrightarrow \rho) \quad +\partial_\nu \partial_\mu \varepsilon_\rho + \partial_\rho \partial_\nu \varepsilon_\mu = (\partial_\nu f)\, \delta_{\mu\rho}$

$(\mu \leftrightarrow \nu) \quad +\partial_\mu \partial_\nu \varepsilon_\rho + \partial_\rho \partial_\mu \varepsilon_\nu = (\partial_\mu f)\, \delta_{\nu\rho}$
~~~~

Adding the last three equations gives:

$$(**) \quad 2\partial_\mu\partial_\nu\varepsilon_\rho = (\partial_\mu f)\delta_{\nu\rho} + (\partial_\nu f)\delta_{\mu\rho} - (\partial_\rho f)\delta_{\mu\nu}$$

Contract w/ $\delta^{\mu\nu}$: $\quad 2\partial^\mu\partial_\mu\varepsilon_\rho = (2-d)(\partial_\rho f)$

Act w/ ∂_ν: $\quad 2\partial^\mu\partial_\mu\partial_\nu\varepsilon_\rho = (2-d)(\partial_\nu\partial_\rho f)$ $\quad$ (sym. in ν & ρ)

Acting w/ $\partial^\mu\partial_\mu$ on (*) gives: $\quad \partial^\mu\partial_\mu(\partial_\nu\varepsilon_\rho + \partial_\rho\varepsilon_\nu) = (\partial^\mu\partial_\mu f)\delta_{\nu\rho}$

It follows that $\quad (2-d)(\partial_\nu\partial_\rho f) = (\partial^\mu\partial_\mu f)\delta_{\nu\rho}$

Contracting w/ $\delta^{\nu\rho}$ gives $\quad (2-d)(\partial^\mu\partial_\mu f) = (\partial^\mu\partial_\mu f)\, d$, or

$$\boxed{(d-1)(\partial^\mu\partial_\mu f) = 0}$$

$d=1$: no constraints (all angles invariant!)

$d=2$: deal w/ separately.

$d \geq 3$: (**) implies: $\partial_\mu \partial_\nu \varepsilon_\rho$ is constant $\Rightarrow$ ε_ρ is at most quadratic.

General form: $\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$, w/ $c_{\mu\nu\rho} = c_{\mu\rho\nu}$

0^{th} order: $\varepsilon_\mu = a_\mu$: no constraints; infinitesimal translations.

1^{st} order: $\varepsilon_\mu = b_{\mu\nu} x^\nu \Rightarrow \partial_\nu \varepsilon_\mu = b_{\mu\nu}$

(*) $\Rightarrow$ $b_{\mu\nu} + b_{\nu\mu} = f(x)\,\delta_{\mu\nu} = \frac{2}{d}(\partial^\rho \varepsilon_\rho)\,\delta_{\mu\nu} = \frac{2}{d} b^\rho{}_\rho\, \delta_{\mu\nu}$

So, b is the sum of a trace and an anti-symmetric piece:

$b_{\mu\nu} = \alpha\,\delta_{\mu\nu} + m_{\mu\nu}$, w/ $m_{\mu\nu} = -m_{\nu\mu}$

scale trans.

inf. rotation.

2^{nd} order: (**) gives the following result: $\varepsilon_\mu = b_\mu x^2 - 2x_\mu (b\cdot x)$, or

$$\frac{x'^\mu}{(x')^2} = \frac{x^\mu}{x^2} + b^\mu$$ [Exercise]

Finite transformations:

$$x'^{\mu} = x^{\mu} + a^{\mu} \qquad \Omega = 1$$

$$x'^{\mu} = \alpha\, x^{\mu} \qquad \Omega = \alpha^{-2}$$

$$x'^{\mu} = M^{\mu}{}_{\nu}\, x^{\nu} \qquad \Omega = 1$$

$$x'^{\mu} = \frac{x^{\mu} + b^{\mu} x^2}{1 + 2(b\cdot x) + b^2 x^2} \qquad \Omega = \left(1 + 2(b\cdot x) + b^2 x^2\right)^2$$

Important relation: $|x_1' - x_2'|^2 = \dfrac{|x_1 - x_2|^2}{\gamma_1 \gamma_2}$ $\qquad \gamma_i = \left(1 + 2(b\cdot x_i) + b^2 x_i^2\right)$

Jacobian: $\left|\dfrac{\partial x'}{\partial x}\right| = \Omega(x)^{-d/2}$

Consequences of conformal invariance

Assume we have a system w/ conf. inv. Hamiltonian:

Correlation functions behave as: $\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} \langle \phi_1(x_1')\phi_2(x_2')\rangle$

where Δ_i are 'scaling dimensions'.

Box: 'Quasi-primary' field: ~~Iam~~ Field that transforms as

$$\phi(x) \longrightarrow \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x)$$

Dilatations: $x \to \lambda x$ $\quad \left(\left|\frac{\partial x'}{\partial x}\right| = \lambda^d\right)$

$$\Rightarrow \quad \langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle$$

Translation + Rotation invariance: $\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1-x_2|)$ w/

$$f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$$

So, we obtain: $\langle \phi_1(x_1)\phi_2(x_2)\rangle = \dfrac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}$

A conformal transformation is a position dependent dilatation:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \lambda^{\Delta_1}(x_1)\,\lambda^{\Delta_2}(x_2)\,\langle \phi_1(x_1')\,\phi_2(x_2')\rangle$$

Rotation over 180° gives:

$$\langle \phi_2(x_1)\,\phi_1(x_2)\rangle = \lambda^{\Delta_2}(x_1)\,\lambda^{\Delta_1}(x_2)\,\langle \phi_2(x_1')\,\phi_1(x_1')\rangle$$

But the rotation leaves $\langle\ \rangle$ invariant, so we find:

$$\Delta_1 \neq \Delta_2 \Rightarrow \langle \phi_1 \phi_2 \rangle = 0, \quad w/\ \Delta_1 = \Delta_2 = \Delta: \quad \langle \phi_1(x_1)\,\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta}}$$

We can also use SCT's:

$$\frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} \underset{\substack{\uparrow \\ \text{inv of corr.} \\ \text{function}}}{=} \frac{C_{12}(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1-x_2|^{\Delta_1+\Delta_2}}\,\frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \Rightarrow \Delta_1 = \Delta_2, \text{ or } C_{12} = 0$$

Using conformal invariance, we can also fix 3-point functions:

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle = \frac{C_{123}}{x_{12}^{a}\, x_{23}^{b}\, x_{13}^{c}} \quad , \text{ with } x_{ij} = |x_i - x_j|$$

* Four point functions are much harder to deal with, they ~~depend~~ can be 'arbitrary' function of $x = \dfrac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)}$: anharmonic ratio, invariant under conf. transformations.
* We also need the coefficients C_{123} (obtained from four point functions).

We need the power of 2-dim, so we can make progress!!