Exercises CFT-course fall 2023, set 1.

The transverse-field Ising model.

The hamiltonian of the transverse-field Ising model in one dimension reads, in terms of the Pauli matrices σ,

$$H = \sum_i \left(-J \sigma_i^x \sigma_{i+1}^x - h \sigma_i^z \right) \ .$$

This model can be solved by means of a Jordan-Wigner transformation, which transforms the spin operators into fermionic ones.
To transform the spin degrees of freedom into (spin-less) fermions, we let a spin-up at site i correspond to an empty site. Conversely, a spin-down corresponds to a site occupied by a fermion. If we consider a single site i, the spin raising operator σ_i^+ corresponds to the fermionic annihilation operator c_i. Conversely, $\sigma_i^- = c_i^\dagger$. These operators indeed satisfy the fermionic anti-commutation relations (with $i = j$)

$$\{c_i, c_j^\dagger\} = \delta_{i,j} \qquad\qquad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \ .$$

However, the spin operators on different sites commute, so the construction above does not work for $i \neq j$.

Jordan and Wigner solved this problem, by introducing

$$\sigma_i^z = 1 - 2c_i^\dagger c_i \qquad \sigma_i^+ = \Big(\prod_{j<i} (1 - 2c_j^\dagger c_j) \Big) c_i \qquad \sigma_i^- = \Big(\prod_{j<i} (1 - 2c_j^\dagger c_j) \Big) c_i^\dagger \ .$$

The inverse is

$$c_i = \Big(\prod_{j<i} \sigma_j^z \Big) \sigma_i^+ \qquad\qquad c_i^\dagger = \Big(\prod_{j<i} \sigma_j^z \Big) \sigma_i^- \ .$$

(a) Show that the σ operators defined above satisfy the correct commutation relations, by using the fermion anti-commutation relations given above.

(b) We will assume that the chain contains L sites, and is closed, so that the sites i and $i+L$ are identified (i.e., we assume that $\sigma_{i+L}^\alpha = \sigma_i^\alpha$). Show that after the Jordan-Wigner transformation, the hamiltonian reads

$$\begin{aligned} H =& h \sum_{i=0}^{L-1} (2c_i^\dagger c_i - 1) - J \sum_{i=0}^{L-2} \left(-c_i c_{i+1} - c_i c_{i+1}^\dagger + c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger \right) \\ &+ (-1)^F J \Big(-c_{L-1} c_0 - c_{L-1} c_0^\dagger + c_{L-1}^\dagger c_0 + c_{L-1}^\dagger c_0^\dagger \Big) \ , \end{aligned}$$

where F is the number of fermions (that is, $F = \sum_{i=0}^{L-1} c_i^\dagger c_i$), which is conserved modulo two.

The boundary term can be taken into account by imposing periodic boundary conditions $c_L = c_0$ when F is odd, and anti-periodic boundary conditions $c_L = -c_0$ when F is even. With these boundary conditions, the hamiltonian is translation invariant, namely

$$H = h\sum_{i=0}^{L-1}(2c_i^\dagger c_i - 1) - J\sum_{i=0}^{L-1}\left(-c_i c_{i+1} - c_i c_{i+1}^\dagger + c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger\right) .$$

(c) To diagonalize the hamiltonian, first transform it to momentum space, via $c_j = \frac{1}{\sqrt{L}}\sum_k c_k e^{iakj}$, with $a = \frac{2\pi}{L}$. Show that the hamiltonian takes the following form:

$$H = \sum_k \Psi_k^\dagger \begin{bmatrix} h - J\cos(2\pi k/L) & iJ\sin(2\pi k/L) \\ -iJ\sin(2\pi k/L) & -h + J\cos(2\pi k/L) \end{bmatrix} \Psi_k \ ,$$

where $\Psi_k^\dagger = (c_k^\dagger, c_{-k})$.

(d) Diagonalize the Hamiltonian, that is, write it in the form

$$H = \sum_k \epsilon_k (2\gamma_k^\dagger \gamma_k - 1) \ .$$

Determine ϵ_k, and specify which values k takes, depending on the parity of the number of fermions. Pay special attention to the cases for which $k = -k$, that is $k = 0$ and $k = L/2$, and specify when these occur.

(e) Plot the spectrum (that is, the 2^L eigenvalues) of the Hamiltonian (for a reasonable system size, say $L = 10$ or so), as a function of the momenta K of the states, which is given by $K = \left(\sum_k k\gamma_k^\dagger \gamma_k\right) \bmod L$. Use a different colour for the even and odd fermion sectors. Pick three characteristic values of (J, h), such as $(1, .5), (1, 1), (1, 2)$.