Why am I interested in CFT? ('What is CFT?' is tricky to answer)

* Symmetry
  ↳ great importance in physics:
  - space-time
  - standard model
  - string theory
  - symmetry breaking & critical phenomena

  ⋮

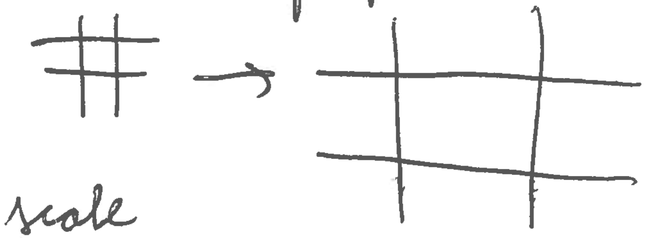
* Interplay between algebra/group theory, analysis (structure of CFT)

* Powerful: 'exactly solvable' in (1+1)d or (2+0)d systems

* Applicable:
  * (2+0)d or (1+1)d systems w/ critical behaviour
  * string theory

What is conformal invariance?

* Consider ~~the~~ 'scale invariance', symmetry of dilatations of space (for instance at critical points, see below)



* Conformal transformations are dilatations w/ a scale factor that depends on position (local scale transformation)

When do we expect it?

Polyakov: consider a system w/ only <u>local</u> interactions. If it's invariant under ~~a~~ global scale transformations, it is natural to expect/it is plausible, that it is also invariant under <u>local</u> scale transformations.

We will make this (somewhat) more precise later on. Note: works <u>very</u> well!

Power of conformal symmetry:

In d-dims.: group of dimension $\frac{1}{2}(d+1)(d+2)$.

In 2-d : 6 dimensional group. But: ∞-dim set of transformations, that are defined locally (not globally invertible)

↳ reason 2d CFT is so powerful!

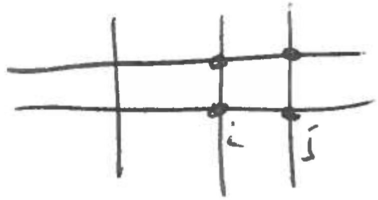
Most physical phenomena are dominated by length-scales (no scale invariance)

If the scales go to 0 or ∞, scale invariance emerges: f.i. at critical phenomena.

RG: set of ideas to describe systems at larger & larger scales, smearing out microscopics ⇝ scale invariant hamiltonians, critical phenomena, etc.

2+0 d examples:

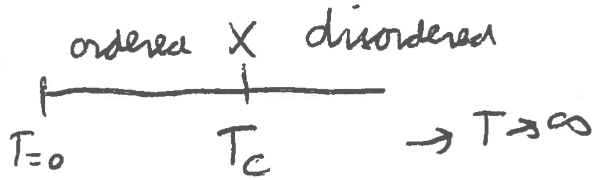
Ising model: classical spins $\sigma_i = \pm 1$ on f.i. square lattice



$$E_{\{\sigma\}} = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j \rightsquigarrow Z = \sum_{\{\sigma\}} e^{-\beta E_{\{\sigma\}}}$$

Two length scales: a: lattice spacing, ξ correlation length: $\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = e^{-|i-j|/\xi}$,

where ξ dep. on temperature.



At T_c, ξ diverges, fluctuations on all length scales. → $\langle \sigma_i \sigma_j \rangle$ behaves as a <u>power</u> law

Critical exponents: $t = \left(\frac{T-T_c}{T_c}\right)$; $h = H/k_B T$

At critical points: $\xi(t) = |t|^{-\nu}$ spon. Magnetization: $\lim_{H\to 0} M \propto (-t)^{\beta}$

spec. heat: $C \sim A\,|t|^{-\alpha}$ Susceptibility $\chi = \left.\frac{\partial M}{\partial H}\right|_{H=0} \propto |t|^{-\gamma}$

correlations: $g(r) = \frac{1}{r^{d-2+\eta}}$ Mag.: $M \propto |h|^{1/\delta}$

Universality of exponents: many different systems have the <u>same</u> exponents, which hence <u>do not</u> depend on microscopics

CFT tries to study / classify / explain different classes of exponents!

(1+1) d quantum example:

Transverse field Ising model: $H = -J \sum_i \left(\sigma_i^x \sigma_{i+1}^x + g\sigma_i^z\right)$

J: sets energy scale; g: tunes phase transition.

$\sigma^x = \begin{pmatrix}0&1\\1&0\end{pmatrix}, \sigma^y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \sigma^z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\sigma_i^z |\uparrow\rangle_i = |\uparrow\rangle_i$

$\sigma_i^z |\downarrow\rangle_i = -|\downarrow\rangle_i$

Set $J=1$; Two regimes: $g \gg 1$: $|0\rangle = \prod_i |\Uparrow\rangle_i$

$|\Uparrow\rangle_i = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle_i + |\leftarrow\rangle_i\right)$

$|\Downarrow\rangle_i = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle_i - |\leftarrow\rangle_i\right)$

Correlations of σ^x are local:

$\langle 0|\sigma_i^x \sigma_j^x|0\rangle = \delta_{ij} \quad (g \to \infty)$

$\sim e^{|i-j|/\xi}$ via perturbation in $1/g$

Regime $g \ll 1$: different g.S: $|0\rangle_1 = |\rightarrow\rangle = \prod_i |\rightarrow\rangle_i$

$|0\rangle_2 = |\leftarrow\rangle = \prod_i |\leftarrow\rangle_i$

H is invariant under $\sigma_i^x \to -\sigma_i^x$: no tunneling matrix element between two ground states.

Two regimes $g \gg 1$ & $g \ll 1$: gapped, and can not be connected analytically.

$\Rightarrow$ There is a phase transition for some finite g_c, where the gap Δ closes. $\Delta = 0$ is necessary for scale invariance, because Δ, or better $\hbar/v\Delta$ is a length.

Show pictures of some spectra!

Model can be solved exactly: $H = \sum_k \varepsilon_k (\gamma_k^\dagger \gamma_k - \frac{1}{2})$; $\varepsilon_k = 2J(1+g^2-2g\cos k)^{\frac{1}{2}}$

$\varepsilon_k \geq 0$

Δ: minimal excitation energy: occurs for $k=0$: $\Delta = 2J|1-g|$, $\Delta = 0$ for $g = g_c = 1$