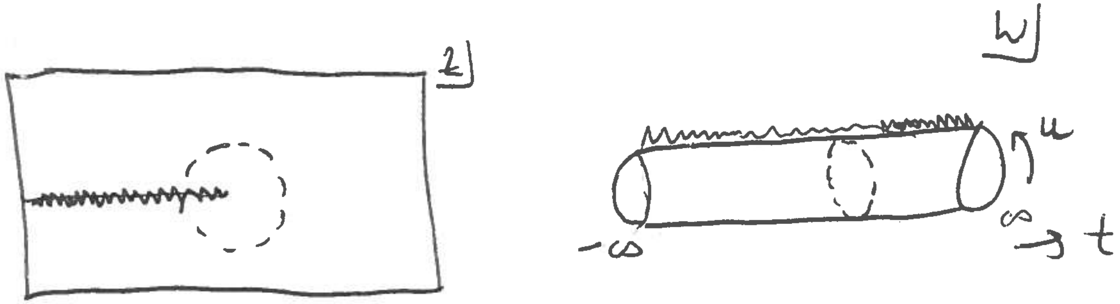
We perform a conf. transformation that introduces a length scale: plane $\to$ cylinder (strip)

$$z \to w = \frac{L}{2\pi} \log(z) \qquad z = e^{\frac{2\pi}{L} w} \qquad w = t + iu$$



One calculates: $\{w; z\} = \frac{1}{2z^2}$ $\quad \left(w' = \frac{L}{2\pi z} \right)$

On the plane, we have $\langle T \rangle_{plane} = 0$ (trans. invariance)

$$\langle T(z) \rangle_{plane} = \left(\frac{L}{2\pi}\right)^2 \frac{1}{z^2} \langle T(w) \rangle_{cyl} + \frac{c}{12} \frac{1}{2z^2}$$

$$\langle T(w) \rangle_{cyl} = \left(\frac{2\pi}{L}\right)^2 \left(z^2 \langle T(z) \rangle_{plane} - \frac{c}{24} \right) \Rightarrow \langle T(w) \rangle_{cyl} = -\frac{c\pi^2}{6L^2}$$

The finite size effects give a contribution to the energy (cf Casimir effect).

$$F_L = \underbrace{f_0 L}_{\text{bulk contribution}} - \underbrace{\frac{\pi c}{6L}}_{\text{Casimir contribution}}$$

F_L: free energy per unit length

(more on this later)

Specific heat : $C_V = \frac{c\pi}{3} \frac{k_B^2 T}{\hbar v_s}$: c measures the 'degrees of freedom';

$c = 1$: one boson

$c = \frac{1}{2}$: one real (majorana) fermion.

Example of a CFT: the free, massless boson.

This time, we have an action:

$$S(\varphi) = \frac{g}{2}\int(\partial_\nu\varphi)(\partial^\nu\varphi)\, d^2x = g\int dz\, d\bar z\, (\partial_z\varphi)(\partial_{\bar z}\varphi)$$

The action is invariant under $z \to z' = f(z)$

$$\partial_z = f'(z)\,\partial_{z'} \;;\; \partial_{\bar z} = \overline{f'(z)}\;\partial_{\bar z'}$$

$$d^2z = |f'(z)|^{-2} d^2z'$$

The eq. of motion: $\left(\frac{\delta \mathcal{L}}{\delta\varphi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\nu\varphi)}\right) = 0$ read: $\partial_{\bar z}\partial_z\varphi(z,\bar z) = 0$, so

$\partial_z\varphi(z)$ is holomorphic $\partial_{\bar z}\varphi(\bar z)$ is anti holomorphic

The stress energy tensor: $T_{\nu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial^\nu\varphi)}(\partial_\nu\varphi) - g_{\nu\nu}\mathcal{L}$ gives:

$$T(z) = -2\pi g (\partial_z \varphi)^2 \quad \text{and} \quad \bar{T}(\bar{z}) = -2\pi g (\partial_{\bar{z}} \varphi)^2$$

Important: $\varphi(z,\bar{z})$ is <u>not</u> a primary field:

The propagator is given by: $\langle \varphi(z,\bar{z})\, \varphi(w,\bar{w}) \rangle = \frac{-1}{4\pi g} \log|z-w|^2 + \text{const.}$

All kinds of problems w/ divergences, etc.

We focus on set of fields by taking derivatives; picking the holomorphic field $\partial_z \varphi(z)$:
$= \partial\varphi$

$$* \quad \langle \partial\varphi(z)\, \partial\varphi(w) \rangle = \frac{-1}{4\pi g} \frac{1}{(z-w)^2}$$

So, we see that $\partial\varphi$ could be primary w/ $h_\varphi = 1$

To def T properly, we need to regularise because of $(*)$:

$$T(z) = -2\pi g :\partial_z\varphi\, \partial_z\varphi: \;= -2\pi g \lim_{w\to z} \left(\partial\varphi(z)\partial\varphi(w) - \langle \partial\varphi(z)\partial\varphi(w) \rangle \right)$$

We can now calculate the OPE of $T(z)$ w/ $\partial\varphi(w)$, by using Wick's theorem.

The OPE will be used in correlators, ~~and~~ so we'll use $\langle :X: \rangle = 0$

stop writing lim

$$\lim_{z \to w} T(z)\, \partial_w \varphi(w) = -2\pi g :\partial_z \varphi\, \partial_z \varphi: \partial_w \varphi(w)$$

$$\sim -4\pi g :\partial\varphi(z)\, \partial\varphi(z): \partial\varphi(w)$$

$$\sim \frac{-4\pi g}{-4\pi g} \frac{\partial\varphi(z)}{(z-w)^2} = \frac{\partial\varphi(z)}{(z-w)^2} = 1 \cdot \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial_w(\partial\varphi(w))}{(z-w)} + \text{reg}$$

So, the OPE is that of a primary field w/T with $h=1$

Now look at $T(z)T(w)$, ~~and~~ check if it takes the form $T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{reg}.$

We need: $\lim_{z \to w} (-2\pi g)^2 :\partial\varphi(z)\partial\varphi(z): \; :\partial\varphi(w)\partial\varphi(w):$

$$\sim 2(-2\pi g)^2 \langle \partial\varphi(z)\partial\varphi(w)\rangle^2 + 4(-2\pi g)^2 \langle \partial\varphi(z)\partial\varphi(w)\rangle :\partial\varphi(z)\partial\varphi(w):$$

double cont.   single cont.

$$= \frac{1/2}{(z-w)^4} - \frac{4\pi g :\partial\varphi(z)\partial\varphi(w):}{(z-w)^2} \Rightarrow \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.}$$

We can construct other primary fields from $\varphi(z)$, namely $V_\alpha(z) = e^{i\alpha\varphi(z)}$, which have scaling dimensions $h_\alpha = \frac{\alpha^2}{8\pi g}$ (exercise)

Other important theory: free, massless fermion: $S = \frac{g}{2}\int d^2x \, \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi$, w/ $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Takes the form $S = g\int d^2z \, (\bar\psi \partial \bar\psi + \psi \bar\partial \psi)$ w/ $\Psi = (\psi, \bar\psi)$.

Gives: eq of motion: $\partial\bar\psi = \bar\partial\psi = 0$; $c = \frac{1}{2}$.

Operator formalism & radial quantisation

Goal: go from fields & path integrals to Hilbert spaces and operators.

Gain: can use algebraic methods, group theory, and analysis (contour integrals

Operator formalism: not manifestly Lorentz invariant, so there's a choice of reference frame:

In euclidian space: can chose time to be '<u>radial</u>'.

⇒ equal time commutators ↔ Contour integrals

⇒ Gen. of conf. transformations → (quantum) operators

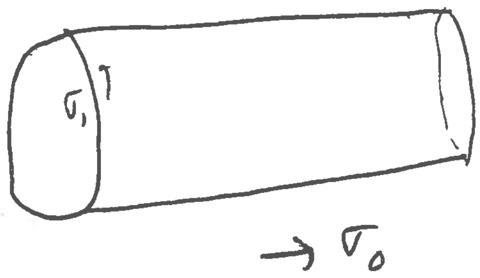
↳ satisfy Virasoro algebra

↳ minimal models.

↳ descendent fields

Lets take cylinder as our 'starting' space-time

~~cross~~ circumference: L

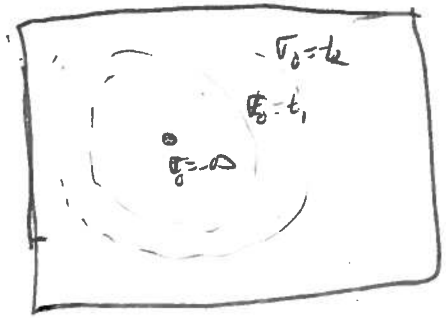


$(\sigma_0, 0) \sim (\sigma_0, L)$: identified

Euclidean space: one complex coordinate $\zeta = \sigma_0 + i\sigma_1$: Map cylinder to plane (Rieman sphere):

$\sigma_0 = it$ ← Minkowski

$$z = e^{2\pi \zeta / L}$$



Time translations on cylinder: $\sigma_0 \to \sigma_0 + a$ are dilatations on plane:

$$z \to e^{2\pi a/L} z$$

Dilatation generator L_0 corresponds to the Hamiltonian on the plane.

We denote the vacuum by $|0\rangle$, and act w/ operators to generate the Hilbert space:

The 'in' states correspond to the fields: $\phi_j = \lim_{t\to-\infty} \phi_j(x,t)$, so

$$|\phi_j\rangle_{in} = \lim_{z,\bar z\to 0} \phi_j(z,\bar z)|0\rangle$$

[Assume: free theory at $t=\pm\infty$]

We need innerproducts, so we need hermitian conjugate.

time t is invariant, so $\sigma_0 = it$ implies that $\sigma_0 \to -\sigma_0$, or $z \to 1/z^*$ under herm. conj.

On the real surface $\bar z = z^*$, we define $\left[\phi(z,\bar z)\right]^\dagger = \bar z^{-2h} z^{-2\bar h} \phi\left(\frac{1}{\bar z}, \frac{1}{z}\right)$

With this def we have that ${}_{out}\langle\phi_j| = |\phi_j\rangle_{in}^\dagger$ has correct inner product w/ $|\phi_j\rangle_{in}$:

$${}_{out}\langle\phi_j|\phi_j\rangle_{in} = \lim_{z,\bar z\to 0} \langle 0|\phi_j\left(\frac{1}{\bar z},\frac{1}{z}\right)\phi_j(0,0)|0\rangle \bar z^{-2h} z^{-2\bar h} = \lim_{\xi,\bar\xi\to\infty} \langle 0|\phi(\bar\xi,\xi)\phi(0,0)|0\rangle \bar\xi^{2h}\xi^{2\bar h},$$

which is independent of ξ so well defined.

<u>Aside</u>: $\hat{\varphi}(w,\bar{w})$ is the oper. in w coor, for which the point at $z=\infty$ corresponds to $w\to 0$

$$z=\frac{1}{w}$$

$$z=f(w)=\frac{1}{w}$$

$$\hat{\varphi}(w,\bar{w})=\varphi(f(w),\bar{f}(\bar{w}))\left(\partial f(w)\right)^{h}\left(\bar{\partial}\bar{f}(\bar{w})\right)^{\bar{h}}$$

$$=\varphi\left(\frac{1}{w},\frac{1}{\bar{w}}\right)(-w^{-2})^{h}(-\bar{w}^{-2})^{\bar{h}}$$

$${}_{out}\langle\phi_j| := \lim_{w,\bar{w}\to 0}\langle 0|\hat{\varphi}(w,\bar{w}) \qquad \text{def.}$$

$$=\lim_{z,\bar{z}\to 0}\langle 0|\,\varphi\left(\frac{1}{z},\frac{1}{\bar{z}}\right)z^{-2h}\bar{z}^{-2\bar{h}} \qquad \text{(up to a phase)} \qquad \text{conf invariance}$$

$$=\lim_{z,\bar{z}\to 0}\langle 0|\left[\varphi(\bar{z},z)\right]^{+} \qquad \text{def of h.c.}$$

$$=\left[\lim_{z,\bar{z}\to 0}\varphi(\bar{z},z)\,|0\rangle\right]^{+}=|\phi_j\rangle_{in}^{+}$$