Time ordering of the operators in correlators in QFT becomes a radial ordering, so in OPE's:

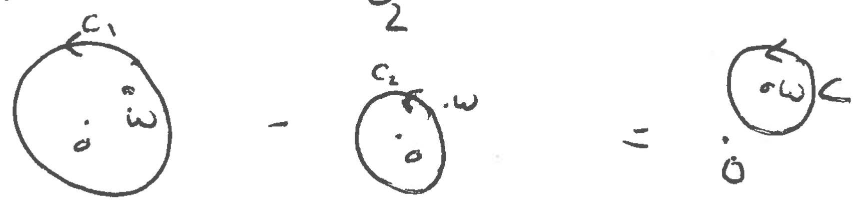
$$R\big(A(z)\,B(w)\big) = \begin{cases} A(z)\,B(w) & |z| > |w| \\ B(w)\,A(z) & |z| < |w| \end{cases} \quad \text{(for bosons, otherwise sign)}$$

In OPE's we'll not write the R all the time, it's implicit.

Relating OPE's to equal time commutators: $a(z)$, $b(z)$: holomorphic fields.

Let $A = \oint_C dz\, a(z)$, then

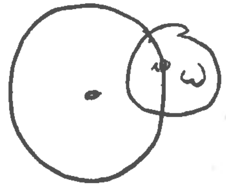
$$[A, b(w)] = \oint_{C_1} dz\, a(z)\, b(w) - \oint_{C_2} dz\, b(w)\, a(z) = \oint_w dz\, R(a(z) b(w))$$



To calculate the integral, one uses the OPE $a(z)\,b(w)$, and pick up the $\frac{1}{z-w}$ term.

Now let $B = \oint_0 dw\, b(w)$, then

$$[A,B] = \oint_0 dw\,[A, b(w)] = \oint_0 dw \oint_w dz\; a(z)\, b(w)$$



Let's define 'conformal charge': $Q_\varepsilon = \frac{1}{2\pi i} \oint dz\; \varepsilon(z)\, T(z)$

Then: $[Q_\varepsilon, \phi(w)] = \frac{1}{2\pi i} \oint_w dz\; \varepsilon(z)\, R[T(z)\, \phi(w))$

$$\underset{\substack{\uparrow \\ \text{OPE}}}{=} \frac{1}{2\pi i} \oint_w dz\; \varepsilon(z) \left[\frac{h\, \phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{(z-w)} + \text{reg} \right]$$

$$= h\big(\partial_w \varepsilon(w)\big)\, \phi(w) + \varepsilon(w)\, \partial_w (\phi(w)) = \delta_\varepsilon\, \phi(w)$$

To utilise the power of radial ordering, etc, we define Mode expansions (Fourier transforms):

We ~~for~~ write conformal fields as: $\phi(z,\bar{z}) = \sum_{m \in \mathbb{Z}-h} \sum_{n \in \mathbb{Z}-\bar{h}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n}$

Summation is such that the exponents of $z, \bar{z}$ are integers

The modes are obtained from the fields via contour integrals:

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz\, z^{m+h-1} \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+\bar{h}-1} \phi(z,\bar{z})$$

The conjugate of the modes follows (assume the real surface condition, $\bar{z} = z^{*}$):

$$\phi^{\dagger}(z,\bar{z}) = \sum_{m \in \mathbb{Z}-h} \sum_{n \in \mathbb{Z}-\bar{h}} \bar{z}^{-m-h} z^{-n-\bar{h}} \phi^{\dagger}_{m,n}$$

Def of conjugation: $\phi^+(z,\bar z) = \bar z^{-2h} z^{-2\bar h} \phi\left(\frac{1}{\bar z}, \frac{1}{z}\right)$

$$= \bar z^{-2h} z^{-2\bar h} \sum_{m\in\mathbb{Z}-h} \sum_{n\in\mathbb{Z}-\bar h} \bar z^{m+h} z^{n+\bar h} \phi_{m,n}$$

[assume $h, \bar h \in \mathbb{Z}$, or $\frac{1}{2}\mathbb{Z}$]
(bosons or fermions)

$$\underset{\substack{m\to -m\\ n\to -n}}{=} \sum_{m\in\mathbb{Z}-h} \sum_{n\in\mathbb{Z}-\bar h} \phi_{-m,-n} \bar z^{-m-h} z^{-n-\bar h}$$

So, we find $\phi^+_{m,n} = \phi_{-m,-n}$

<u>Def of states</u>:

We demand that the states $\phi(z)|0\rangle = \sum_{m\in\mathbb{Z}-h} \phi_m z^{-m-h} |0\rangle$ are regular at $z=0$: This implies that $\phi_m|0\rangle = 0$ for $m \geq -h+1$

The 'in state' $\phi(0)|0\rangle := |\phi\rangle$ is ~~gen~~ created as $\phi_{-h}|0\rangle$.

The hamiltonian corresponds to scale transformations,

$$D = L_0 + \bar{L}_0 = \frac{1}{2\pi i} \oint_0 z\, T(z)\, dz + \text{c.c.}$$

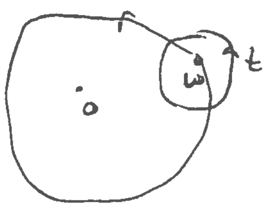
In general, we define the modes of $T(z)$ as : $L_n := \frac{1}{2\pi i} \oint_0 z^{n+1}\, T(z)\, dz$, or

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n$$

To calculate the commutators $[L_m, L_n]$, we need to evaluate commutators of type $\left[\oint dz, \oint dw\right]$

In particular:

$$[L_m, L_n] = \frac{1}{(2\pi i)^2} \oint_0 dw \oint_w dz\, z^{m+1}\, w^{n+1}\, T(z)\, T(w)$$



We use the (radially ordered!) OPE of T with itself:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.}$$

Here, we concentrate on the $\frac{c/2}{(z-w)^4}$ term. we need to expand z^{m+1} around w to third order: $z^{m+1} \underset{\substack{3^{rd}\text{ order}\\ \text{term}}}{\rightsquigarrow} (m+1)m(m-1)\frac{1}{3!} w^{m-2} (z-w)^3$

Doing the z integral first results in: $\frac{1}{2\pi i}\oint_0 dw \underbrace{w^{n+1} w^{m-2}}_{= w^{m+n-1}} \left(\frac{c}{12}(m+1)(m)(m-1)\right) = \frac{c}{12} m(m^2-1)\delta_{m+n,0}$

Together with the other terms, we have:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$ ← Virasoro algebra

$$[L_m, \bar{L}_n] = 0$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}.$$

This is the 'quantum version' of the Witt algebra of the gen. of the 'inf.' conf. transformations $l_n = -z^{n+1}\partial_z$. The l_n act on functions, the L_n of states of a CFT, with the 'central extension'.

Again, the $L_0, L_{\pm 1}$ form a subalgebra: $[L_{\pm 1}, L_0] = \pm L_{\pm 1}$; $[L_{+1}, L_{-1}] = 2L_0$.

We go back to the operators $L_n = \frac{1}{2\pi i}\oint_C z^{n+1}\, T(z)\, dz$, and consider

$L_n |0\rangle$; assume no fields ~~by~~ lie within C (which is ~~a~~ around the origin).

For $n \geq -1$, C can be shrunk to zero: $L_n |0\rangle = 0$ for $n \geq -1$

If there is a field ϕ_j (with scaling dimension h_j) at the origin, we have:

$$L_0 |\phi_j\rangle = \left[\frac{1}{2\pi i}\oint_C z\, T(z)\, \phi_j(0)\right] |0\rangle$$

The $\frac{1}{z^2}$ term ~~reads~~ of the OPE reads: $T(z)\, \phi_j(0) \sim \frac{h_j}{z^2}\phi_j(0) + \frac{1}{z}\partial_{z'}\phi_j(z')\Big|_{z'=0} + \text{reg}$

(~~if~~ here it's assumed ϕ_j is primary, see below).

So we have $L_0 |\phi_j\rangle = h_j\, |\phi_j\rangle$. Because $L_n^\dagger = L_{-n}$, in part. or $L_0^\dagger = L_0$, L_0 has a real spectrum.

So, the h_j's are <u>real</u>.