Short recap: Vir algebra: $[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$

$L_m^+ = L_{-m}$

States in a Verma module: $L_{-n_p} \cdots L_{-n_1} |h\rangle$, w/ $|h\rangle$ a highest weight state:

$L_n |h\rangle = 0 \quad n > 0.$

Unitarity (no neg. norm states): $\| L_{-n} |h\rangle \|^2 \geq 0 \quad : c \geq 0 \ ; \ h \geq 0$

States w/ zero norm (null states): $L_n |\chi\rangle = 0 \quad n > 0$

Null states give rise to diff eq^{n}s for the correlators of the associated primary fields.

General structure: Find reps of Vir that are highest weight representations.

$H = L_0 + \bar{L}_0$ should be bounded from below. L_n, w/ n positive lowers the scaling dimension. So if $|\psi\rangle$ is an L_0 eigenstate, there should be a state $|\phi\rangle = L_p |\psi\rangle$ s.t. $L_n |\phi\rangle = 0$ for $n > 0$

This state is the ~~highest~~ lowest weight (primary) state, denoted by $|h\rangle$.

The Verma module is spanned by $L_{-n_k} \cdots L_{-n_1} |h\rangle$.

The inner products follow from $L_m^\dagger = L_{-m}$; ($L_{+n} |h\rangle = 0$ & $L_{-n} \neq 0$ $n > 0$)

$|\psi_1\rangle = L_{-n_k} \cdots L_{-n_1} |h\rangle$

$|\psi_2\rangle = L_{-m_p} \cdots L_{-m_1} |h\rangle$ $\quad \langle \psi_2 | \psi_1 \rangle = \langle h | L_{m_1} \cdots L_{m_p} L_{-n_k} \cdots L_{-n_1} | h \rangle$

$\langle \psi_2 | \psi_1 \rangle \neq 0 \Rightarrow \sum_i m_i = \sum_i n_i$

To show this, commute $L_{m_i>0}$'s to the right. (*) implies that the inner product is a sum of terms of the form $\langle h | L_{k_q} \cdots L_{k_1} | \chi \rangle = 0$

So, we also have that all the descendants of $|\chi\rangle$ have zero norm.

An irrep is constructed by 'quotienting out' all null-states i.e. states w/ differ by a null state are identified!

These irreps for the 'basis' / main ingredients of minimal models.

We are interested in 'irreducible' representations:
no subspace of the representation should be a representation on ~~of~~ it's own.

Say we have such a rep representation: w/ 'highest' weight $|\chi\rangle$: $L_n|\chi\rangle = 0$ $n>0$,
~~and its descendants~~ but also: $|\chi\rangle = L_{-n_k}\cdots L_{-n_1}|h\rangle$, for some n_i.

$|\chi\rangle$ is orthogonal to whole Verma module: $\langle\chi| L_{-m_p}\cdots L_{-m_1}|h\rangle \underset{h.c.}{=} \langle h| L_{m_1}\cdots L_{m_p}|\chi\rangle^{*} = 0$
w/ $m_i > 0$

Thus, we also have $\langle\chi|\chi\rangle = 0$. $|\chi\rangle$ is a <u>null</u> state.

The descendents of $|\chi\rangle$ are also orthogonal to the Verma module

Let N be the level of $|\chi\rangle$, and ~~$\langle h|$~~ $^{(*)}$ $\sum_i m_i = N + \sum_i n_i$, then

$$\langle h| L_{m_p}\cdots L_{m_1} L_{-n_k}\cdots L_{-n_1}|\chi\rangle = 0.$$

Minimal model: set of fields correspond to representations of the Vir. algebra at certain value of c.

The representations are constructed from the Verma modules $L_{-n_p} \cdots L_{-n_1} |h\rangle$, which are highest weight representations.
(lowest)

Irreducible representations: no descendant weights are also the highest weight of a sub representation.

$|\chi\rangle = L_{-n_p} \cdots L_{-n_1} |h\rangle$, w/ $L_n |\chi\rangle = 0$ for $n > 0$ has zero norm, and overlap zero w/ any state in the Verma module. Same is true for descendants of $|\chi\rangle$. These states $|\chi\rangle$ have to be 'quotiented out'.

(And give rise to diff eqns for primary fields).

$$\det\left(M^{(\ell)}\right) = \alpha_\ell \prod_{\substack{r,s\geq 1 \\ rs\leq \ell}} \left(h - h_{r,s}^{(c)}\right)^{p(\ell - rs)} \qquad \alpha_\ell > 0$$

$p(n) = \#$ of partition of n ; $p(0) = 1$; $p(n) = 0$ if $n<0$

$$\alpha_\ell = \prod_{\substack{r,s \\ rs\leq \ell}} \left((2r)^s s! \right)^{p(\ell - rs) - p(\ell - r(s+1))}$$

A null state that appears first at level rs gives rise to $p(\ell - rs)$ null states at level $(\ell - rs)$, because the descendants of a null state are also null.

We need: * unitary irreducible highest weight representations of Vir.
* to find (c, h) such that there are <u>no</u> neg. norm states.
* to quotient out zero norm states (~~all~~ also called <u>null states</u>)

We construct the 'Gramm matrix' for the states at a given level:

$$M^{(0)} = \langle h | h \rangle = \mathbb{1}$$

$$M^{(1)} = \langle h | L_1 L_{-1} | h \rangle = 2h \qquad (\Rightarrow h \geq 0 \,;\; L_{-1}|0\rangle \text{ is a null state})$$

$$M^{(2)} = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + c/2 \end{pmatrix} \qquad \text{ordering: } (L_{-1})^2, L_{-2}$$

$\vdots$

$\dim(M^{(\ell)}) = p(\ell) = \#$ partitions of ℓ into positive integers.

A common representation for $h_{r,s}(c)$ is:

$$c = 1 - \frac{6}{m(m+1)} \; ; \; h_{r,s} = \frac{\left[(m+1)r - ms\right]^2 - 1}{4m(m+1)} \; , \text{ where } m = -\frac{1}{2} \pm \sqrt{\frac{25-c}{1-c}}$$

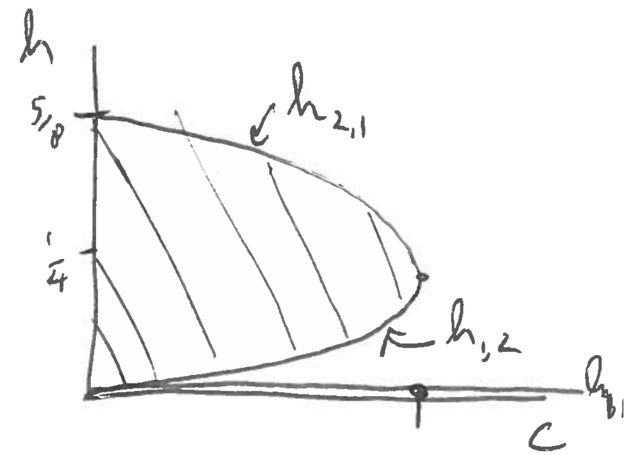
Results: $c \geq 1$; $h \geq 0$:
* highest weight reps are unitary & irreducible
* no Virasoro null states, ∞ # of primary fields
* classification is hard, one needs additional symmetry!

$0 < c < 1$
* Unitarity is harder to prove here
* set of 'minimal models' labeled by $m = 3, 4, \ldots$, which have a finite ~~not~~ number of primaries.
* Null states exist giving a handle on the correlation functions

The determinant of $M^{(2)}$: $\det\left(M^{(2)}\right) = 32\,(h-h_{1,1})(h-h_{1,2})(h-h_{2,1})$

with $h_{1,1} = 0$; $h_{1,2} = \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right)$

$$h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right)$$



$\det\left(M^{(2)}\right) < 0$ in region ▥, and zero on the lines $h_{1,2}$ and $h_{2,1}$. When $\det\left(M^{(2)}\right) < 0$, there is an odd number of neg. eigenvalues, so at least one negative norm state, implying non-unitarity.

Kac '79 gave an explicit expression for $\det\left(M^{(\ell)}\right)$, proven by Feigin & Fuchs 1982