Exercises CFT-course fall 2018, set 5.

Due on december 7, 2018.

1. The Virasoro algebra.

   Show that the Virasoro algebra has the following form

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \ ,$$

   where $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$.

2. Mode expansion of primary fields.

   In analogy to the stress-energy tensor, we define a mode expansion of (holomorphic) primary fields $\phi(z)$ as

$$\phi(z) = \sum_{n\in\mathbb{Z}-h} \phi_n z^{-n-h} \ , \qquad \phi_n = \oint \frac{dz}{2\pi i} z^{h+n-1}\phi(z) \ .$$

a. Show that regularity requires $\phi_n|0\rangle = 0$ for $n \geq -h+1$.

b. Calculate the commutator $[L_n, \phi_m]$ by first calculating $[L_n, \phi(w)]$. Show that $L_0\phi_{-h}|0\rangle = h|h\rangle$, as expected.

3. The OPE of descendent fields.

   Show that the OPE of $T(z)$ with $\phi^{(-1)}(w) = \partial\phi(w)$, where $\phi(w)$ is a primary field of dimension h, takes the form

$$T(z)\partial\phi(w) = \frac{2h\phi(w)}{(z-w)^3} + \frac{(h+1)\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w} + \text{reg} \ .$$