Structure of Virasoro representations

Minimal model $\mathcal{M}(p, p')$: $c = 1 - \frac{6(p'-p)^2}{pp'}$; $h_{r,s} = \frac{(pr - p's)^2 - (p'-p)^2}{4pp'}$ $\quad 1 \le r \le p'-1$, $1 \le s \le p-1$

$h_{r,s} = h_{p'-r, p-s}$

Without null states, the states in a highest (lowest) weight module would look like:

$$L_{-p}^{n_p} \cdots L_{-2}^{n_2} L_{-1}^{n_1} |\phi_{r,s}\rangle$$

The partition function would look like: $\chi_{r,s} \overset{?}{\propto} \prod_{l=1}^{\infty} (1-q^l)^{-1}$ (gen. function of all partitions).

However, the state $|\phi_{r,s}\rangle$ has null descendants at level $l = rs$ and $l = (p'-r)(p-s)$, so we should remove them: $\chi_{r,s} \overset{?}{\propto} \prod_{l=1}^{\infty}(1-q^l)^{-1}\left[1 - q^{rs} - q^{(p'-r)(p-s)}\right]$

In writing this, we assumed that the subtracted modules do not have null states themselves.

We know: $h_{r,-s} - h_{r,s} = rs$; $h_{r+p',s+p} = h_{r,s}$ $\Rightarrow$

The null state: $h_{r,s} + rs = h_{p'+r,\,p-s} = h_{p'-r,\,p+s}$

is itself in the Kac table. So it has two null states itself.

The null state $h_{r,s} + (p'-r)(p-s) = h_{r,\,2p-s} = h_{2p'-r,\,s}$

is also in the Kac table, with the same null states!

There is a tower of sub-modules:

$$\begin{tikzcd}
 & (p'+r,\,p-s) \arrow[r] \arrow[dr] & (2p'+r,\,s) \arrow[r] \arrow[dr] & (3p'+r,\,p-s) \cdots \\
(r,s) \arrow[ur] \arrow[dr] & & & \\
 & (r,\,2p-s) \arrow[r] \arrow[uur] & (r,\,2p+s) \arrow[r] \arrow[uur] & (r,\,4p-s) \cdots
\end{tikzcd}$$

to be subtracted

subtracted twice, to be added back

In the end, one finds:

$$\chi_{r,s} = q^{h_{r,s}} \left(\prod_{l=1}^{\infty} (1-q^{l})^{-1} \right) K_{r,s} \qquad K_{r,s} = \sum_{n=-\infty}^{\infty} \left(q^{n(npp' + rp - sp')} - q^{(np+s)(np'+r)} \right)$$

<u>Ising case: $M(3,4)$:</u> $\prod_{n\geq 1}^{\infty} (1-q^{n})^{-1} = 1+q+2q^2+3q^3+5q^4+7q^5+11q^6+\cdots$

<u>$p'=4;\ p=3$</u>

$h_{1,1} = h_{2,3} = 0$. $\quad \cancel{\chi_{0,0}}$ $\quad 1 + \underline{\underline{0}}\, q^1 + q^2 + q^3 + 2q^4 + 2q^5 + \underline{\underline{3}}\, q^6 + \cdots$

$h_{1,3} = h_{2,1} = \frac{1}{2}$ $\quad 1 + q + \cancel{2}1 q^2 + \cancel{2}1 q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots$

$h_{2,2} = h_{1,2} = \frac{1}{16}$ $\quad 1 + q + \underline{\underline{1q^2}} + 2q^3 + \underline{\underline{2q^4}} + 3q^5 + 4q^6 + \cdots$

Characters of $\mathbb{1}, \psi$ can be interpreted in another way!

Expand $\psi(z)$ in modes: $\psi(z) = \sum_{n\in\mathbb{Z}} z^{-n-\frac{1}{2}} \psi_n$, w/ $\{\psi_n, \psi_m\} = \delta_{n+m,0}$

Pauli principle forbids double occupancy of the same mode!

Densest n-fermion state: $\psi_{-\frac{(2n-1)}{2}} \cdots \psi_{-\frac{3}{2}} \psi_{-\frac{1}{2}} |0\rangle$ level: $\sum_{p=1}^{n} \frac{1}{2}(2p-1) = \frac{1}{2}n^2$

Most general state: $\psi_{-\frac{(2n-1)}{2}-p_n} \cdots \psi_{-\frac{3}{2}-p_2} \psi_{-\frac{1}{2}-p_1} |0\rangle$; $p_i \in \mathbb{Z}_{\geq 0}$; $0 \leq p_1 \cdots \leq p_n$

p's form a partition in n parts.

↳ generating function: $\frac{1}{(q)_n} = \prod_{l=1}^{n} (1-q^l)^{-1}$

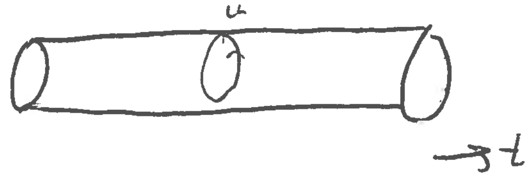
Acting w/ even # of ψ's: $\mathbb{1}$ sector

" " odd # of ψ's: ψ sector

$\chi_{1,1} \propto \sum_{\substack{n\geq 0\\ \text{even}}} \frac{q^{n^2/2}}{(q)_n}$; $\chi_{2,1} \propto \sum_{\substack{n\geq 0\\ \text{odd}}} \frac{q^{n^2/2}}{(q)_n}$ Note: $\sum_{n\geq 0} \frac{q^{n^2/2}}{(q)_n} = \prod_{n=1}^{\infty} (1+q^{n+\frac{1}{2}})$

Also: $\chi_{1,2} \propto \sum_{\substack{n\geq 0\\ \text{even}}} \frac{q^{n(n-1)/2}}{(q)_n} = \sum_{\substack{n\geq 0\\ \text{odd}}} \frac{q^{n(n-1)/2}}{(q)_n} = \prod_{n=1}^{\infty} (1+q^n)$

CFT on torus & cylinder



Map from z) plane to $w = t + iu$ cylinder: $w = \frac{L}{2\pi} \log z$

$$T(w)_{cyl} = \left(\frac{dz}{dw}\right)^2 T_{pl}(z) + \frac{c}{12}\{z, w\} = \left(\frac{2\pi}{L}\right)^2 \left(z^2 T_{pl}(z) - \frac{c}{24}\right)$$

t-translations: gen. by $H = \frac{1}{2\pi}\int_0^L T_{tt}(u)\, du = \frac{1}{2\pi}\int_0^L \left(T(u) + \bar{T}(\bar{u})\right) du$

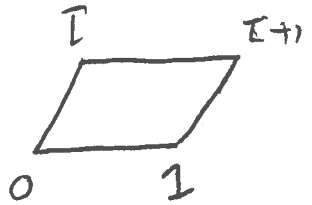
On cylinder: $H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0\right) - \frac{\pi c}{6L}$

u-translation correspond to rotations: $P = \frac{2\pi}{L}\left(L_0 - \bar{L}_0\right)$

Putting a CFT on a torus gives constraints on the fields that can appear!

Key word: modular invariance (of full partition function)

Torus: parallelogram w/ vertices in $\mathbb{C}$:
(using scale & rot. invariance): $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$



How to obtain: take a cylinder w/ $L=1$ (circ.), length $\mathrm{Im}(\tau)$; twist one side by $\mathrm{Re}(\tau)$, and glue together.

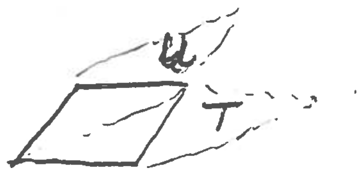
τ is not a unique parametrisation of the torus! $T: \tau \to \tau+1$

$S: \tau \to -1/\tau$

leave torus invariant! T, S generate: $\tau \to \frac{a\tau+b}{c\tau+d}$, w/ $a,b,c,d \in \mathbb{Z}$, $ad-bc=1$

This is the modular group: $SL(2,\mathbb{Z})/\mathbb{Z}_2$ $\quad S^2 = 1$; $(ST)^3 = 1$

S: interchanges space & time:



$U = TST$ $\quad U: \tau \to \frac{\tau}{\tau+1}$