Kac determinant:

$$\det\left(M^{(l)}\right) = \alpha_l \prod_{\substack{r,s \geq 1 \\ rs \leq l}} \left(h - h_{r,s}(c)\right)^{p(l-rs)}$$

$\alpha_l > 0$

$p(n) =$ # part. of l into pos. integers

$$h_{r,s}(c) = \frac{\left[(m+1)r - ms\right]^2 - 1}{4m(m+1)} \quad ; \quad m = -\frac{1}{2} \pm \sqrt{\frac{25-c}{1-c}}$$

Use Kac determinant to prove existence of ~~unitarity~~ unitary irreps of Virasoro algebra. Note: unitarity implies $c, h \geq 0$

Two separate regions: $0 < c < 1$; $c \geq 1$

Unitarity of highest (lowest) weight modules for $c \geq 1$; $h \geq 0$.

Sketch of the proof: * There are no vanishing curves, $\Rightarrow \det M^{(l)} \neq 0$

$1 \leq c < 25$: $h_{r,r} < 0$; $h_{r,s\neq r} \notin \mathbb{R}$

$c \geq 25$: $h_{r,s} < 0$

* $\det M^{(l)} > 0$. ~~...~~ Take $h \gg \max |h_{r,s}|$ for given l, then $\det M^{(l)} = \underset{\vee 0}{\alpha_l}\, h^q$, for some $q > 0$. $\det M^{(l)}$ does not vanish in this region, so it is positive everywhere.

* Show that <u>all</u> eigenvalues are positive for some h. (or, $M^{(l)}$ pos. def. for some h).

$n(\alpha)$ = length of a state $|\alpha\rangle$: # L_{-n_i}'s acting on $|h\rangle$. $\Rightarrow$ $\langle\alpha|\alpha\rangle = \underset{\wedge 0}{C_\alpha}\, h^{n(\alpha)}\left[1 + O(1/h)\right]$

$\langle\alpha|\beta\rangle = O\left(h^{(n(\alpha)+n(\beta))/2 - 1}\right)$

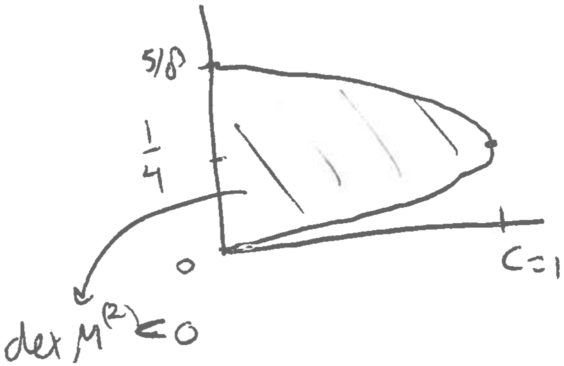
$\Rightarrow$ Eigenvalues of $M^{(l)}$ are given by eigenval. of $M^{(l)}_n$ (w/ length fixed), which are positive.

Unitary irreps for $0 < c < 1$

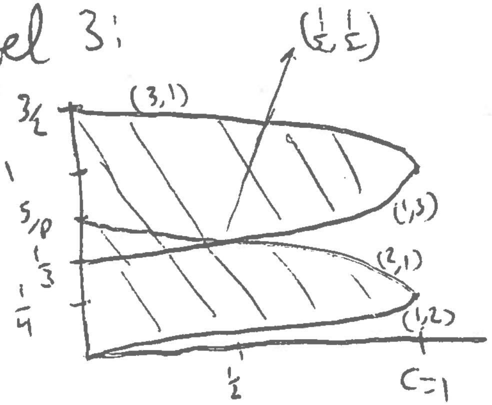
Bit more tricky to do the proofs, see also Friedan, Qiu, Shenker, 1984.

$$c=0 \Rightarrow h_{r,s} = \frac{(3r-2s)^2-1}{24} \qquad c=1 : h_{r,s} = \frac{(r-s)^2}{4}$$

Vanishing curves at level 2:

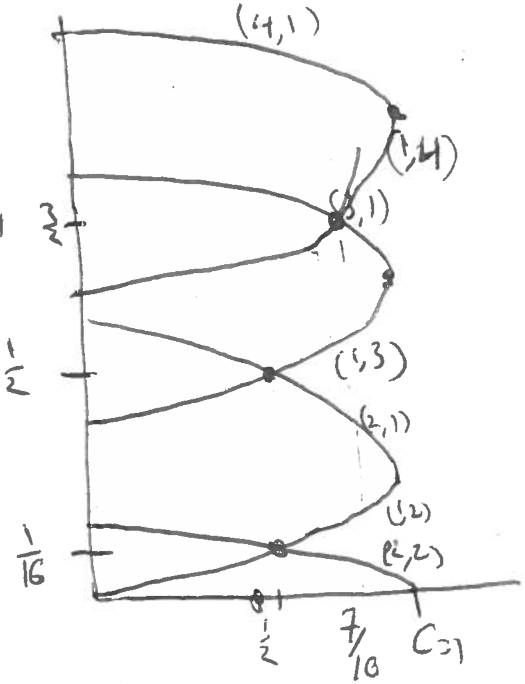


Level 3:



Excluded region is enlarged!

Level 4



Even bigger excluded region.

Result: all points w/ $0 < c < 1$ are ~~not~~ excluded, except possibly on the vanishing curves, where $\det M^{(l)}$ changes sign.

On the vanishing curves, there are typically also ~~neg~~ neg. norm states.

Exception: the <u>intersection points</u>.

The representations "at the intersection points" (ie, $|h\rangle$ and appropriate c)

correspond to ~~the~~ unitary representations of Vir. [Null states have to
(not ~~proven~~ here!) be 'modded' out]

<u>Unitary minimal models:</u> $c(m) = 1 - \frac{6}{m(m+1)}$, w/ $m = 3, 4, 5, \ldots$

Fields: $\phi_{r,s}$, w/ $h_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4(m)(m+1)}$, where $r = 1, 2, \ldots, m-1$
$s = 1, 2, \ldots, m$

Note: $h_{r,s} = h_{m-r, m+1-s}$; the fields $\phi_{r,s} \cong \phi_{m-r, m+1-s}$ are 'identified'!
(same intersection point).

One can show that these ~~~~ rep. are indeed unitary (not done here).

$c = \frac{1}{2}$: Ising ; $c = 7/10$: Tri-critical Ising ; $c = \frac{4}{5}$: tetra critical Ising / 3 state Potts

etc. They describe critical 2d stat mech models, or 1D quantum chains at criticality.

Kac table of scaling dimensions $h_{r,s}$:

$M(3,4) = \text{Ising} \quad c = \frac{1}{2}$

$m = p = 3$
$p' = m + 1 = 4$

$r \uparrow$		
	$\frac{1}{2}$	0
	$\frac{1}{16}$	$\frac{1}{16}$
	0	$\frac{1}{2}$
		$\to s$

$M(4,5)$, tri-critical Ising; $c = \frac{7}{10}$

4	$\frac{3}{2}$	$\frac{7}{16}$	0
3	$\frac{3}{5}$	$\frac{3}{80}$	$\frac{1}{10}$
2	$\frac{1}{10}$	$\frac{3}{80}$	$\frac{3}{5}$
$r \uparrow$ 1	0	$\frac{7}{16}$	$\frac{3}{2}$
$s \to$	1	2	3

Possible way of restricting the fields:

$s \le r; \quad r + s = 0 \mod 2$

$p r < p' s$

Correlators of minimal model primary fields satisfy diff eq^{n}s, due to the null vectors. Not all 3-point functions are non-zero:

Example (exercise)

$C_3 = \langle \phi_{(2,1)}(z_1)\, \phi_{(r,s)}(z_2)\, \phi_{(r',s')}(z_3) \rangle$ is only nonzero for $(r',s') = (r\pm1, s)$

To show this, consider the null vector condition:

$$\left[\mathcal{L}_{-2} - \frac{3}{2(2h_{2,1}+1)} \left(\mathcal{L}_{-1}\right)^2 \right] C_3 = 0 ,$$

where the form of C_3 is fixed by conformal invariance

If C_3' is obtained by changing $\phi_{2,1}(z_1) \longrightarrow \phi_{(1,2)}(z_1)$, we find the constraint $(r',s') = (r, s\pm1)$ instead.