Exercises CFT-course fall 2018, set 9.

Due on january 18, 2019.

1. Character formulas for the Ising model.

a. Calculate the character of the vacuum and ψ sector by calculating the partition function of states of the form

$$\psi_{-(n-1)/2-p_n} \cdots \psi_{-3/2-p_2} \psi_{-1/2-p_1} |0\rangle \ ,$$

where $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$.

b. Repeat exercise a. for the σ sector, by considering

$$\psi_{-(n-1)-p_n} \cdots \psi_{-1-p_2} \psi_{-p_1} |\sigma\rangle \ ,$$

again with $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$.

Answer:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} = \prod_{n=1}^{\infty} (1+q^n) = \sum_{n\geq 0 \text{ even}} \frac{q^{n(n-1)/2}}{(q)_n} = \sum_{n \geq 1 \text{ odd}} \frac{q^{n(n-1)/2}}{(q)_n} \ . \tag{1}$$

c. Obtain the last two equalities in (1), by making use of an identity due to Cauchy (which you don't have to prove):

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} x^n}{(q)_n} = \prod_{n=0}^{\infty} (1 + x q^n) \ ,$$

and considering the role of the zero mode ψ_0.

2. Modular transformation properties of the Ising characters.

The following definitions and identities are given ($q = e^{2\pi i \tau}$):

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2} = 2q^{1/8} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2 \qquad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-1/2})^2 \qquad \eta(\tau)^3 = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)$$

$$\theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-1/2})^2$$

a. Show, by making use of the results of exercise 1., that the characters of the Ising model are given by

$$\chi_0 = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \chi_{1/2} = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \quad \chi_{1/16} = \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}$$

b. Calculate the transformation properties of the Ising characters under the transformation $\tau \to -1/\tau$, by making use of the Poisson resummation formula:

$$\sum_{n\in\mathbb{Z}} e^{-\pi a n^2 + bn} = \frac{1}{\sqrt{a}} \sum_{k\in\mathbb{Z}} e^{(b+2\pi i k)^2/(4\pi a)}$$