The Stress-energy tensor & conformal Ward identity.

Takes into account the effects of conf. transformations that are not globally defined.

T is defined in terms of the variation δ of the action under infinitesimal conf. transformations

$x^\mu \to x^\mu + \varepsilon^\mu$: $\delta S = \frac{-1}{2\pi} \int T_{\mu\nu} \partial^\nu \varepsilon^\mu \, d^2x$ (*)

(**) We assume S is invariant under translations, rotations and dilatations

Consider rotations: $\varepsilon^1 = \alpha x^2$; $\varepsilon^2 = -\alpha x^1$

$$0 = \delta S = \frac{-\alpha}{2\pi} \int (T_{12} - T_{21}) \, d^2x \Rightarrow T_{\mu\nu} \text{ is symmetric.}$$

Dilatations : $T_{11} + T_{22} = 0 \Rightarrow T^\mu{}_\mu = 0$, T is traceless

Translations : $\partial^\mu T_{\mu\nu} = 0$

(*) & (**) imply conformal invariance: take $\varepsilon^\nu = b^\nu x^2 - 2x^\nu (b \cdot x)$ This gives

$$T_{\mu\nu} \partial^\mu \varepsilon^\nu = T_{\mu\nu} (2 b^\nu x^\mu - 2 b^\mu x^\nu) - 2 T^\mu{}_\mu (b \cdot x) = 0 \Rightarrow \delta S = 0$$

(*): Not valid for systems with long range interactions

Express T in complex coordinates. $z = x^1 + i x^2$; $\bar{z} = x^1 - i x^2$

$$T_{zz} = \frac{\partial x^\mu}{\partial z}\frac{\partial x^\nu}{\partial z} T_{\mu\nu} = \frac{1}{4}\left(T_{11} - i\left(T_{12} + T_{21}\right) - T_{22}\right)$$

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}\left(T_{11} + i\left(T_{12} - T_{21}\right) + T_{22}\right) = 0$$

Conservation $\partial^\mu T_{\mu\nu} = 0$ gives

$$0 = g^{\alpha\mu}\partial_\alpha T_{\mu\nu} \overset{\nu = z}{\Rightarrow} 2\big(\partial_z \underbrace{T_{\bar{z}z}}_{=0} + \partial_{\bar{z}} T_{zz}\big) = 2\partial_{\bar{z}} T_{zz} = 0 \Rightarrow \underbrace{T_{zz}(z,\bar{z}) = T_{zz}(z) = T(z)}_{\text{holomorphic part}}$$

$$g^{\alpha\mu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}_{\alpha\mu}$$

Similarly, we have: $T_{\bar{z}\bar{z}}(z,\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z}) = \bar{T}(\bar{z})$: anti-holomorphic part.

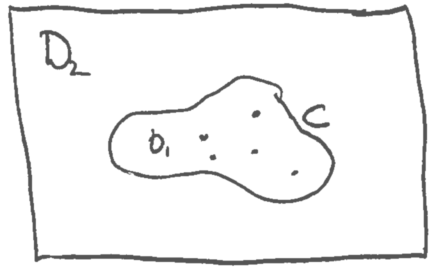
On the level of 2d quantum conf. field theory, we will assume these properties to hold on the level of the correlation functions.

We consider an arbitrary, inf. conformal transformation; (can not be small everywhere!)

Total space: D. Transformation acts on D_1, but is the identity on D_2, ($D = D_1 \cup D_2$)

On the boundary C, there is an infinitesimal discontinuity.

D



The action transforms as:

$$\delta S = -\frac{1}{2\pi}\int_{D_1} T_{\mu\nu}\,\partial^\nu \varepsilon^\mu\, d^2x = \frac{1}{2\pi}\int_{D_1} \big(\underbrace{\partial^\nu T_{\mu\nu}}_{0 \text{ on } D_1}\big)\varepsilon^\mu\, d^2x + \frac{1}{2\pi}\oint_C T_{\mu\nu}\varepsilon^\mu n^\nu\, dl$$

$\hookrightarrow$ outward normal to C

$$= \dots = \frac{1}{2\pi i}\oint_C \varepsilon(z)\, T(z)\, dz + c.c.$$

We now concentrate on the correlation functions of <u>primary</u> <u>fields</u> in region D_1

$\varphi_i(z_i)$; w/ $z_i \in D_1$

Recall: $\varphi(z) \to \varphi'(z') = \left(\frac{\partial f(z)}{\partial z}\right)^h \varphi(f(z))$ (holomorphic part)

$z' = f(z)$

(***) or: $\langle \varphi_1(z_1) \dots \varphi_n(z_n)\rangle_S = \prod_i \left(\frac{\partial f(z_i)}{\partial z_i}\right)^{h_i} \langle \varphi_1(f(z_1)) \cdots \varphi_n(f(z_n))\rangle_{S+\delta S}$

Variation of the primary fields: $\delta_\varepsilon\ \varphi(z) = [h(\partial_z\varepsilon) + \varepsilon\partial_z]\,\varphi(z)$

For global conf transformations: $\delta S = 0$, so we get: $\delta_\varepsilon \langle \varphi_1(z_1)\cdots\varphi_n(z_n)\rangle_S = 0$

In general, arb. conf transformations: need to expand $e^{-S-\delta S} = (1-\delta S)\, e^{-S}$ when calculating correlation functions:

$$\delta_\varepsilon \langle \varphi_1(z_1)\cdots\varphi_n(z_n)\rangle_S = \frac{1}{2\pi i}\oint_C dz\, \varepsilon(z) \langle T(z)\, \varphi_1(z_1)\cdots\varphi_n(z_n)\rangle_S$$

This is the conformal ward identity. By deforming the contour:

we have (RHS):

$$= \sum_j \langle \varphi_1(z_1)\cdots\left(\frac{1}{2\pi i}\oint_{z_j} dz\, \varepsilon(z)\, T(z)\, \varphi_j(z_j)\right)\cdots\varphi_n(z_n)\rangle$$

For LHS, we use Cauchy: $f(a) = \frac{1}{2\pi i}\oint_a dz \frac{f(z)}{z-a}$

$$\sum_j \langle \varphi_1 \cdots \delta_\varepsilon\varphi_j \cdots \varphi_n\rangle = \sum_j \frac{1}{2\pi i}\oint_{z_j} dz\, \varepsilon(z) \langle \varphi_1 \cdots \left(\frac{h_j\varphi_j(z_j)}{(z-z_j)^2} + \frac{\partial_{z_j}\varphi_j(z_j)}{(z-z_j)}\right)\cdots\varphi_n\rangle$$

We find the following local behavior of $T(z)\,\varphi_j(z_j)$:

$$\lim_{z\to z_j} T(z)\,\varphi_j(z_j) = \frac{h_j\,\varphi_j(z_j)}{(z-z_j)^2} + \frac{\partial_{z_j}\varphi_j(z_j)}{(z-z_j)} + \text{reg}$$
. This is an 'operator product expansion'.

This ~~op~~ OPE is often used to define a primary field.

Using that the conf. ward identity is valid for arbitrary ε, we obtain:

$$\langle T(z)\,\varphi_1(z_1)\cdots\varphi_n(z_n)\rangle = \sum_j \left(\frac{h_j}{(z-z_j)^2} + \frac{\partial_{z_j}}{(z-z_j)}\right)\langle\varphi_1(z_1)\cdots\varphi_n(z_n)\rangle$$

We see that $T(z)$ has scaling dimension $2 \doteq h_T$; $\bar{h}_T = 0$

For $\varepsilon = 1, z, z^2$; we have that $\delta_\varepsilon\langle\varphi_1\cdots\varphi_n\rangle = 0$; the conf ward identity then gives that: $T(z) \sim \frac{1}{z^4}$ for $z\to\infty$. The correlator $\langle T(z)\,T(z_1)\rangle = \dfrac{c/2}{(z-z_1)^4}$.

The coefficient c is fixed, because T_{zz} is defined by $(*)$, so we can not rescale T_{zz}.

So, c is not fixed by conf symmetry it depends on the microscopics of the models.

T itself is <u>not</u> a primary field. Using the conf ward ident on $\delta_\varepsilon \langle T\, \varphi_1 \cdots \varphi_n \rangle$,

we get: $$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{\overset{h=h_T}{2}\, T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{reg}$$

c is called the central charge, or conf. anomaly number.

The transformation behaviour of T gives a clue about the interpretation of c:

$$\delta \langle T(z_1) \cdots \rangle = \frac{1}{2\pi i} \oint_C \varepsilon(z) \langle T(z)\, T(z_1) \cdots \rangle\, dz + \text{c.c.}$$

$$= \frac{1}{2\pi i} \oint_C \varepsilon(z) \left\langle \left(\frac{c/2}{(z-z_1)^4} + \frac{2T(z_1)}{(z-z_1)^2} + \frac{\partial_{z_1} T(z_1)}{(z-z_1)} + \text{reg} \right) \cdots \right\rangle + \text{c.c.}$$

which gives: $$\delta_\varepsilon T(z) = \underset{\substack{\uparrow \\ \text{additional} \\ \text{contribution}}}{\frac{c}{12} \frac{\partial^3 \varepsilon(z_1)}{\partial z_1}} + \underbrace{2\,T(z_1)\left(\partial_{z_1} \varepsilon(z_1)\right) + \varepsilon(z_1)\, \partial_{z_1} T(z_1)}_{\text{transformation behaviour of a primary field}}$$

This inf. transformation can be integrated up to $z \to z' = f(z)$:

$$\square \quad T(z) \to (\partial f)^2\, T(f(z)) + \frac{c}{12}\{f;z\}$$

$$\{f;z\} = \frac{(\partial_z f)^3}{(\partial_z f)} - \frac{3}{2}\left(\frac{\partial_z^2 f}{\partial_z f}\right)^2$$

Schwartzian derivative:

* vanishes for global conf. transformations
* has correct composition property under $z \to w \to u$

$$\{u;z\} = \{w;z\} + \left(\frac{\partial w}{\partial z}\right)^2 \{u;w\}$$

$\square$: One can check that $\square$ gives the correct inf. transformation.