Exercises CFT-course fall 2023, set 3.

1. The conformal ward identity.

a. Carefully derive the conformal ward identity

$$\delta\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle = \frac{1}{2\pi i}\oint_C dz\varepsilon(z)\langle T(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle + c.c.\ .$$

b. Show that for primary fields, this gives rise to the equation

$$\langle T(z)\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle = \sum_i\Big(\frac{h_i}{(z-z_i)^2}+\frac{\partial_{z_i}}{(z-z_i)}\Big)\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle\ .$$

2. Show that the n point functions of quasi-primary fields satisfy

$$\sum_i \partial_{z_i}\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0$$
$$\sum_i (z_i\partial_{z_i}+h_i)\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0$$
$$\sum_i (z_i^2\partial_{z_i}+2h_iz_i)\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0\ ,$$

by considering the infinitesimal form of the global conformal transformations.

3. The Schwartzian derivative.

a. Show that the Schwartzian derivative

$$\{w;z\} = \frac{w'''}{w'} - \frac{3}{2}\Big(\frac{w''}{w'}\Big)^2\ ,$$

(with $w' = \frac{dw}{dz}$ etc), is zero if and only if $w(z) = \frac{az+b}{cz+d}$. Start by solving the first order differential equation for (w''/w'), and proceed by solving for $v = w'$.

b. Show that for successive transformations $z \to w \to u$, one has the following composition rule

$$\{u;z\} = \{w;z\} + \big(\frac{dw}{dz}\big)^2\{u;w\}$$