Conformal invariance in 2 dimensions.

The transformation of the metric: $g'_{\mu\nu}(x') = \dfrac{\partial x^\alpha}{\partial x'^\mu} \dfrac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$

Line element: $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$; $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu$

Conformal transformations: $g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x)$

We found the condition: $\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}(\partial \cdot \varepsilon)\, g_{\mu\nu}$; take $g_{\mu\nu} = \delta_{\mu\nu}$.

then, in $d=2$: $\mu = \nu = 1$: $2\partial_1 \varepsilon_1 = \partial_1 \varepsilon_1 + \partial_2 \varepsilon_2 \Rightarrow$ $\boxed{\begin{array}{l} \partial_1 \varepsilon_1 = \partial_2 \varepsilon_2 \\ \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1 \end{array}}$ Cauchy-Riemann! eq^n for holomorphic functions!

$\mu \neq \nu$

So we write $\varepsilon(z) = \varepsilon^1 + i\varepsilon^2$; $\bar{\varepsilon}(\bar{z}) = \varepsilon^1 - i\varepsilon^2$ with $z = x^1 + ix^2$ $\quad x^1 = \frac{1}{2}(z+\bar{z})$

$\bar{z} = x^1 - ix^2 \quad x^2 = \frac{1}{2i}(z-\bar{z})$

So we have: $\partial_z = \frac{1}{2}(\partial_{x^1} - i\partial_{x^2})$ $\quad \partial_{x^1} = \partial_z + \partial_{\bar{z}}$

$\partial_{\bar{z}} = \frac{1}{2}(\partial_{x^1} + i\partial_{x^2})$ $\quad \partial_{x^2} = i(\partial_z - \partial_{\bar{z}})$

And:

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} ; \quad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \Rightarrow ds^2 = dx^1dx^1 + dx^2dx^2 = dz\,d\bar{z}$$

$\mu, \nu = z, \bar{z}$

Under conformal transformations, i.e., analytic functions, we have

$z \to f(z)$; $\bar{z} \to \bar{f}(\bar{z})$: $ds^2 = dz\,d\bar{z} \to \left|\frac{\partial f}{\partial z}\right|^2 dz\,d\bar{z}$, with

$$\Omega(z) = \left|\frac{\partial f}{\partial z}\right|^2$$

Which analytic maps are globally well defined?

$$f(z) = \frac{az+b}{cz+d}.$$ Invertibility requires: $ad-bc \neq 0$. By rescaling of a, b, c, d, we can set: $ad-bc=1$

Why?

$f(z)$ can not have branch points ($f(z)$ needs to be well defined)

" " " essential singularities ($f(z)$ sweeps over $\mathbb{C}$ in arb. small neighbourhood around z_c, $\Rightarrow$ not invertible)

So, we have: $f(z) = \frac{P(z)}{Q(z)}$, w/ $P(z)$ & $Q(z)$ polynomials.

$P(z)$ can not have 2 or more zeros (invertibility), nor higher order zeros (say n): (their neighbourhood around zero wraps around 0 n times).

Same is true for $Q(z)$ around ∞; so both $P(z)$, $Q(z)$ are first order polynomials!

Set: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Translations: $*\ z' = z + a$, w/ $a = a^1 + i a^2$: $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

Rotation $*\ z' = e^{i\alpha} z \qquad \alpha \in \mathbb{R} \qquad A = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$

Dilatation $*\ z' = \lambda z \qquad \lambda \in \mathbb{R} \qquad A = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}$

SCT: $*\ \vec{x}' = \dfrac{\vec{x} + \vec{b}\,|\vec{x}|^2}{1 + 2(\vec{b}\cdot\vec{x}) + b^2 |\vec{x}|^2}$, take $b = b^1 - i b^2$

$$z' = \frac{z + \bar{b} z \bar{z}}{1 + \bar{b}\bar{z} + bz + b\bar{b} z\bar{z}} = \frac{z(1+\bar{b}\bar{z})}{(1+\bar{b}\bar{z})(1+bz)} = \frac{z}{1+bz} \rightsquigarrow A = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

Local conformal transformations:

$z \to z + \varepsilon_n(z)$; $\bar{z} \to \bar{z} + \bar{\varepsilon}_n(\bar{z})$, with $\varepsilon_n(z) = -z^{n+1}$

Then, ~~we~~ fields $\phi(z,\bar{z})$ transform as:

$$\phi(z,\bar{z}) \to \phi(z,\bar{z}) - z^{n+1}\partial_z \phi(z,\bar{z}) - \bar{z}^{n+1}\partial_{\bar{z}} \phi(z,\bar{z})$$

So, the inf. generators are given by:

$$l_n = -z^{n+1}\partial_z$$

$$\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$$

Commutation relations:

$$[l_n, l_m] = -z^{n+1}\partial_z(-z^{m+1}\partial_z) = (m+1)z^{n+m+1}\partial_z + z^{n+m+2}\partial_z^2$$

$$- (n+1)z^{n+m+1}\partial_z - z^{n+m+2}\partial_z^2$$

$$= -(n-m)z^{n+m+1}\partial_z = (n-m)\,l_{n+m}$$

$$[l_n, \bar{l}_m] = 0$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\,\bar{l}_{n+m}$$

What are the <u>non-singular</u> conf. transformations?

Gen. transfon: $v(z) = -\sum_n a_n l_n = \sum_n a_n z^{n+1} \partial_z$

$v(z)$ is nonsingular at $z=0 \Rightarrow a_n = 0$ for $n \leq -2$

Around infinity: $z = -\frac{1}{\omega}$; $v(z) = \sum_n a_n \left(-\frac{1}{\omega}\right)^{n+1} \left(\frac{dz}{d\omega}\right)^{-1} \partial_\omega = \sum_n a_n \left(-\frac{1}{\omega}\right)^{n-1} \partial_\omega$

So we need: $a_n = 0$ for $n \geq 2$

Only l_{-1}, l_0, l_1 are globally well-defined, and form a sub algebra!

$l_{-1}, \bar{l}_{-1}$: translations

$l_0 + \bar{l}_0$: dilatations, w/ scale dimension $\Delta = h + \bar{h}$

$i(l_0 - \bar{l}_0)$: rotation $\rightarrow$ spin $s = h - \bar{h}$

$l_{+1}, \bar{l}_{+1}$: special conf. transformation

Because $[l_n, \bar{l}_m] = 0$, we will treat the z's and $\bar{z}$'s as independent complex variables, so instead of $(x^1, x^2) \in \mathbb{R}^2$, we consider $(z, \bar{z}) \in \mathbb{C}^2$. In $\mathbb{C}^2$, the 'real surface' is defined as $\bar{z} = z^*$, which we impose at the end of calculations. But, we often focus on the holomorphic dependence $(z_1, \dots, z_n)$ of correlation functions!

Characterization of a CFT

A CFT is specified by the correlators of an (∞) set of (local) operators $A(z,\bar{z})$

Properties: 1) $A(z,\bar{z})$ is local, and so are its derivatives

2) Set of 'quasi-primary' fields $\{\Phi(z,\bar{z})\} \subset \{A(z,\bar{z})\}$ transform under globally def. conf. transformations $z \to w(z) = \frac{az+b}{cz+d}$

$$\Phi(z,\bar{z}) \to \left(\frac{\partial w}{\partial z}\right)^{h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(w(z), \bar{w}(\bar{z})) \qquad (*)$$

For correlators, we have:

$$\langle \Phi_1(z_1,\bar{z}_1) \cdots \Phi_n(z_n,\bar{z}_n)\rangle = \prod_{i=1}^{n} \left(\frac{\partial w_i}{\partial z_i}\right)^{h_i} \left(\frac{\partial \bar{w}_i}{\partial \bar{z}_i}\right)^{\bar{h}_i} \langle \Phi_1(w_1(z_1), \bar{w}_1(\bar{z}_1)) \cdots \Phi_n(w_n(z_n), \bar{w}_n(\bar{z}_n))\rangle \qquad (**)$$

3) The other fields in $\{A\}$ are combinations of the fields $\{\Phi\}$ and their derivatives

4) There is a vacuum state $|0\rangle$

5) The set of 'primary' fields $\{\phi(z,\bar{z})\} \subset \{\Phi(z,\bar{z})\}$ are the fields that transform as

(*) under ~~any~~ any conf. transformation (so primary $\Rightarrow$ quasi-primary)

The number of primary fields is often finite. One calculates all correlation functions of the primaries, to 'solve' the theory. From those, one obtains the other correlators via the stress-energy tensor T. (next lecture)

Properties of quasi-primary fields: global conf. sym. are gen. by $z \to f(z) = z + \varepsilon(z)$, w/

$\varepsilon(z) = 1, z, z^2$

$$\Phi(z,\bar z) \to \left(\frac{\partial f}{\partial z}\right)^{h}\left(\frac{\partial \bar f}{\partial \bar z}\right)^{\bar h} \Phi(f(z), \bar f(\bar z))$$ or infinitesimally:

$$\delta\Phi(z,\bar z) = \Phi'(z,\bar z) - \Phi(z,\bar z) = \left[h(\partial_z \varepsilon(z)) + \varepsilon(z)\partial_z\right]\Phi(z,\bar z) + \left[\bar h(\partial_{\bar z}\bar\varepsilon(\bar z)) + \bar\varepsilon(\bar z)\partial_{\bar z}\right]\Phi(z,\bar z)$$

The infinitesimal form of (**) ~~reads~~ (holomorphic part):

$$0 = \delta_\varepsilon \langle \Phi_1 \cdots \Phi_n \rangle = \sum_{i=1}^{n}\left[h_i(\partial_{z_i}\varepsilon(z_i)) + \varepsilon(z_i)\partial_{z_i}\right]\langle \Phi_1 \cdots \Phi_n \rangle = 0$$

For $\varepsilon = 1$:

$$\sum_i \partial_{z_i} \langle \Phi_1(z_1) \cdots \Phi_n(z_n)\rangle = 0$$

$\varepsilon = z$:

$$\sum_i (h_i + z_i \partial_{z_i}) \langle \Phi_1(z_1) \cdots \Phi_n(z_n)\rangle = 0$$

$\varepsilon = z^2$:

$$\sum_i (2 z_i h_i + z_i^2 \partial_{z_i}) \langle \Phi_1(z_1) \cdots \Phi_n(z_n)\rangle = 0$$